Banach and quasi-Banach spaces of vector-valued sequences with special properties

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Introduction

- In this work we continue the research initiated in [1, 3] on the existence of infinite dimensional closed subspaces of Banach or quasi-Banach sequence spaces formed by sequences with special properties.
- Given a Banach space X, in [3] the authors introduce a large class of Banach or quasi-Banach spaces formed by X-valued sequences, called *invariant sequences spaces*, which encompasses several classical sequences spaces as particular cases.

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- Roughly speaking, the main results of [1, 3] prove that, for every invariant sequence space E of X-valued sequences and every subset Γ of $(0, \infty]$, there exists a closed infinite dimensional subspace of E formed, up to the null vector, by sequences not belonging to $\bigcup_{q \in \Gamma} \ell_q(X)$; as well as a closed infinite dimensional subspace of E formed, up to the null vector, by sequences not belonging to $c_0(X)$.

- In other words we can say that $E - \bigcup_{q \in \Gamma} \ell_q(X)$ and $E - c_0(X)$ are spaceable. Remember that a subset A of a topological vector space V is *spaceable* if $A \cup \{0\}$ contains a closed infinite dimensional subspace of V.

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- In this talk we consider the following much more general situation: given Banach spaces X and Y, a map $f\colon X \longrightarrow Y$, a set $\Gamma \subseteq (0,+\infty]$ and an invariant sequence space E of X-valued sequences, we investigate the existence of closed infinite dimensional subspaces of E formed, up to the origin, by sequences $(x_j)_{j=1}^{\infty} \in E$ such that either

$$(f(x_j))_{j=1}^{\infty} \notin \bigcup_{q\in\Gamma} \ell_q(Y) \text{ or}$$

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- As usual, $\ell_p(X)$ and $\ell_p^w(X)$ are the Banach spaces (p-Banach spaces if $0 < p < 1$) of p-summable and weakly p-summable X-valued sequences, respectively, and $c_0(X)$ is the Banach space of norm null X-valued sequences. Letting f be the identity on X, the cases of sequences $(x_j)_{j=1}^\infty \in E$ such that $(f(x_j))_{j=1}^\infty \notin \bigcup\limits_{q\in\Gamma} \ell_q(Y)$ or $(f(x_j))_{j=1}^\infty \notin c_0(Y)$ recover the situation investigated in [1, 3]. So, the results proved here generalize the previous results in two directions: we consider f belonging to a large class of functions and we consider spaces formed by sequences $(x_j)_{j=1}^\infty \in E$ such that $(f(x_j))_{j=1}^\infty$ does not belong to $\bigcup\limits_{q\in\Gamma} \ell_q^w(Y)$, a condition much more restrictive than not to belong to $\bigcup\limits_{q\in\Gamma} \ell_q(Y)$.

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Main result

Definition

Let $X \neq \{0\}$ be a Banach space.

(a) Given $x \in X^{\mathbb{N}}$, by x^0 we mean the zerofree version of x, that is: if x has only finitely many non-zero coordinates, then $x^0 = 0$; otherwise, $x^0 = (x_j)_{j=1}^{\infty}$ where x_j is the j-th non-zero coordinate of x.

(b) By an *invariant sequence space over X* we mean an infinite-dimensional Banach or quasi-Banach space E of X-valued sequences enjoying the following conditions:

(b1) For $x \in X^{\mathbb{N}}$ such that $x^0 \neq 0$, $x \in E$ if and only if $x^0 \in E$, and in this case $\|x\|_E \leq K\|x^0\|_E$ for some constant K depending only on E.

(b2) $\|x_j\|_X \leq \|x\|_E$ for every $x = (x_j)_{j=1}^{\infty} \in E$ and every $j \in \mathbb{N}$.

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Example

(a) For $0 < p \leq \infty$, $\ell_p(X)$, $\ell_p^w(X)$, $\ell_p^u(X)$ (unconditionally p-summable X-valued sequences) and $\ell_{m(s;p)}(X)$ (mixed sequence space) are invariant sequence spaces over X with their respective usual norms (p-norms if $0 < p < 1$).

(b) The Lorentz sequence spaces, Orlicz sequence space, ...

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Definition

Let X and Y be Banach spaces, E be an invariant sequence space over X, $\Gamma \subseteq (0, +\infty]$ and $f\colon X \longrightarrow Y$ be a function. We define the sets:

$$C(E, f, \Gamma) = \left\{ (x_j)_{j=1}^{\infty} \in E : (f(x_j))_{j=1}^{\infty} \notin \bigcup_{q \in \Gamma} \ell_q(Y) \right\},$$

$$C^w(E, f, \Gamma) = \left\{ (x_j)_{j=1}^{\infty} \in E : (f(x_j))_{j=1}^{\infty} \notin \bigcup_{q \in \Gamma} \ell_q^w(Y) \right\} \text{ and}$$

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A map $f\colon X \longrightarrow Y$ between normed spaces is said to be:

(a) *Non-contractive* if $f(0) = 0$ and for every scalar $\alpha \neq 0$ there is a constant $K(\alpha) > 0$ such that

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for every $x \in X$.

(b) *Strongly non-contractive* if $f(0) = 0$ and for every scalar $\alpha \neq 0$ there is a constant $K(\alpha) > 0$ such that

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- Subhomogeneous functions (with $f(0) = 0$) are non-contractive;
- bounded and unbounded linear operators are strongly non-contractive (hence non-contractive); and
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Theorem

Let X and Y be Banach spaces, E be an invariant sequence space over X, $f\colon X \longrightarrow Y$ be a function and $\Gamma \subseteq (0, +\infty]$.

(a) *If f is non-contractive, then $C(E, f, \Gamma)$ and $C(E, f, 0)$ are either empty or spaceable.*

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Sketch of the proof for the case $C(E, f, \Gamma)$.

Remember that

$$C(E, f, \Gamma) = \left\{ (x_j)_{j=1}^{\infty} \in E : (f(x_j))_{j=1}^{\infty} \notin \bigcup_{q \in \Gamma} \ell_q(Y) \right\}.$$

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Assume that $C(E, f, \Gamma)$ is non-empty and choose $x \in C(E, f, \Gamma)$. Since E is an invariant sequence space, then $x^0 \in E$ and the condition $f(0) = 0$ guarantees that $x^0 \in C(E, f, \Gamma)$. Writing $x^0 = (x_j)_{j=1}^{\infty}$ we have that $x_j \neq 0$ for every j.

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In both cases the series $\sum_{i=1}^{\infty} a_i y_i$ converges in E, hence the operator is well defined. It is easy to see that T is linear and injective. Thus $\overline{T(\ell_{\tilde{s}})}$ is a closed infinite-dimensional subspace of E.

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Applications

Definition

Let X and Y be Banach spaces and $0 < p \leq q < +\infty$. We say that

- a linear operator $u\colon X \longrightarrow Y$ is *absolutely* $(q;p)$*-summing* if $(u(x_j))_{j=1}^{\infty} \in \ell_q(Y)$ for each $(x_j)_{j=1}^{\infty} \in \ell_p^w(X)$.
- an n-homogeneous polynomial $P\colon X \longrightarrow Y$ is p*-dominated* if $(P(x_j))_{j=1}^{\infty} \in \ell_{p/n}(Y)$ for each $(x_j)_{j=1}^{\infty} \in \ell_p^w(X)$.

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Let X and Y be Banach spaces.
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Proof. It is enough to note that linear operators and homogeneous polynomials are (strongly) non-contractive maps and that $\ell_p^w(X)$ is an invariant sequence space over X.

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By $c_0^w(X)$ we denote the closed subspace of $\ell_\infty(X)$ formed by weakly null X-valued sequences. It is easy to check that $c_0^w(X)$ is an invariant sequence space over X, so by last Theorem we have:

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Let X and Y be Banach spaces.
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is spaceable. In particular, if X lacks the Schur property, then there exists an infinite dimensional Banach space formed, up to the origin, by weakly null but non-norm null X-valued sequences.
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Recall that a subset A of a topological vector space E is *α-lineable* if $A \cup \{0\}$ contains an α-dimensional linear subspace of E. And A is *maximal dense-lineable* if $A \cup \{0\}$ contains a dense linear subspace V of E with $\dim(V) = \dim(E)$.

Proposition

Let X and Y be Banach spaces and $p > 0$. Then the sets

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Theorem (L. Bernal-González and M. Cabrera - JFA 2014)

Assume that X is a metrizable and separable topological vector space. Let $A \subset X$ and α be an infinite cardinal number such that A is α-lineable. If there exists a subset $B \subset X$ such that $A + B \subset A$, $A \cap B = \emptyset$ and B is dense-lineable, then $A \cup \{0\}$ contains a dense vector space D with $\dim(D) = \alpha$.

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Proof. It is not difficult to see that the spaceability of both sets follows from the main theorem. We shall apply the Bernal-Cabrera Theorem to prove the second assertion. Assume that X is separable and $p < +\infty$. It is clear that $c_0(X)$ and $\ell_p(X)$ are separable as well. Let A be either $C(c_0(X), u, 0)$ or $C^w(\ell_p(X), u, \{p\})$. By the spaceability of A we have that $A \cup \{0\}$ contains a $\mathfrak{c}$-dimensional subspace, where $\mathfrak{c}$ is the cardinality of the continuum. Let $c_{00}(X)$ denote the space of eventually null X-valued sequences. It is clear that $A + c_{00}(X) \subseteq A$, $A \cap c_{00}(X) = \emptyset$ and $c_{00}(X)$ is a dense infinite dimensional subspace of $c_0(X)$ and $\ell_p(X)$. By the Bernal-Cabrera Theorem, $A \cup \{0\}$ contains a $\mathfrak{c}$-dimensional dense subspace, and the result follows because $c_0(X)$ and $\ell_p(X)$ are $\mathfrak{c}$-dimensional (remember that they are separable infinite dimensional Banach or quasi-Banach spaces). $\square$

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Proof. It is not difficult to see that the spaceability of both sets follows from the main theorem. We shall apply the Bernal-Cabrera Theorem to prove the second assertion. Assume that X is separable and $p < +\infty$. It is clear that $c_0(X)$ and $\ell_p(X)$ are separable as well. Let A be either $C(c_0(X), u, 0)$ or $C^w(\ell_p(X), u, \{p\})$. By the spaceability of A we have that $A \cup \{0\}$ contains a $\mathfrak{c}$-dimensional subspace, where $\mathfrak{c}$ is the cardinality of the continuum. Let $c_{00}(X)$ denote the space of eventually null X-valued sequences. It is clear that $A + c_{00}(X) \subseteq A$, $A \cap c_{00}(X) = \emptyset$ and $c_{00}(X)$ is a dense infinite dimensional subspace of $c_0(X)$ and $\ell_p(X)$. By the Bernal-Cabrera Theorem, $A \cup \{0\}$ contains a $\mathfrak{c}$-dimensional dense subspace, and the result follows because $c_0(X)$ and $\ell_p(X)$ are $\mathfrak{c}$-dimensional (remember that they are separable infinite dimensional Banach or quasi-Banach spaces). $\square$

Example

Let X be an infinite dimensional Banach space and $0 < p < +\infty$. We know that $\ell_p^w(X) - \ell_p(X)$ is spaceable, that is, there exists an infinite dimensional Banach/quasi-Banach space formed, up to the origin, by X-valued sequences $(x_j)_{j=1}^{\infty}$ such that $\sum\limits_{j=1}^{\infty} |\varphi(x_j)|^p < +\infty$ for every bounded linear functional $\varphi \in X'$ and $\sum\limits_{j=1}^{\infty} \|x_j\|^p = +\infty$. Considering an unbounded linear functional φ on X, the last Proposition yields the following dual result: there exists an infinite dimensional Banach/quasi-Banach space formed, up to the origin, by X-valued sequences $(x_j)_{j=1}^{\infty}$ such that $\sum\limits_{j=1}^{\infty} \|x_j\|^p < +\infty$ and $\sum\limits_{j=1}^{\infty} |\varphi(x_j)|^p = +\infty$.

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Thank You very much!!!