Geometry of Spaces of Non-Homogeneous Polynomials

C.B. (UCD) (Thesis of Anthony Brown)

July 2014 Workshop

Duality in Spaces of Linear Operators

$$c_o \to \ell_1 \to \ell_\infty$$

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H Hilbert space

$$\mathcal{K}(H) \to \mathcal{N}(H) \to \mathcal{B}(H).$$

Spaces of Polynomials

E Banach space, $n \in \mathbb{N}$

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$\mathscr{P}(^{n}E)$ the space of all n-homogeneous polynomials
$P(\lambda x) = \lambda^n P(x)$

Duality in Spaces of Homogeneous Polynomials

$$
\begin{array}{ccccc}
\mathscr{P}_w({}^nE) & \overset{\cong}{\longrightarrow} & \mathscr{P}_A({}^nE) & \overset{\cong}{\longrightarrow} & \widehat{\bigotimes}_{s,n,\epsilon} E'_b \\
 & & & & \downarrow \\
\widehat{\bigotimes}_{s,n,\pi} E''_{bb} & \overset{\cong}{\longrightarrow} & \mathscr{P}_N({}^nE'_b) & \overset{\cong}{\longrightarrow} & \mathscr{P}_I({}^nE'_b) \\
\downarrow & & & & \\
\mathscr{P}({}^nE''_{bb}) & & & &
\end{array}
$$

Duality in Spaces of Polynomials of degree at most n

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 & & \downarrow & & \\
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 & & & & \downarrow \\
 & & & & \mathscr{P}({}^{\leq n}E''_{bb})
\end{array}
$$

Norm on $\mathscr{P}(^{\leq n}E)$

The ‘natural’ norm on $\mathscr{P}(^{\leq n}E)$ (and hence $\mathscr{P}_A(^{\leq n}E)$) is

$$\|P\| = \sup_{\|x\|\leq 1} |P(x)|.$$

Approximable Polynomials

Space of approximable polynomials is closure of finite rank polynomials in the operator norm

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Denoted by $\mathscr{P}_A(^{\leq n}E)$

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$P \in \mathscr{P}_I({}^nE)$ if

$$P(x) = \int_{B_{E'}} \varphi(x)^n \, d\mu(\varphi)$$

for some measure μ on $(B_{E'}, \sigma(E', E))$

Spaces of Integral Polynomials

$P \in \mathscr{P}_I(^{\leq n}E)$ (P is integral) if

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Truncation of idea of integral holomorphic function (Dimant, Galindo, Maestre & Zalduendo)

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E real Banach space (C.B. & Ryan, Dimant, Galicer & R. García) The set of extreme points of unit ball of $\mathscr{P}_I(^nE)$ is

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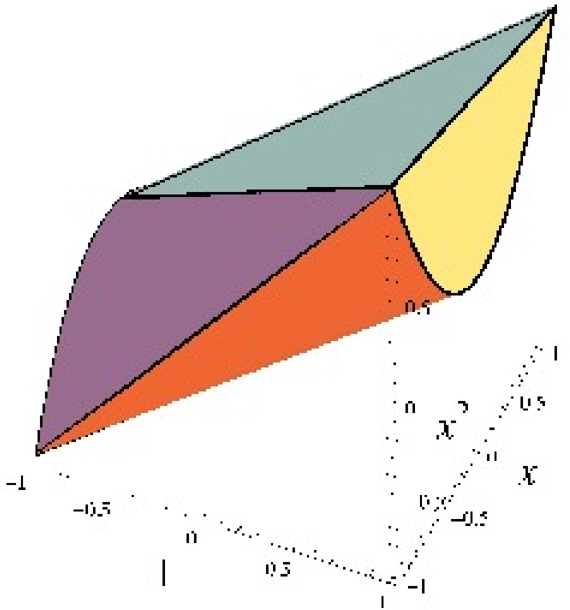
$$\{\phi^n : \phi \in E', \|\phi\| \leq 1\}$$

Unit Ball of $\mathscr{P}({}^{\leqslant 2}\mathbb{R})$



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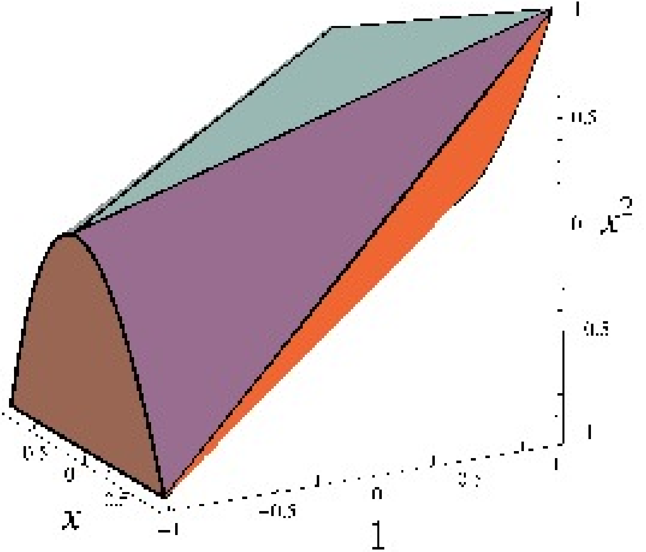

Isometries of Spaces of Approximable Polynomials (Real Spaces)

(C.B. & S. Lassalle)

E and F be real Banach spaces, let $n \geq 2$

$$T\colon \mathscr{P}_A(^nE) \to \mathscr{P}_A(^nF)$$

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There exists an isometric isomorphism $r\colon E' \to F'$ such that

$$T(P)(y) = \pm\overline{P} \circ r^t \circ J_F(y) \text{ for all } P \in \mathscr{P}_A({}^nE),$$

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where $\overline{P}$ is the Aron-Berner extension of P, r^t is the transpose of r and J_F is the canonical inclusion of F in F''

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E and F be complex Banach spaces, let n be a positive integer

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There exists an isometric isomorphism $r\colon E' \to F'$ such that

$$T(P)(y) = \overline{P} \circ r^t \circ J_F(y) \text{ for all } P \in \mathscr{P}_A(^nE),$$

Isometries of Spaces of Non-Homogeneous Approximable Polynomials (Complex Spaces)

E be a complex Banach spaces, let $n \in \mathbb{N}$

Then

$$T(P)(y) = \overline{P} \circ r^t \circ J_F(y) \text{ for all } P \in \mathscr{P}_A({}^{\leqslant n}E),$$

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$$T(\mathscr{P}_A({}^{j}E)) = \mathscr{P}_A({}^{j}E)$$

New Type of Isometry

E complex Banach space then $\mathscr{P}_A({}^{\leqslant n}E) = \mathscr{P}_A({}^{n}E \oplus_\infty \mathbb{C})$

$$R\colon \sum_{j=0}^{n} P_j \mapsto \tilde{P}$$

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$\tilde{P}(x, \lambda) = \sum_{j=0}^{n} \lambda^{n-j} P_j(x)$

New Type of Isometry

$$\begin{CD} \mathscr{P}_A({}^{\leqslant n}E) @>T>> \mathscr{P}_A({}^{\leqslant n}E) \\ @V R VV @A R^{-1} AA \\ \mathscr{P}_A({}^{n}E \oplus_\infty \mathbb{C}) @>\widetilde{T}>> \mathscr{P}_A({}^{n}E \oplus_\infty \mathbb{C}) \end{CD}$$

New Type of Isometry

$$\begin{array}{ccc} \mathscr{P}_A({}^{\leqslant n}E) & \xrightarrow{\;T\;} & \mathscr{P}_A({}^{\leqslant n}E) \\ \downarrow R & & \uparrow R^{-1} \\ \mathscr{P}_A({}^{n}E \oplus_\infty \mathbb{C}) & \xrightarrow{\;\widetilde{T}\;} & \mathscr{P}_A({}^{n}E \oplus_\infty \mathbb{C}) \end{array}$$

$$\widetilde{T}(R(P)) = \overline{R(P)} \circ s' \circ J_{E\oplus_\infty \mathbb{C}}$$

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$$E = X \oplus_\infty \mathcal{C}$$

write x as (x_1, μ) with $x_1 \in X$, $\mu \in \mathbb{C}$

consider the isometry

$$s(x_1, \mu, \lambda) = (r(x_1), \lambda, \mu)$$

New Type of Isometry

For $j = 1, 2, \ldots, n-1$:

$$T((\phi_2,\mu)^j) = \sum_{q=n-j+1}^{n} \frac{\binom{j}{n-q}}{q\binom{q}{q+j-n}} \sum_{k=1}^{q} e^{\frac{-2\pi(n-j)k}{q}i}\left(\phi_2 \circ r, e^{\frac{2\pi k}{q}i}\mu^{\frac{n-q}{n-j}}\right)^q + \left(0, \mu^{\frac{j}{n-j}}\right)^{n-j},$$

where r is an isometry of X.
for $j = 0$:

$$T((\phi_2,\mu)^0) = (0,1)^n.$$

for $j = n$

$$T(\phi^n)(x,\lambda) = \phi^n(x)\lambda^{n-n} = \phi(x)^n = (\phi,0)^n(x,\lambda),$$

Isometries of $\mathscr{P}({}^{\leqslant 8} l_\infty^5)$

Consider $T\colon \mathscr{P}({}^{\leqslant 8} l_\infty^5) \to \mathscr{P}({}^{\leqslant 8} l_\infty^5)$ defined by

$$T(P)(x_1, \dots, x_5) = \sum_{j=0}^{8} \sum_{k_1+\cdots+k_5=j} \alpha_{k_1,k_2,8-j,k_4,k_5} x_1^{k_1} x_2^{k_2} x_3^{k_3} x_4^{k_4} x_5^{k_5}.$$

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If $P_1(x_1, x_2, x_3, x_4, x_5) = 1$ then

$$T(P_1)(x_1, \ldots, x_5) = x_3^8,$$

Isometries of $\mathscr{P}({}^{\leqslant 8} l_{\infty}^{5})$

If $P_2(x_1, \ldots, x_5) = x_1$ then

$$T(P_2)(x_1, \ldots, x_5) = x_1 x_3^7,$$

Isometries of $\mathscr{P}({}^{\leqslant 8} l_\infty^5)$

If $P_2(x_1, \ldots, x_5) = x_1$ then

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If $P_3(x_1, \ldots, x_5) = x_3$ then

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Isometries of $\mathscr{P}(^{\leqslant 8} l_\infty^5)$

If $P_4(x_1, \ldots, x_5) = x_1 x_2$ then

$$T(P_4)(x_1, \ldots, x_5) = x_1 x_2 x_3^6,$$

Isometries of $\mathscr{P}({}^{\leqslant 8}l_\infty^5)$

If $P_4(x_1, \ldots, x_5) = x_1 x_2$ then

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If $P_5(x_1, \ldots, x_5) = x_3 x_5$ then

$$T(P_5)(x_1, \ldots, x_5) = x_3^6 x_5,$$

Isometries of $\mathcal{P}({}^{\leqslant 8} l_\infty^5)$

If $P_4(x_1, \ldots, x_5) = x_1 x_2$ then

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If $P_5(x_1, \ldots, x_5) = x_3 x_5$ then

$$T(P_5)(x_1, \ldots, x_5) = x_3^6 x_5,$$

If $P_6(x_1, \ldots, x_5) = x_3^2$ then

$$T(P_6)(x_1, \ldots, x_5) = x_3^6,$$

New Type of Isometry

For isometries, T, of the second type we have

$$T(\mathscr{P}_A(^jE)) \subset \mathscr{P}_A(^{\leqslant n-j}E)$$