

Holomorphic functions in high dimensions and primes numbers

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joint work with
Bayart, Carando, Frerick, García, Maestre, Schlüters, Sevilla

Buenos Aires, 2014

Dirichlet series

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$$\Leftrightarrow \forall \sum_n a_n \frac{1}{n^s} \in \mathcal{H}_\infty : \sum_n |a_n b_n| < \infty$$

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$$b_n = \frac{1}{n^{\frac{1}{2}}}$$

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① $b_n = \frac{1}{n^{\frac{1}{2}+\varepsilon}}$ YES Bohr 1914

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Queffélec 1995,
Konyagin-Queffélec 2001,
Balasubramanian-Calado-Queffélec 2006,
de la Bretèche 2008,
D.-Frerick-Ortega-Ounaïes-Seip =: DFOOS 2011

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Problem

Give a reasonable description of **all** ℓ_1 -multipliers of \mathcal{H}_∞ .

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... this problem seems to be too difficult right now!

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(b_n) (completely) multiplicative

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Two examples

- 1 **b** Dirichlet character

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Two examples

① b Dirichlet character

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Our problem

Give a description of **all** multiplicative ℓ_1 -multipliers (b_j) of \mathcal{H}_∞ in terms of the asymptotic decay of the subsequence (b_{p_j}) .

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We know from previous work that—at least philosophically—this set of all multiplicative multipliers should be close to $\ell_2 \dots$

Content

- An almost satisfying solution ...

Content

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- Dirichlet series and holomorphic functions in high dimensions

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- From polydiscs to ℓ_r -spaces

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- From scalar values to vector values

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The Problem

$$? \subset \text{Set of all multiplicative } \ell_1\text{-multipliers of } \mathcal{H}_\infty \subset ??$$

Definition

$$\mathbf{B} := \left\{ b \in \mathbb{D}^{\mathbb{N}} \text{ multiplicative} \mid \limsup_{n \rightarrow \infty} \frac{1}{\log n} \sum_{j=1}^n b_{p_j}^{*2} < 1 \right\}$$

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Theorem, Bayart, D., Frerick, Maestre, Sevilla Peris 2014

$$\mathbf{B} \subset \text{Set of all multiplicative } \ell_1\text{-multipliers of } \mathcal{H}_\infty \subsetneq \overline{\mathbf{B}}$$

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The theorem is up to some point isometric, which explains why its proof is somewhat sophisticated ...

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Test 1

$$b \in \mathbb{D}^{\mathbb{N}} \cap \ell_2 \text{ multiplicative}$$

YES

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Test 2

$$b = \left(\frac{1}{\sqrt{n}} \right) \quad \boxed{\text{YES}}$$

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Reformulation

$$b_n = \left(\frac{1}{\sqrt{p_j}} \right)^\alpha \text{ if } n = p^\alpha \quad \boxed{\text{YES}}$$

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Problem

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Test 3

$$b_n = \begin{cases} \left(\frac{c}{\sqrt{n}}\right)^\alpha & \text{if } n = p^\alpha \\ \left(\frac{c}{\sqrt{j}}\right)^\alpha & \text{if } n = p^\alpha \end{cases}$$

YES for $c < 1$
NO for $c > 1$

Again

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$c = 1?$

Content

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- Dirichlet series and holomorphic functions in high dimensions
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-

Bohr's vision

Dirichlet series



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$$\sum_n a_n \frac{1}{n^s}$$

formal power series



$$\sum_{\alpha} c_{\alpha} z^{\alpha}$$

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\mathfrak{D}

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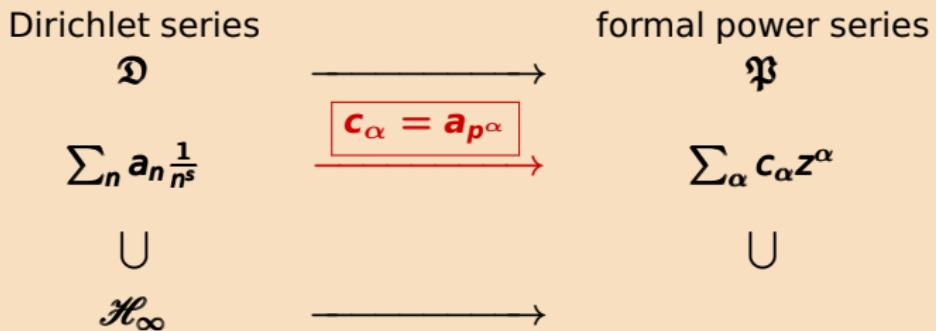
formal power series

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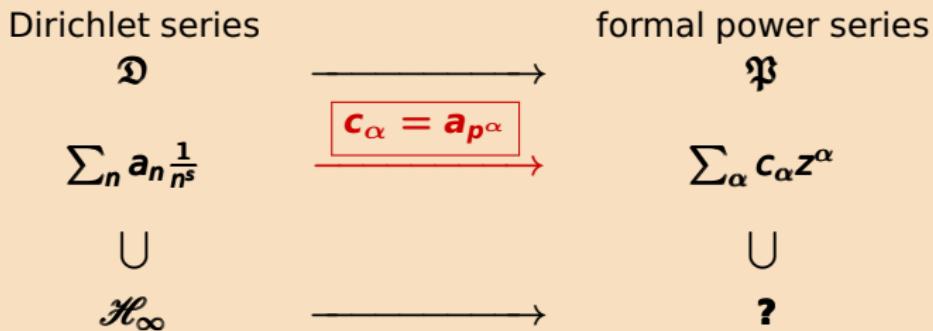
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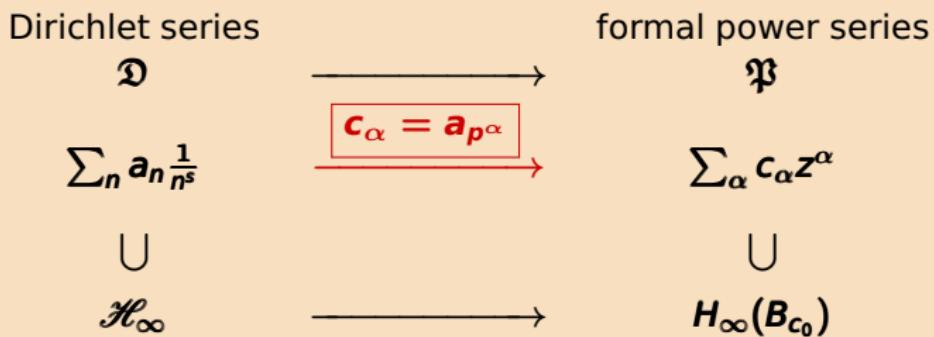
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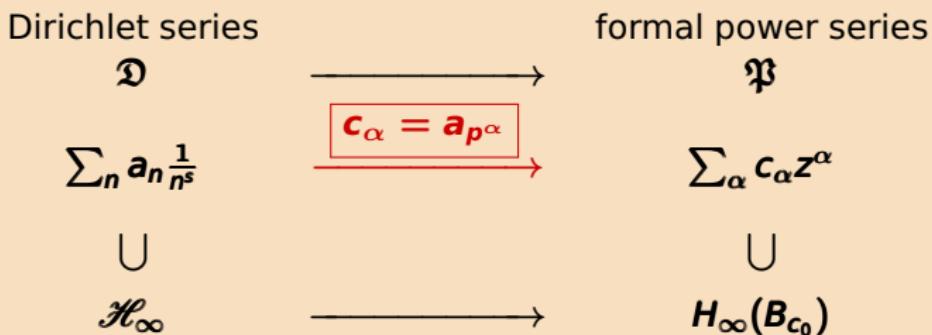
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Definition

$H_\infty(B_{c_0}) :=$ all bdd. and holo. fcts $f : B_{c_0} \rightarrow \mathbb{C}$

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Definition

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In which sense is $H_\infty(B_{c_0})$ in fact a subspace of \mathfrak{P} ?

Monomial coefficients

$$f \in H_\infty(B_{c_0})$$

$$\alpha = (\alpha_1, \dots, \alpha_N, 0, \dots) \in \mathbb{N}_0^{(\mathbb{N})}$$

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Easy

$\sum_\alpha c_\alpha(f) z^\alpha$ determines $f \in H_\infty(B_{c_0})$ uniquely

Recall ...

Dirichlet series

 \mathfrak{D}

$$\sum_n a_n \frac{1}{n^s}$$

 \cup \mathcal{H}_∞ 

$$\xrightarrow{c_\alpha = a_{p^\alpha}}$$



formal power series

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Theorem

Bohr 1913, Hedenmalm-Lindqvist-Seip 1997

$$\mathcal{H}_\infty = H_\infty(B_{c_0})$$

Clearly:

$$f(z) = \sum_{\alpha} c_{\alpha}(f) z^{\alpha} \quad \text{for all finite sequences } z \in B_{c_0}$$

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$$\text{mon } H_{\infty}(B_{c_0}) := \left\{ z \in B_{c_0} \mid \forall f \in H_{\infty}(B_{c_0}) : f(z) = \sum_{\alpha} c_{\alpha}(f) z^{\alpha} \right\}$$

Then our original problem ...

$$? \subset \text{Set of all multiplicative } \ell_1\text{-multipliers of } \mathcal{H}_\infty \subset ??$$

... transfers into

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Bridge

For each multiplicative sequence \mathbf{b} the following are equivalent:

- ① (b_n) is an ℓ_1 -multiplier for \mathcal{H}_∞
- ② $(b_{p_j}) \in \text{mon } H_\infty(B_{c_0})$

Recall, BDFMS 2014

$$\mathbf{B} := \left\{ b \in \mathbb{D}^{\mathbb{N}} \text{ multiplicative} \mid \limsup_{n \rightarrow \infty} \frac{1}{\log n} \sum_{j=1}^n b_{p_j}^{*2} < 1 \right\}$$

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This result improves work of

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D.-Prengel-M. 2009

Idea of proof: Given $z \in \mathbb{Z}$ and $f \in H_\infty(B_{c_0})$ the aim is to control the following sum uniformly in n

$$\sum_{\alpha \in \mathbb{N}_0^n} |c_\alpha(f)z^\alpha|$$

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$$= \left| \sum_{m=1}^{\infty} \sum_{1 \leq j_1 \leq \dots \leq j_m \leq n} |c_{j_1, \dots, j_m}(f) z_{j_1} \dots z_{j_m}| \right|$$

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$$= \sum_{m=1}^{p-1} \sum_{\mathbf{j} \in \mathcal{J}(m,n)} |c_j(f) z_{j_1} \dots z_{j_m}| + \sum_{m=p}^{\infty} \sum_{\mathbf{j} \in \mathcal{J}(m,n)} |c_j(f) z_{j_1} \dots z_{j_m}|$$

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$$\times \sum_{m \geq p} \rho^m \left[\sum_{\mathbf{j} \in \mathcal{J}(p,n)} \left(\sum_{\substack{\mathbf{i} \in \mathcal{J}(m-p,n) \\ i_{m-p} \leq j_1}} |c_{(\mathbf{i},\mathbf{j})}|^2 \right)^{\frac{1}{2} \times \frac{2p}{p+1}} \right]^{\frac{p+1}{2p}}$$

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DFOOS 2011, Bayart-Pellegrino-Seoane 2013 ...

Content

- From polydiscs to ℓ_r -spaces
-

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- ③ $1 < r < \infty$: NO Next theorem:

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Next theorem:

Theorem, B., D., Schlüters 2014

For each $1 \leq r \leq \infty$

$$\frac{1}{p^{1-\frac{1}{\min\{r,2\}}}} \cdot B_{\ell_r} \subset \text{mon } H_\infty(B_{\ell_r}),$$

and the exponent in the primes p is optimal.

Content

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-
- From scalar values to vector values

New object of desire

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Problem ...

Given X , describe **all** multiplicative ℓ_1 -multipliers of $\mathcal{H}_{\infty}(X)$.

... or equivalently

$$\text{mon } H_{\infty}(B_{c_0}, X) = ?$$

Local Banach space theory enters the game ...

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