Gleason's problem in infinite dimension

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Gleason's problem was raised in 1964 by Andrew Gleason [8] when he asked whether in the uniform algebra $A_u(B^n)$ of functions defined on the open unit ball $B^n \subset \mathbb{C}^n$ that are holomorphic and uniformly continuous the coordinate functions generate the maximal ideal of the functions that vanish at the origin.

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The affirmative answer to Gleason's original question was given by Leibenzon who proved the following:

If $a \in B^n$ and $f \in A_u(B^n)$, then there are $f_1, \dots, f_n \in A_u(B^n)$ such that

$$f(z) - f(a) = \sum_1^n (z_i - a_i) f_i(z). \quad (1.1)$$

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This problem and some variants of it have been studied by many authors, see for instance [12] VII §4 or [1, 2, 4, 9]. However the setting has been, to my knowledge, always finite dimensional.

Notation

For an open subset B of a complex Banach space E, and a complex Banach space Y, we put

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The Banach space of all bounded mappings in $H(B;Y)$ endowed with the norm $\|f\| := \sup_{z\in B} \|f(z)\|$ is denoted by $H^\infty(B;Y)$ and $A_u(B;Y)$ is the closed subspace of $H^\infty(B;Y)$ generated by the Y-valued polynomials on E.

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Remark

If the underlying space has a basis (e_n), the dual basis (π_n) does not generate (finitely) the ideal $\mathcal{I}$ of vanishing functions at 0 :

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Remark

If the underlying space has a basis (e_n), the dual basis (π_n) does not generate (finitely) the ideal $\mathcal{I}$ of vanishing functions at $0 :$ For the function $f(z) = \sum_{i=1}^{\infty} \left(\frac{\pi_i(z)}{2}\right)^i$, $\|z\| < 1$, there is no m such that $f = \sum_1^m \pi_i f_i$ since $f(e_{m+1}) \neq 0$.

Recall that for a convex subset B of the normed space (E, n) and a normed space F, a function $\kappa : B \to F$ is said to be *Hölder continuous of order* $\epsilon > 0$ if there is a constant $M > 0$ such that

$$\|\kappa(u) - \kappa(v)\| \leq Mn(u - v)^{\epsilon} \quad \forall\, u, v \in B.$$

Lemma

If the norm of the space (E, n) *is twice differentiable, except perhaps at* 0, *and its second derivative is bounded in every annulus* $\{x \in E : \frac{1}{2} \leq n(x) \leq \rho\}$, *then there is a convex function* $q : E \to \mathbb{R}$ *such that* $q^{-1}(]-\infty, 0[) = B_E$ *whose derivative is Hölder continuous of order* 1 *on every ball centered at* 0.

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The following is most likely well-known.

Example

The $\| \|_p$-norm of $L_p(\mu)$, $2 \leq p < \infty$, is twice differentiable, except perhaps at 0, and its second derivative is bounded on any annulus centered at the origin.

Theorem

Let E be a complex Banach space and let q be a convex function on E whose derivative is Hölder continuous on every ball. Put $B := q^{-1}(]-\infty, 0[)$ and suppose B is bounded.

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$$f(z) - f(a) = T(f)(a, z)(z - a) \qquad \forall z \in B \qquad \forall f \in H^\infty(B).$$

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In particular if E has a basis, given $a \in B$ and $f \in H^\infty(B)$, there is a bounded sequence $(f_n) \in H^\infty(B)$ such that for $z \in B$,

$$f(z) - f(a) = \sum (z_i - a_i) f_i(z) \tag{3.1}$$

where (z_i) is the sequence of coordinates of z with respect to the basis.

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A similar statement holds for $A_u(B)$.

Corollary

The above theorem holds for $B = B_{L_p(\mu)}$, *the unit ball of the* $L_p(\mu)$ *space*, $2 \leq p < \infty$.

Remark

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Indeed, the space $P_{wsc}(E)$ of weakly sequentially continuous polynomials is a closed subspace of $A_u(B_E) \subset H^\infty(B_E)$. Would that series be weakly convergent, P would belong to the weak closure of the finite type polynomials, hence $P \in \overline{P_{wsc}(E)}^w = P_{wsc}(E)$, so it would be weakly sequentially continuous on B_E.

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Recall that the convergence of the series

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is uniform on compact subsets of $E \times E^*$.Then since $(L - a) \times T_a(f)(L)$ is a compact set in $E \times E^*$, it follows that

$$< z - a, T_a(f)(z) >= \sum T_a(f)(z)(e_i)(z_i - a_i) = \sum f_i(z)(z_i - a_i)$$

is uniformly convergent on L.

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Recall that there is a Banach space $G^\infty(B)$ whose dual is isometrically isomorphic to $H^\infty(B)$ [11], and this is the duality used.

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The partial sums form a bounded subset in $H^\infty(B_E)$ because

$$|\sum_1^k f_i(z)(z_i - a_i)| = |\sum_1^k T(f)(a,z)(e_i)(z_i - a_i)| =$$

$$|T(f)(a,z)\left(\sum_1^k (z_i - a_i)e_i\right)|$$

$$\leq \|T(f)(a,z)\| \|\sum_1^k (z_i - a_i)e_i\| \leq 2K\|T_a(f)\|,$$

where K is the constant of the basis $\{e_i\}$.

Remark

The series (3.1) is $w^*(H^\infty(B), G^\infty(B))$ convergent .

Since the series converges on the total subset of $G^\infty(B)$ of the evaluations at points in B, and is a bounded, hence equicontinuous set of $H^\infty(B) = G^\infty(B)^*$, it is also pointwise, i.e., weak* convergent.

Definition

Let A be a vector subspace of a given algebra of holomorphic functions on B, $\mathcal{A}(B)$. We say that A is an **s-ideal** *in $\mathcal{A}(B)$ if for all open subsets D of arbitrary complex Banach spaces Y and for every $F \in H(B \times D)$ such that the functions $z \in B \overset{F_y}{\mapsto} F_y(z) := F(z, y)$ belong to A for all $y \in D$, one has that for any $g \in H(B; D)$ the function $z \mapsto F(z, g(z))$ is in A provided it is in $\mathcal{A}(B)$.*

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- Notice that in $\mathcal{A}(B)$ every S-ideal A is an ideal. Indeed:

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Let $g \in \mathcal{A}(B)$ and $f \in A$. Consider $F : B \times \mathbb{C} \to \mathbb{C}$ given by $F(z, y) := yf(z)$. Since $F_y = yf$ belongs to A for all $y \in D = \mathbb{C}$, it follows that $F(z, g(z)) = g(z)f(z)$ is a function in A.

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- For each subset $S \subset B$ the (ideal) $S^{\perp} := \{f \in \mathcal{A}(B) : f_{|_S} = 0\}$ is also an s-ideal in $\mathcal{A}(B)$:

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Let $F \in H(B \times D)$ be such that for all $y \in D$, the functions $z \in B \overset{F_y}{\mapsto} F_y(z) := F(z,y)$ belong to $S^{\perp}$, that is, $F(z,y) = 0 \; \forall z \in S$. If $g \in H(B;D)$ is such that $F(\cdot, g(\cdot)) \in \mathcal{A}(B)$ and $z \in S$, $F(z,g(z)) = F_{g(z)}(z) = 0$, thus $F(\cdot, g(\cdot)) \in S^{\perp}$.

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- Not every maximal ideal is an s-ideal:

Definition

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Consider the algebra $A_u(B_{\ell_2})$. The sequence of evaluations $\{\delta_{\frac{n}{n+1}e_n}\}$ has a cluster point χ in the spectrum when endowed with the Gelfand topology. The kernel $A := Ker(\chi)$ is not an s-ideal: Let $F : B_{\ell_2} \times B_{\ell_2} \to \mathbb{C}$ be given by $F(z, y) = \sum_i z_i y_i$. Clearly, F_y is a continuous linear map, $F_y \in A_u(B_{\ell_2})$, and $\lim_n F_y(\frac{n}{n+1}e_n) = \lim_n \frac{n}{n+1}y_n = 0$, thus $\chi(F_y) = 0$, hence $F_y \in A$. However, for $g(z) = z$, we have that $F(z, g(z)) = \sum_i z_i^2$ is a polynomial and $F(\frac{n}{n+1}e_n, g(\frac{n}{n+1}e_n)) = (\frac{n}{n+1})^2$. Hence, $\chi(F(\cdot, g(\cdot))) = \lim_n \delta_{\frac{n}{n+1}e_n}[F(\cdot, g(\cdot))] = \lim_n (\frac{n}{n+1})^2 = 1$, thus $F(\cdot, g(\cdot)) \notin A$.

Corollary

Let B be as in Theorem 3.1 and let μ belong to the spectrum of $H^\infty(B)$, resp. $A_u(B)$. Assume that μ belongs to the fiber of some $a \in B$, that is, $\mu_{|_{E^*}} = a$. Then $\mu = \delta_a$ if and only if $Ker(\mu)$ is an s-ideal in $H^\infty(B)$, resp. $A_u(B)$.

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Proof.

Let $f \in Ker(\mu)$. By Theorem 3.1, $f(z) - f(a) = T(f)(a,z)(z)$. Consider the function $F(z,y) := T(f)(a,y)(z) \;\; z, y \in B$. It is an analytic function. Since $\mu(F_y) = F_y(a) = T(f)(a,y)(a) = 0$, it turns out that $F_y \in Ker(\mu)$. Now, being $Ker(\mu)$ an s-ideal, it implies that $F(z,z) \in Ker(\mu)$, that is,
$0 = \mu\left(F(z,z)\right) = \mu(f - f(a)) = \mu(f) - f(a)$. Hence $\mu(f) = f(a)$ and so $Ker(\mu) \subset Ker(\delta_a)$. Therefore, $\mu = \delta_a$. □

Corollary

Let E be a strictly convex reflexive Banach space such that $B = B_E$ fulfills the assumptions in Theorem 3.1. Let T be a continuous endomorphism of $H^\infty(B_E)$ that maps at least one linear functional, $\lambda \in E^*$, into a nonconstant function. If moreover, $T^{-1}(A)$ is an s-ideal for every maximal s-ideal $A \subset H^\infty(B_E)$, then T is a composition operator. The same statement holds for $A_u(B)$.

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For every $z \in B_E$, $Ker(\delta_z)$ is an s-ideal that is also maximal, hence $Ker(\delta_z \circ T) = T^{-1}(Ker(\delta_z))$ is an s-ideal.

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Next we want to apply Corollary 3.11, so we check that the restriction of the homomorphism $\delta_z \circ T$ to the dual space has norm less than 1.

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Consider the mapping $\varphi : z \in B_E \mapsto (\delta_z \circ T)_{|_{E^*}} \in E^{**} = E$. It is analytic and actually, $\|\varphi(z)\| \leq 1$, so φ ranges into $\overline{B_E}$. If for some $z_0 \in B_E$, $\|\varphi(z_0)\| = 1$, then the mapping φ would be constant by the properties of E, and so, $T(\lambda)(z) = (\delta_z \circ T)(\lambda) = \lambda(\varphi(z)) = \lambda(\varphi(z_0))$, would be a constant function. This contradiction shows that $\varphi(B_E) \subset B_E$, i.e., $\|(\delta_z \circ T)_{|_{E^*}}\| < 1$ as desired.

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Hence, there is $a \in B_E$ such that $\delta_z \circ T = \delta_a$. A fortiori, $a = \varphi(z)$. Now, it is immediate that $T = C_\varphi$.

Theorem

Let B be the open unit ball of c_0. Given $a \in B$, there is a linear operator $f \in H^\infty(B) \rightsquigarrow (f_i) \in H^\infty(B, \ell^\infty)$ such that for all $z \in B$,

$$f(z) - f(a) = \sum_i (z_i - a_i) f_i(z).$$

For the algebra $A_u(B)$ the sequence (f_i) can be chosen in $A_u(B)$ and further, the convergence of the series also holds for the weak topology.

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Since the partial sums sequence defines a bounded subset of $H^\infty(B)$, we also have

Remark

The series $\sum z_i f_i(z)$ converges in the $w^*(H^\infty(B), G^\infty(B))$ topology to $f - f(0)$.

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Remark

The convergence of the series in Theorem 4.1 for the $H^\infty(B)$ case may fail for the weak topology.

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It suffices to choose $f \in H^\infty(B)$ that is not weakly sequentially continuous at 0.

Recall that the spectrum of the algebra $A_u(B)$ is the closed unit ball B^{**} of ℓ_∞, or in other words, the maximal ideals in $A_u(B)$ are the kernels of evaluations at points in B^{**}.

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The maximal ideals in $A_u(B)$ are "weakly countably generated" by the sequence of the canonical projections.

That is, for the maximal ideal $\mathcal{M}$ and $a = (a_i) \in B^{**}$, such that $\mathcal{M} = Ker\delta_a$, it turns out that every $f \in \mathcal{M}$, can be written as $\tilde{f}(z) = \sum_{i=1}^{\infty} \tilde{f}_i(z)(z_i - a_i)$, being the convergence in the weak topology of $A_u(B)$.

Recall that the spectrum of the algebra $A_u(B)$ is the closed unit ball B^{**} of ℓ_∞, or in other words, the maximal ideals in $A_u(B)$ are the kernels of evaluations at points in B^{**}.

Corollary

Every continuous endomorphism T of $A_u(B)$ that maps none of the canonical projections into a nonconstant function arises from some analytic mapping $\varphi : B \to int(B^{**})$ in such a way that $Tf = \tilde{f} \circ \varphi$.

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