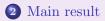
Banach and quasi-Banach spaces of vector-valued sequences with special properties

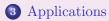
Vinícius Vieira Fávaro

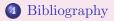
Workshop on Infinite Dimensional Analysis Buenos Aires, 22-25 July 2014

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G. Botelho, V. V. Fávaro Buenos Aires 07/25/2014

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• Roughly speaking, the main results of [1, 3] prove that, for every invariant sequence space E of X-valued sequences and every subset Γ of $(0, \infty]$, there exists a closed infinite dimensional subspace of E formed, up to the null vector, by sequences not belonging to $\bigcup_{q \in \Gamma} \ell_q(X)$; as well as a closed infinite dimensional subspace of E formed, up to the null

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• In other words we can say that $E - \bigcup_{q \in \Gamma} \ell_q(X)$ and $E - c_0(X)$ are spaceable. Remember that a subset A of a topological vector space V is *spaceable* if $A \cup \{0\}$ contains closed infinite dimensional subspace of V.

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Let $X \neq \{0\}$ be a Banach space.

(a) Given $x \in X^{\mathbb{N}}$, by x^0 we mean the zerofree version of x, that is: if x has only finitely many non-zero coordinates, then $x^0 = 0$; otherwise, $x^0 = (x_j)_{j=1}^{\infty}$ where x_j is the *j*-th non-zero coordinate of x.

(b) By an *invariant sequence space over* X we mean an infinite-dimensional Banach or quasi-Banach space E of X-valued sequences enjoying the following conditions: (b1) For $x \in X^{\mathbb{N}}$ such that $x^0 \neq 0, x \in E$ if and only if $x^0 \in E$, and in this case $||x||_E \leq K ||x^0||_E$ for some constant K depending only on E. (b2) $||x_j||_X \leq ||x||_E$ for every $x = (x_j)_{j=1}^{\infty} \in E$ and every $j \in \mathbb{N}$. An *invariant sequence space* is an invariant sequence space over some Banach space X.

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$$C(E, f, \Gamma) = \left\{ (x_j)_{j=1}^{\infty} \in E : (f(x_j))_{j=1}^{\infty} \notin \bigcup_{q \in \Gamma} \ell_q(Y) \right\},$$

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Remember that

$$C(E, f, \Gamma) = \left\{ (x_j)_{j=1}^{\infty} \in E : (f(x_j))_{j=1}^{\infty} \notin \bigcup_{q \in \Gamma} \ell_q(Y) \right\}.$$

Let us fix a notation. For $\alpha = (\alpha_n)_{n=1}^{\infty} \in \mathbb{K}^{\mathbb{N}}$ and $w \in X$ we denote

 $w \otimes \alpha = \alpha \otimes w := (\alpha_n w)_{n=1}^{\infty} \in X^{\mathbb{N}}.$

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Sketch of the proof for the case $C(E, f, \Gamma)$.

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Assume that $C(E, f, \Gamma)$ is non-empty and choose $x \in C(E, f, \Gamma)$. Since E is an invariant sequence space, then $x^0 \in E$ and the condition f(0) = 0 guarantees that $x^0 \in C(E, f, \Gamma)$. Writing $x^0 = (x_j)_{j=1}^{\infty}$ we have that $x_j \neq 0$ for every j. **Step 1:** Split \mathbb{N} into countably many infinite pairwise disjoint subsets $(\mathbb{N}_i)_{i=1}^{\infty}$. For every $i \in \mathbb{N}$ set $\mathbb{N}_i = \{i_1 < i_2 < ...\}$ and define

$$y_i = \sum_{j=1}^{\infty} x_j \otimes e_{i_j} \in X^{\mathbb{N}}.$$

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Observe that $y_i^0 = x^0$, so $0 \neq y_i^0 \in E$, hence $y_i \in E$ for every ibecause E is an invariant sequence space. For $q \in \Gamma$, $q < +\infty$, we have $\sum_{j=1}^{\infty} ||f(x_j)||_Y^q = +\infty$ because $x^0 \in C(E, f, \Gamma)$. If $+\infty \in \Gamma$, by the same reason we have $\sup_i ||f(x_i)||_Y = +\infty$. It follows that each $y_i \in C(E, f, \Gamma)$.

Step 2: Define $\tilde{s} = 1$ if E is a Banach space and $\tilde{s} = s$ if E is an s-Banach space, 0 < s < 1. We need to prove that the operator

$$T\colon \ell_{\delta}\longrightarrow E^-, \ \ T\left((a_i)_{i=1}^{\infty}
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In both cases the series $\sum_{i=1}^{\infty} a_i y_i$ converges in E, hence the operator is well defined. It is easy to see that T is linear and injective. Thus $\overline{T(\ell_s)}$ is a closed infinite-dimensional subspace of E.

Step 3: We have to show that if $z = (z_n)_{n=1}^{\infty} \in T(\ell_{\bar{s}}), z \neq 0$, then $(f(z_n))_{n=1}^{\infty} \notin \bigcup_{q \in \Gamma} \ell_q(Y)$. Given such a z, there are sequences $\left(a_i^{(k)}\right)_{i=1}^{\infty} \in \ell_{\bar{s}}, k \in \mathbb{N}$, such that $z = \lim_{k \to \infty} T\left(\left(a_i^{(k)}\right)_{i=1}^{\infty}\right)$ in E. Using the convergence below and the fact that f is non-contractive, it is possible to prove (hardwork) that there is a subsequence $(z_{m_j})_{i=1}^{\infty}$ of $z = (z_n)_{n=1}^{\infty}$ satisfying:

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Applications

Definition

Let X and Y be Banach spaces and 0 . We say that

- a linear operator $u: X \longrightarrow Y$ is absolutely (q; p)-summing if $(u(x_j))_{j=1}^{\infty} \in \ell_q(Y)$ for each $(x_j)_{j=1}^{\infty} \in \ell_p^w(X)$.
- an *n*-homogeneous polynomial $P: X \longrightarrow Y$ is *p*-dominated if $(P(x_j))_{j=1}^{\infty} \in \ell_{p/n}(Y)$ for each $(x_j)_{j=1}^{\infty} \in \ell_p^w(X)$.

Corollary

Let X and Y be Banach spaces. (a) Let $1 \le p \le q < +\infty$ and let $u: X \longrightarrow Y$ be a non-absolutely (q, p)-summing linear operator. Then the set

 $\{(x_j)_{j=1}^{\infty} \in \ell_p^w(X) : (u(x_j))_{j=1}^{\infty} \notin \ell_q(Y)\}$

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Proof. It is enough to note that linear operators and homogeneous polynomials are (strongly) non-contractive maps and that $\ell_p^w(X)$ is an invariant sequence space over X.

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Proof. It is enough to note that linear operators and homogeneous polynomials are (strongly) non-contractive maps and that $\ell_p^w(X)$ is an invariant sequence space over X.

Let X and Y be Banach spaces. We say that

- a linear operator $u: X \longrightarrow Y$ is completely continuous if $u(x_j) \longrightarrow u(x)$ in Y whenever $x_j \stackrel{w}{\longrightarrow} x$ in X.
- an *n*-homogeneous polynomial $P: X \longrightarrow Y$ is weakly sequentially continuous at the origin if $P(x_j) \longrightarrow 0$ in Y whenever $x_j \xrightarrow{w} 0$ in X.

By $c_0^w(X)$ we denote the closed subspace of $\ell_{\infty}(X)$ formed by weakly null X-valued sequences. It is easy to check that $c_0^w(X)$ is an invariant sequence space over X, so by last Theorem we have:

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Let X and Y be Banach spaces. (a) Let $u: X \longrightarrow Y$ be a non-completely continuous linear operator. Then the set

$\left\{ (x_j)_{j=1}^{\infty} \in c_0^w(X) : (u(x_j))_{j=1}^{\infty} \notin c_0(Y) \right\}$

is spaceable. In particular, if X lacks the Schur property, then there exists an infinite dimensional Banach space formed, up to the origin, by weakly null but non-norm null X-valued sequences.

(b) Let $P: X \longrightarrow Y$ be an n-homogeneous polynomial that fails to be weakly sequentially continuous at the origin. Then the set

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Recall that a subset A of a topological vector space E is α -lineable if $A \cup \{0\}$ contains an α -dimensional linear subspace of E. And A is maximal dense-lineable if $A \cup \{0\}$ contains a dense linear subspace V of E with dim $(V) = \dim(E)$.

Proposition

Let X and Y be Banach spaces and p > 0. Then the sets

 $\{(x_j)_{j=1}^\infty \in c_0(X) : (u(x_j))_{j=1}^\infty \notin c_0(Y)\}$ and

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Theorem (L. Bernal-González and M. Cabrera - JFA 2014)

Assume that X is a metrizable and separable topological vector space. Let $A \subset X$ and α be an infinite cardinal number such that A is α -lineable. If there exists a subset $B \subset X$ such that $A + B \subset A, A \cap B = \emptyset$ and B is dense-lineable, then $A \cup \{0\}$ contains a dense vector space D with dim $(D) = \alpha$.

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Bibliography

Thank You very much!!!

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