INFINITE DIMENSIONAL BANACH SPACES HAVE DIMENSION $\geq \mathfrak{c}$

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This is a (small part of a) joint work with

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D. Pellegrino (João Pessoa),

J. B. Seoane-Sepúlveda (Madrid).

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The idea is to show you a computation of the dimension of arbitrary L_p-spaces we proved for other purposes, and then to apply this computation to prove the result stated in the title.

Computing the dimension of L_p-spaces

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(ii) Two sets $A, B \in \Sigma_{fin}$ are equivalent, denoted $A \sim B$, if $\mu((A - B) \cup (B - A)) = 0$. The elements of $\frac{\Sigma_{fin}}{\sim}$ are denoted by $[B]$, for $B \in \Sigma_{fin}$.

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(iii) The cardinal number $\#\frac{\Sigma_{fin}}{\sim}$ is called the entropy of the measure space (Ω, Σ, μ) and is denoted by $ent(\Omega)$.

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(iii) The cardinal number $\#\frac{\Sigma_{fin}}{\sim}$ is called the entropy of the measure space (Ω, Σ, μ) and is denoted by $ent(\Omega)$.

(iv) A set $A \in \Sigma$ is an atom if $0 < \mu(A)$ and there is no $B \in \Sigma$ such that $B \subset A$ and $0 < \mu(B) < \mu(A)$.

Lemma. If $ent(\Omega) \geq \aleph_0$, then there are sets $(B_i)_{i \in \mathbb{N}}$ in Σ_{fin} such that $\mu(B_i) > 0$ for every $i \in \mathbb{N}$ and $\mu\left(B_i \cap B_j\right) = 0$ whenever $i \neq j$.

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(c) If $ent(\Omega) \in \mathbb{N}$, then there is $k \in \mathbb{N}$ such that $ent(\Omega) = 2^k$ and $\dim\left(L_p(\Omega)\right) = k$.

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It follows from the Cantor-Bernstein-Schröder Theorem that $\dim\left(L_p(\Omega)\right) = \mathfrak{c}$.

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It is easy to see that $ent(\Omega) < +\infty \Longrightarrow U$ is finite, say $U = \{A_1, \ldots, A_k\}$.

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The next step is to use the maximality of U to show that every set $B \in \Sigma_{fin}$ satisfies

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Now, we know that $L_p(X)$ is the closure of

$$W = \left\{ \sum_{i=1}^{n} b_i \chi_{B_i} : n \in \mathbb{N},\ b_i \in \mathbb{K},\ B_i \in \Sigma_{fin} \right\}.$$

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Since any finite-dimensional subspace of a topological vector space is closed, it follows that

$$\dim \left(L_p(\Omega) \right) = \dim \overline{W} = \dim \left\{ \sum_{i=1}^{k} a_i \chi_{A_i} : a_i \in \mathbb{K} \right\} = k. \qquad \square$$

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Next we show how our computation of the dimension of L_p-spaces can be used to give a new CH-free proof of this fact:

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THANK YOU VERY MUCH!