Bishop-Phelps-Bollobás version of Lindenstrauss properties A and B

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Definition

A pair of Banach spaces (X, Y) *is said to have the Bishop-Phelps-Bollobás property (BPBp for short) if for every* $\epsilon \in (0, 1)$ *there is* $\eta(\epsilon) > 0$ *such that for every* $T_0 \in L(X, Y)$ *with* $\|T_0\| = 1$ *and every* $x_0 \in S_X$ *satisfying*

$$\|T_0(x_0)\| > 1 - \eta(\epsilon),$$

there exist $S \in L(X, Y)$ *and* $x \in S_X$ *such that*

$$1 = \|S\| = \|Sx\|, \qquad \|x_0 - x\| < \epsilon \qquad \text{and} \qquad \|T_0 - T\| < \epsilon.$$

In this case, we will say that (X, Y) *has the BPBp with function* $\epsilon \longmapsto \eta(\epsilon)$.

Universal BPB Space

J. Lindenstrauss introduced and studied the following two properties.

A Banach space X is said to have *Lindenstrauss property A* if $\overline{NA(X,Z)} = L(X,Z)$ for every Banach space Z.

A Banach space Y is said to have *Lindenstrauss property B* if $\overline{NA(Z,Y)} = L(Z,Y)$ for every Banach space Z.

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Definition

Let X and Y be Banach spaces.

We say that X is a universal BPB domain space if for every Banach space Z, the pair (X,Z) has the BPBp.

We say that Y is a universal BPB range space if for every Banach space Z, the pair (Z,Y) has the BPBp.

Lindenstrauss property A and *RNP*

Positive Result
[J. Bourgain, 1977]
A Banach space has *RNP* if and only if it has *Lindenstrauss property A* in every equivalent norm.

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[W. Schachamayer, 1983]

Property (α) was introduced, which implies *Lindenstrauss property A*.
For instance, ℓ_1 has *Property* (α) .

Property (α) is satisfied in many Banach spaces.
For example, every WCG Banach space can be equivalently renormed to have *Property* (α).

Lindenstrauss property A is stable under arbitrary ℓ-sums [C and Song, 2008]

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The basic field $\mathbb{R}$ clearly has *Lindenstrauss property B*, which is just the Bishop-Phelps theorem. However, we don't know if the two-dimensional Euclidean space $\mathbb{R}^2$ has *Lindenstrauss property B*.

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Lindenstrauss property B and *Property* (β)

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[J. Partington, 1982]

Every Banach space can be equivalently renormed to have *Property* (β), hence *Lindenstrauss property B*.

Lindenstrauss property B is stable under arbitrary c_0-sums [Acosta, Aguirre and Payá, 1996]

Counterexamples of *Lindenstrauss property A or B*

The following spaces do not have *Lindenstrauss property A*:

c_0,

Non-atomic $L_1(\mu)$ spaces,

$C(K)$ spaces for infinite and metrizable K,

$d_*(w, 1)$ with $w \in \ell_2 \setminus \ell_1$

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The following spaces do not have *Lindenstrauss property B*:

ℓ_p for $1 < p < \infty$,

Infinite-dimensional $L_1(\mu)$-spaces,

$C[0,1]$, $d(w, 1)$ with $w \in \ell_2 \setminus \ell_1$,

Infinite-dimensional strictly convex spaces

Universal BPB Space

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The converse of (a) is known to be false:

the space ℓ_1 has Lindenstrauss property A, but fails to be a universal BPB domain space.

Even ℓ_1^2 fails to be a universal BPB domain space. (We will prove this later.)

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[Kim, Lee, Canadian J. Math, To appear]

A two-dimensional real space is a universal BPB domain space if and only if it is uniformly convex.

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We will provide an example of a Banach space having Lindenstrauss property B, but failing to be a universal BPB range space

One of the main tools in this study is to compare the function $\eta(\epsilon)$ appearing in the definition of the BPBp for different pairs of spaces.

Notations

Fix a pair (X, Y) of Banach spaces and let

$$\Pi(X, Y) = \{(x, T) \in X \times L(X, Y) \ : \ \|T\| = \|x\| = \|Tx\| = 1\}.$$

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$\eta(X,Y)(\epsilon) = \inf\{1 - \|Tx\|\}$, where the infimum is taken over the set

$$\big\{(x,T) \ : \ x \in S_X, \ \ T \in L(X), \ \|T\| = 1, \ dist\big((x,T), \Pi(X,Y)\big) \geq \epsilon\big\}.$$

Here,
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It is clear that the pair (X, Y) has the *BPBp* if and only if $\eta(X,Y)(\epsilon) > 0$ for every $\epsilon \in (0,1)$.

Properties of $\eta(X, Y)(\epsilon)$

If a function $\epsilon \to \eta(\epsilon)$ is valid in the definition of the *BPBp* for the pair (X, Y), then $\eta(\epsilon) \leq \eta(X, Y)(\epsilon)$.

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That is, $\eta(X, Y)(\epsilon)$ is the best function (i.e. the largest) we can find to ensure that (X, Y) has the *BPBp*.

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Theorem

Let $\{X_i \ : \ i \in I\}$ and $\{Y_j \ : \ j \in J\}$ be families of Banach spaces,

let X be the c_0-, ℓ_1-, or ℓ_∞-sum of $\{X_i\}$ and let Y be the c_0-, ℓ_1-, or ℓ_∞-sum of $\{Y_j\}$.

If the pair (X, Y) has the BPBp with $\eta(\epsilon)$,
then the pair (X_i, Y_j) also has the BPBp with $\eta(\epsilon)$ for every $i \in I$, $j \in J$.

In other words,

$$\eta(X, Y) \le \eta(X_i, Y_j) \qquad \big(i \in I,\ j \in J\big).$$

Properties of $\eta(X, Y)(\epsilon)$

If the pair (X, Y) has the *BPBp*, then every pair (X_i, Y_j) has also the *BPBp* with a function $\eta(\epsilon) \geq \eta(X, Y)(\epsilon) > 0$.

Properties of $\eta(X, Y)(\epsilon)$

If the pair (X, Y) has the *BPBp*, then every pair (X_i, Y_j) has also the *BPBp* with a function $\eta(\epsilon) \geq \eta(X, Y)(\epsilon) > 0$.

We will see from this that every universal BPB space has a "universal" function η.

That is, if X is a universal BPBp domain space, then

$$\inf\{\eta(X, Z)(\epsilon) : Z \text{ Banach space}\} > 0 \qquad \big(\epsilon \in (0, 1)\big);$$

if Y is a universal BPBp range space, then

$$\inf\{\eta(Z, Y)(\epsilon) : Z \text{ Banach space}\} > 0 \qquad \big(\epsilon \in (0, 1)\big).$$

Converse Result only for Range Spaces

Proposition

Let X be a Banach space and let $\{Y_j : j \in J\}$ be a family of Banach spaces.

Then, for both $Y = \left[\bigoplus_{j\in J} Y_j\right]_{c_0}$ and $Y = \left[\bigoplus_{j\in J} Y_j\right]_{\ell_\infty}$, one has

$$\eta(X, Y) = \inf_{j\in J} \eta(X, Y_j).$$

Consequently, the following four conditions are equivalent:

(I) $\inf_{j\in J} \eta(X, Y_j)(\epsilon) > 0$ *for all* $\epsilon \in (0, 1)$,

(II) *every pair* (X, Y_j) *has the BPBp with a common function* $\eta(\epsilon) > 0$,

(III) *the pair* $\left(X, \left[\bigoplus_{j\in J} Y_j\right]_{\ell_\infty}\right)$ *has the BPBp,*

(IV) *the pair* $\left(X, \left[\bigoplus_{j\in J} Y_j\right]_{c_0}\right)$ *has the BPBp.*

No counterpart even finite ℓ_1-sums of domain spaces

Indeed, this Proposition has no counterpart even for finite ℓ_1 or ℓ_∞-sums of domain spaces.

This follows from the fact that ℓ_1^2 (it is isometric to ℓ_∞^2) fails to be a universal BPB domain space, which we will show very soon.

Results on Domain Spaces

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The base field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is the unique Banach space which is a universal BPBp domain space in any equivalent renorming.

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The theorem is a direct result of the following lemma.

Lemma

Let X be a Banach space containing a non-trivial L-summand (i.e. $X = X_1 \oplus_1 X_2$ for some non-trivial subspaces X_1 and X_2) and let Y be a strictly convex Banach space. If the pair (X, Y) has the BPBp, then Y is uniformly convex.

(Proof of Theorem)/Lemma

Results on Domain Spaces

Corollary

Let Y be a strictly convex Banach space. If (ℓ_1^2, Y) has the BPBp, then Y is uniformly convex.

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A nice consequence of the above corollary is the following example.

Example

There exists a reflexive Banach space X such that the pair (X, X) fails the BPBp. Indeed, let Y be a reflexive strictly convex space which is not uniformly convex and consider the reflexive space $X = \ell_1^2 \oplus_1 Y$. If the pair (X, X) had the BPBp, then so would (ℓ_1^2, Y), a contradiction.

Results on Domain Spaces

Recall that if a Banach space X has Lindenstrauss property A, then the following hold [Lindenstrauss, Israel J. Math, 1963]:

(1) if X is isomorphic to a strictly convex space, then S_X is the closed convex hull of its extreme points,

(2) if X is isomorphic to a locally uniformly convex space, then S_X is the closed convex hull of its strongly exposed points.

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These results have been strengthened to the case of universal BPBp domain spaces X by [Kim, Lee], but with the additional hypothesis

$$\inf\{\eta(X,Z)(\epsilon) : Z \text{ Banach space}\} > 0 \qquad \big(\epsilon \in (0,1)\big),$$

but we showed that it is always true.

Corollary (Kim, Lee)

Let X be a universal BPB domain space. Then,

(a) *in the real case, there is no face of S_X which contains a non-empty relatively open subset of S_X;*

(b) *if X is isomorphic to a strictly convex Banach space, then the set of all extreme points of B_X is dense in S_X;*

(c) *if X is superreflexive, then the set of all strongly exposed points of B_X is dense in S_X.*

In particular, if X is a real 2-dimensional Banach space which is a universal BPB domain space, then X is uniformly convex.

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Question.

We don't know if a universal BPB domain space has to be uniformly convex.

Example

For $k \in \mathbb{N}$, consider $Y_k = \mathbb{R}^2$ endowed with the norm

$$\|(x,y)\| = \max\{|x|, |y| + \tfrac{1}{k}|x|\} \quad (x, y \in \mathbb{R}).$$

Observe that B_{Y_k} is the absolutely convex hull of the set $\left\{(0,1),\ (1, 1-\frac{1}{k}),\ (-1, 1-\frac{1}{k})\right\}$, so Y_k is polyhedral and, therefore, it is a universal BPB range space. Then, we have that $\inf\limits_{k\in\mathbb{N}} \eta(\ell_1^2, Y_k)(\epsilon) = 0$ for every $\epsilon \in (0, 1/2)$. Therefore, if we consider

$$\mathcal{Y} = \Big[\bigoplus_{i=1}^{\infty} Y_k\Big]_{c_0}, \qquad \mathcal{Z} = \Big[\bigoplus_{i=1}^{\infty} Y_k\Big]_{\ell_1} \quad \text{and} \quad \mathcal{W} = \Big[\bigoplus_{i=1}^{\infty} Y_k\Big]_{\ell_\infty},$$

then none of the pairs $(\ell_1^2, \mathcal{Y})$, $(\ell_1^2, \mathcal{Z})$ and $(\ell_1^2, \mathcal{W})$ has the BPBp.

Results on Range Spaces

Counterexample of a Banach space having Lindenstrauss property B which is not a universal BPB range space:

We recall that finite-dimensional real polyhedral spaces are universal BPB range spaces, because they have property β.

(Proof of Example)

Results on Range Spaces

Counterexample of a Banach space having Lindenstrauss property B which is not a universal BPB range space:

We recall that finite-dimensional real polyhedral spaces are universal BPB range spaces, because they have property β.

(Proof of Example)

Assume that for some $1/2 > \epsilon > 0$ we have

$$\inf_{k\in\mathbb{N}} \eta(\ell_1^2, Y_k)(\epsilon) > 0$$

and take $\eta(\epsilon)$ so that $\inf_{k\in\mathbb{N}} \eta(\ell_1^2, Y_k)(\epsilon) > \eta(\epsilon) > 0$.

For sufficiently large k, we have $1 - \frac{1}{k} > 1 - \eta(\epsilon)$, and

$$\left\| T_k\left(\frac{e_1 + e_2}{2}\right)\right\| = 1 - \frac{1}{k} > 1 - \eta(\epsilon).$$

Proof of Example

There exist $S_k \in L(\ell_1^2, Y_k)$ with $\|S_k\| = 1$ and $u_k \in S_{\ell_1^2}$ such that

$$\|S_k u_k\| = 1, \quad \|T_k - S_k\| < \epsilon \quad \text{and} \quad \left\|u_k - \left(\tfrac{1}{2}e_1 + \tfrac{1}{2}e_2\right)\right\| < \epsilon.$$

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Clearly, $u_k \in (e_1, e_2)$, hence $\|S_k(u_k)\| = 1$ implies that

$$[S_k(e_1), S_k(e_2)] \subset S_{Y_k}.$$

Hence $S_k(e_1)$ and $S_k(e_2)$ lies in the same face of S_{Y_k}.

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This implies that either $\|T_k(e_1) - S_k(e_1)\| > 1$ or $\|T_k(e_2) - S_k(e_2)\| 1 >$.
Therefore, $\|T_k - S_k\| \geq 1$, a contradiction.

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Therefore, $\|T_k - S_k\| \geq 1$, a contradiction.

Lindenstrauss property B does not imply being a universal BPB range space.

Thank

Thank you for your attention.