Holomorphic functions in high dimensions and primes numbers

Andreas Defant

joint work with
Bayart, Carando, Frerick, García, Maestre, Schlüters, Sevilla

Buenos Aires, 2014

Dirichlet series

$D = \sum_n a_n \frac{1}{n^s}$ with coefficients $a_n \in \mathbb{C}$ and variable $s \in \mathbb{C}$

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$$:\Leftrightarrow \quad \forall \quad \sum_n a_n \frac{1}{n^s} \in \mathscr{H}_\infty \ : \quad \sum_n |a_n b_n| < \infty$$

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$$b_n = \frac{1}{n^{\frac{1}{2}}}$$

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History:

1. $b_n = \dfrac{1}{n^{\frac{1}{2}+\varepsilon}}$ YES Bohr 1914

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1. $b_n = \frac{1}{n^{\frac{1}{2}+\varepsilon}}$ **YES** Bohr 1914
2. $b_n = \frac{1}{n^{\frac{1}{2}-\varepsilon}}$ **NO** Bohnenblust-Hille 1931

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Queffélec 1995,
Konyagin-Queffélec 2001,
Balasubramanian-Calado-Queffélec 2006,
de la Bretèche 2008,
D.-Frerick-Ortega-Ounaïes-Seip =: DFOOS 2011

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Give a reasonable description of **all** ℓ_1-multipliers of $\mathscr{H}_\infty$.

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. . . this problem seems to be too difficult right now!

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Two examples

1. b Dirichlet character

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2. $b = \left(\frac{1}{n^{\sigma}}\right)$

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Our problem

Give a description of **all** multiplicative ℓ_1-multipliers (b_j) of $\mathscr{H}_\infty$ in terms of the asymptotic decay of the subsequence (b_{p_j}).

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We know from previous work that–at least philosophically–this set of all multiplicative multipliers should be close to ℓ_2 ...

Content

- An almost satisfying solution . . .

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- Dirichlet series and holomorphic functions in high dimensions

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- From polydiscs to ℓ_r-spaces

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- From scalar values to vector values

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The Problem

? $\subset$ **Set of all multiplicative ℓ_1-multipliers of $\mathscr{H}_\infty$** $\subset$ **??**

An almost satisfying solution

Definition

$$\mathbf{B} := \left\{ b \in \mathbb{D}^{\mathbb{N}} \text{ multiplicative} \;\middle|\; \limsup_{n\to\infty} \frac{1}{\log n} \sum_{j=1}^{n} b_{p_j}^{*2} < 1 \right\}$$

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Theorem, Bayart, D., Frerick, Maestre, Sevilla Peris 2014

$$\mathbf{B} \subset \text{Set of all multiplicative } \ell_1\text{-multipliers of } \mathscr{H}_\infty \subsetneq \overline{\mathbf{B}}$$

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The theorem is up to some point isometric, which explains why its proof is somewhat sophisticated . . .

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$\mathbf{B} \subset$ **Set of all multiplicative ℓ_1-multipliers of $\mathscr{H}_\infty$** $\subsetneq \overline{\mathbf{B}}$

Test 1

$b \in \mathbb{D}^{\mathbb{N}} \cap \ell_2$ **multiplicative** **YES**

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Test 2

$$b = \left(\frac{1}{\sqrt{n}}\right) \qquad \boxed{\text{YES}}$$

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Reformulation

$$b_n = \left(\frac{1}{\sqrt{p_j}}\right)^{\alpha} \text{ if } n = p^{\alpha}$$

YES

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Problem

$$b_n = \left(\frac{1}{\sqrt{j}}\right)^{\alpha} \text{ if } n = p^{\alpha} \qquad \boxed{?}$$

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Test 3

$$b_n = \begin{cases} \left(\frac{c}{\sqrt{j}}\right)^{\alpha} & \text{if } n = p^{\alpha} \\ \left(\frac{c}{\sqrt{j}}\right)^{\alpha} & \text{if } n = p^{\alpha} \end{cases}$$

YES for $c < 1$

NO for $c > 1$

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Content

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- Dirichlet series and holomorphic functions in high dimensions
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Bohr's vision

Dirichlet series

$\mathfrak{D}$

$\sum_n a_n \frac{1}{n^s}$

Bohr's vision

Dirichlet series	formal power series
$\mathfrak{D}$	$\mathfrak{P}$
$\sum_n a_n \frac{1}{n^s}$	$\sum_\alpha c_\alpha z^\alpha$

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Dirichlet series $\longrightarrow$ formal power series

$$\mathfrak{D} \longrightarrow \mathfrak{P}$$

$$\sum_n a_n \frac{1}{n^s} \qquad\qquad \sum_\alpha c_\alpha z^\alpha$$

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$\mathscr{H}_\infty$	$\longrightarrow$	$H_\infty(B_{c_0})$

Definition

$$H_\infty(B_{c_0}) := \text{ all bdd. and holo. fcts } f : B_{c_0} \to \mathbb{C}$$

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In which sense is $H_\infty(B_{c_0})$ in fact a subspace of $\mathfrak{P}$?

Monomial coeffcients

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Easy

$\sum_\alpha c_\alpha(f) z^\alpha$ determines $f \in H_\infty(B_{c_0})$ uniquely

Recall . . .

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Theorem
Bohr 1913, Hedenmalm-Lindqvist-Seip 1997

$$\mathscr{H}_\infty = H_\infty(B_{c_0})$$

Clearly:

$$f(z) = \sum_{\alpha} c_\alpha(f) z^\alpha \quad \text{for all finite sequences } z \in B_{c_0}$$

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Definition

$$\operatorname{mon} H_\infty(B_{c_0}) := \left\{ z \in B_{c_0} \;\middle|\; \forall f \in H_\infty(B_{c_0}) : f(z) = \sum_{\alpha} c_\alpha(f) z^\alpha \right\}$$

Then our original problem . . .

$$\textbf{?} \subset \text{Set of all multiplicative } \ell_1\text{-multipliers of } \mathscr{H}_\infty \subset \textbf{??}$$

. . . transfers into

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Bridge

For each multiplicative sequence $\boldsymbol{b}$ the following are equivalent:

1. (b_n) is an ℓ_1-multiplier for $\mathscr{H}_\infty$
2. $(b_{p_j}) \in \operatorname{mon} H_\infty(B_{c_0})$

Recall, BDFMS 2014

$$\mathbf{B} := \left\{ b \in \mathbb{D}^{\mathbb{N}} \text{ multiplicative} \;\middle|\; \limsup_{n\to\infty} \frac{1}{\log n} \sum_{j=1}^{n} b_{p_j}^{*2} < 1 \right\}$$

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Idea of proof: Given $z \in \mathbf{Z}$ and $f \in H_\infty(B_{c_0})$ the aim is to control the following sum uniformly in n

$$\sum_{\alpha \in \mathbb{N}_0^n} |c_\alpha(f) z^\alpha|$$

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$$= \boxed{\sum_{m=1}^{\infty} \sum_{1 \leq j_1 \leq \ldots \leq j_m \leq n} |c_{j_1,\ldots,j_m}(f) z_{j_1} \ldots z_{j_m}|}$$

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DFOOS 2011, Bayart-Pellegrino-Seoane 2013 ...

Content

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- From polydiscs to ℓ_r-spaces
-

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Theorem, B., D., Schlüters 2014

For each $1 \leq r \leq \infty$

$$\frac{1}{p^{1-\frac{1}{\min\{r,2\}}}} \cdot B_{\ell_r} \subset \operatorname{mon} H_\infty(B_{\ell_r}),$$

and the exponent in the primes p is optimal.

Content

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-
- From scalar values to vector values

New object of desire

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Problem ...

Given X, describe **all** multiplicative ℓ_1-multipliers of $\mathscr{H}_\infty(X)$.

... or equivalently

$$\operatorname{mon} H_\infty(B_{c_0}, X) = \textbf{?}$$

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