Abel's functional equation and eigenvalues of composition operators on spaces of real analytic functions

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Aim of the talk

- $\varphi : \mathbb{R} \to \mathbb{R}$ is a non-constant real analytic map.
- $\mathscr{A}(\mathbb{R})$ denotes the space of real analytic functions defined on $\mathbb{R}$.
- Each symbol $\varphi : \mathbb{R} \to \mathbb{R}$ defines a composition operator $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ by $C_\varphi(f) := f \circ \varphi, f \in \mathscr{A}(\mathbb{R})$.
- When $\mathscr{A}(\mathbb{R})$ is endowed with its natural locally convex topology (see below), C_φ is a continuous linear operator on $\mathscr{A}(\mathbb{R})$.

Our purpose is to determine the eigenvalues and eigenvectors of composition operators $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$.

Schröder equation

Schröder equation

The problem is to find a real analytic solution $f \in \mathscr{A}(\mathbb{R})$ of the equation

$$C_\varphi(f) = \lambda f \qquad \text{for } \lambda \in \mathbb{C}. \tag{1}$$

The equation appeared probably for the first time already in 1871 in a paper of **Schröder** and was partially solved in 1884 in a paper of **Königs** also for real analytic functions.

Notation:

- $\mathrm{id}\,(x) = x, x \in \mathbb{R}$, and $I : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is the identity operator on $\mathscr{A}(\mathbb{R})$.
- For a map $\varphi : \mathbb{R} \to \mathbb{R}$, we write $\varphi^{[0]} = \mathrm{id}$ and $\varphi^{[n]}$ for the n-times composition of φ, $n \in \mathbb{N}$.

The space $\mathscr{A}(\mathbb{R})$. Martineau

- The space $\mathscr{A}(\mathbb{R})$ is equipped with the unique locally convex topology such that for any $U \subset \mathbb{C}$ open, $\mathbb{R} \subset U$, the restriction map $R : H(U) \longrightarrow \mathscr{A}(\mathbb{R})$ is continuous and for any compact set $K \subset \mathbb{R}$ the restriction map $r : \mathscr{A}(\mathbb{R}) \longrightarrow H(K)$ is continuous. In fact,

$$\mathscr{A}(\mathbb{R}) = \mathrm{proj}_{N\in\mathbb{N}}\ H([-N,N]) = \mathrm{proj}_{N\in\mathbb{N}}\ \mathrm{ind}_{n\in\mathbb{N}}\ H^\infty(U_{N,n}).$$

- $(f_n)_{n\in\mathbb{N}}$ in $\mathscr{A}(\mathbb{R})$ tends to f if and only if there is a complex neighbourhood W of $\mathbb{R}$ such that each f_n and f extend to W and $f_n \to f$ uniformly on compact subsets of W.

- $\mathscr{A}(\mathbb{R})$ is complete, separable, bounded sets are relatively compact and it satisfies the assumptions of the open mapping and the closed graph theorems. **Domański, Vogt, 2000,** proved that the space $\mathscr{A}(\mathbb{R})$ has no Schauder basis.

Spectrum and point spectrum

Let T be a continuous linear operator on a locally convex space E

- The kernel and image of T are denoted respectively by $\ker T$ and $\operatorname{im} T$.
- The **point spectrum** $\sigma_p(T)$ of T is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective. Elements of $\sigma_p(T)$ are called **eigenvalues** of T. The **eigenspace** of $\lambda \in \sigma_p(T)$ is $\ker(T - \lambda I)$.
- The **spectrum** $\sigma(T)$ of T is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not a topological isomorphism from E onto E. By the open mapping theorem, $\lambda \notin \sigma(C_\varphi)$ if and only if $C_\varphi - \lambda I : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is bijective.

A (trivial) apéritif

Proposition

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a non-constant real analytic map.

- 0 is never an eigenvalue of C_φ. In particular, C_φ is injective.
- 1 is always an eigenvalue of C_φ and the constant functions are eigenvectors.
- C_φ is surjective if and only if it is bijective if and only if $\varphi : \mathbb{R} \to \mathbb{R}$ is bijective and its inverse is real analytic, i.e. φ is a real analytic diffeomorphism.
- $0 \in \sigma(C_\varphi)$ if and only if φ is not a real analytic diffeomorphism.

Self map with fixed points

Theorem 1

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic function with a fixed point u and let us consider the map $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$.

(a) If $\varphi'(u) = 1$, then 1 is the only eigenvalue and

  (i) either $\varphi = \mathrm{id}$ and in this case the eigenspace of C_φ is equal to $\mathscr{A}(\mathbb{R})$

  (ii) or $\varphi \neq \mathrm{id}$ and the eigenspace is one dimensional.

(b) If $\varphi'(u) = -1$ then

  (i) either $\varphi^{[2]} = \mathrm{id}$ but $\varphi \neq \mathrm{id}$ and in this case there are two eigenvalues ± 1 and $\mathscr{A}(\mathbb{R})$ is a direct sum of two eigenspaces

  (ii) or $\varphi^{[2]} \neq \mathrm{id}$, 1 is the only eigenvalue and its eigenspace is one-dimensional.

Self map with fixed points

Theorem 1 continued

(c) If $\varphi'(u) = 0$ then 1 is the only eigenvalue and its eigenspace is one-dimensional.

(d) If $0 < |\varphi'(u)| < 1$ then

   (i) either $\varphi^{[2]}$ has at least two fixed points and then 1 is the only eigenvalue and its eigenspace is one-dimensional

   (ii) or $((\varphi'(u))^n)_{n\in\mathbb{N}}$ is the sequence of eigenvalues and all of them have one-dimensional eigenspaces.

(e) If $1 < |\varphi'(u)|$ then

   (i) either $\varphi^{[2]}$ has at least two fixed points or φ has a critical point and then in both cases 1 is the only eigenvalue and its eigenspace is one-dimensional

   (ii) or $((\varphi'(u))^n)_{n\in\mathbb{N}}$ is the sequence of eigenvalues and all of them have one-dimensional eigenspaces.

Self map with fixed points

Proposition

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map with a fixed point $u \in \mathbb{R}$.

(1) If -1 is an eigenvalue of C_φ, then $\varphi^{[2]} = \mathrm{id}$.

(2) If $\varphi^{[2]} = \mathrm{id}$ but $\varphi \neq \mathrm{id}$, then C_φ has only two eigenvalues 1 and -1 and $\varphi'(u) = -1$. In this case each $f \in \mathscr{A}(\mathbb{R})$ can be decomposed as $f = f_1 + f_2$, where f_1 (resp. f_2) is an eigenvector with eigenvalue 1 (resp. -1). In this case both eigenspaces are isomorphic to the space of even real analytic functions $\mathscr{A}_+(\mathbb{R})$ on $\mathbb{R}$.

Self map with fixed points. Examples

Examples

- $\varphi(x) = ax, a \notin \{0, 1\}$. Eigenvalues $\{a^n, n = 0, 1, 2, ...\}$, with eigenvector x^n of a^n.
- $\varphi(x) = x^n, n = 2, 3, 4, ...$ In this case $\varphi^{[2]}$ has at least two fixed points, and 1 is the only eigenvalue and its eigenspace is one-dimensional.
- $\varphi(x) = \sin(ax), a > 0$. If $0 < a \leq 1$, then 0 is the only fixed point of φ and $\varphi'(0) = a$; hence $(a^n)_{n\in\mathbb{N}}$ is the sequence of eigenvalues and all of them have one-dimensional eigenspaces. If $a > 1$, then φ has 3 fixed points and 1 is the only eigenvalue and its eigenspace is one-dimensional.
- $\varphi(x) = e^x - 1$. In this case 0 is the only fixed point of φ and $\varphi'(0) = 1$, 1 is the only eigenvalue and the eigenspace is one dimensional.

Self map with fixed points and spectrum of C_φ

It follows from Theorem 1 that the values $(\varphi'(u))^n$ are sometimes eigenvalues and sometimes they are not. They are always elements of the spectrum by a result that is proved with a technique due to Hammond, 2003.

Theorem

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic function and let $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ be the associated composition operator. If u is a fixed point of φ such that $|\varphi'(u)| \neq 1, 0$, then $\varphi'(u)^n \in \sigma(C_\varphi)$ for each $n \in \mathbb{N}_0$.

If $\varphi(x) = x^4$, then $4^n \in \sigma(C_\varphi) \setminus \sigma_p(C_\varphi)$ for all $n \in \mathbb{N}$.

Self map without fixed points and the Abel equation

Abel equation

$$f(\varphi(x)) = f(x) + 1.$$

Clearly, if φ has a fixed point, there is no solution of the Abel equations.

If φ has no fixed points, then either $\varphi > \mathrm{id}$ or $\varphi < \mathrm{id}$.

The Abel equation is another classical subject. It was probably mentioned for the first time by Abel in a note published posthumously. There is also a broad literature about the equation in various function classes.

Self map without fixed points and the Abel equation

- The Abel equation was solved in real analytic functions globally on $\mathbb{R}$ for $\varphi = \exp$ by **Kneser in 1949**.
- Belitskii and Lyubich obtained in 1999 a characterization of **real analytic diffeomorphisms** φ for which the Abel equation is solvable (iff φ has no fixed point).
- In 1998 they had shown that a necessary condition for real analytic solvability of the Abel equation is that all compact sets $K \subset \mathbb{R}$ are **wandering**, i.e., that there is $\nu \in \mathbb{N}$ such that for $n, m \in \mathbb{N}$, $|n - m| > \nu$ holds $\varphi^{[n]}(K) \cap \varphi^{[m]}(K) = \emptyset$.

A consequence of Belitskii and Lyubich's result

Theorem. Belitskii and Lyubich, 1999

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic diffeomorphism without fixed points. Then φ is real analytic conjugate to the shift $x \to x + 1$.

Corollary

Let $\varphi : I \to I$ be a real analytic diffeomorphism without fixed points on an open interval I in $\mathbb{R}$. Then $C_\varphi : \mathscr{A}(I) \to \mathscr{A}(I)$ is a hypercyclic operator; i.e. there is $f \in \mathscr{A}(I)$ with a dense orbit in $\mathscr{A}(I)$.

In particular, if $\varphi(x) = x^2$ in $I =]0, 1[$, then C_φ is hypercyclic. This solves a problem asked by Bonet and Domański in 2012, that had been answered already by A. Peris in a different way.

Abel equation and eigenvalues of C_φ

Proposition. Kneser, 1949

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map such that the Abel equation $f \circ \varphi = f + 1$ has a real analytic solution f_0. Then each $\lambda \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ and this operator has an infinite dimensional eigenspace for the eigenvalue λ. Moreover, for every $\lambda \neq 0$ there is an eigenvector f which does not vanish at any point.

Proposition. Kneser, 1949

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map such that some $\lambda \in \mathbb{C} \setminus \{0, 1\}$ is an eigenvalue of $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ with a never vanishing eigenvector $f_0 \in \mathscr{A}(\mathbb{R})$. Then the Abel equation $f \circ \varphi = f + 1$ has a real analytic solution f.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map.

- For any $x \in \mathbb{R}$ we denote by $O(x)$ the **full orbit** of x via φ, i.e.,

$$O(x) := \{y : \ \exists\, k, l \in \mathbb{N} : \varphi^{[k]}(x) = \varphi^{[l]}(y)\}.$$

  The full orbits form a partition of $\mathbb{R}$.

- The quotient topological space with respect to that partition is denoted by $\mathbb{R}/\varphi$ and the corresponding (continuous) canonical quotient map by $\pi_\varphi : \mathbb{R} \to \mathbb{R}/\varphi$.

- Our study of the natural manifold structure on $\mathbb{R}/\varphi$ for non-diffeomorphic φ is inspired by the method presented by Belitskii and Lyubich in 1999, but it requires further analysis and work.

Main Result

Theorem

The following assertions are equivalent for a real analytic function $\varphi : \mathbb{R} \to \mathbb{R}$:

(a) Every complex $\lambda \neq 0$ is an eigenvalue of C_φ with at least one real analytic eigenvector non-vanishing at any point.

(a') Every complex $\lambda \neq 0$ is an eigenvalue of C_φ with an infinite dimensional eigenspace.

(b) There is a complex eigenvalue $\lambda \neq 1$ for C_φ with at least one real analytic eigenvector non-vanishing at any point.

(b') There is a complex eigenvalue $\lambda \neq 1$ for C_φ and φ has no fixed point.

(c) There is a non-constant eigenvector for the eigenvalue 1 and $\varphi^{[2]} \neq \mathrm{id}$.

Main Result continued

Theorem continued

The following assertions are equivalent to (a)-(c):

(d) The space $\mathbb{R}/\varphi$ of full orbits of φ is a manifold homeomorphic to $\mathbb{T}$ which has a real analytic structure making the canonical map $\pi_\varphi : \mathbb{R} \to \mathbb{R}/\varphi$ real analytic (and, of course, then $\mathbb{R}/\varphi$ is real analytic diffeomorphic to $\mathbb{T}$).

(e) Either $\varphi > \mathrm{id}$ and the set of critical points of φ is bounded from above or $\varphi < \mathrm{id}$ and the set of critical points of φ is bounded from below.

(f) The Abel equation $f \circ \varphi = f + 1$ has a real analytic solution f.

Main Result continued again

Theorem continued

If these conditions hold then for $\lambda > 0$ there is at least one strictly positive eigenvector. Moreover, there is a real analytic solution f_0 of the Abel equation with real values such that the set of critical points is bounded from above (in case $\varphi > \mathrm{id}$) or bounded from below (in case $\varphi < \mathrm{id}$).

In that case for every complex $\lambda \neq 0$, $e^{\mu} = \lambda$, the map

$$T_\lambda : \mathscr{A}(\mathbb{T}) \to \ker(C_\varphi - \lambda I), \qquad T_\lambda(g) := [\exp \circ (\mu f_0)] \cdot [g \circ q \circ f_0],$$

is a topological isomorphism of $\mathscr{A}(\mathbb{T})$ onto the eigenspace of C_φ for λ (here $q : \mathbb{R} \to \mathbb{T}$, $q(x) := \exp(2\pi i x)$).

Self map without fixed points. Examples

Examples

- $\varphi(x) = e^{ax}, a > 0$ has no fixed points. Abel equation $f(\varphi(x)) = f(x) + 1$ has a real analytic solution $f \in \mathscr{A}(\mathbb{R})$. In this case $\sigma_p(C_\varphi) = \mathbb{C} \setminus \{0\}$ and $\sigma(C_\varphi) = \mathbb{C}$.
- $\varphi(x) = x + 1 + a\sin(a^{-1}x), 0 < a < 1$, is real analytic, it is a (continuous) homeomorphism on $\mathbb{R}$, has no fixed points, but it has an unbounded sequence of critical points, namely $\varphi'(x) = 0$ if and only if $x = 2s\pi a, s \in \mathbb{Z}$. By our main Theorem, Abel equation $f(\varphi(x)) = f(x) + 1$ has no real analytic solution $f \in \mathscr{A}(\mathbb{R})$. The only eigenvalue of C_φ is 1.
- (Belitskii, Lyubich, 1999) If φ is a real analytic diffeomorphism without fixed points, then Abel equation $f(\varphi(x)) = f(x) + 1$ has a real analytic solution $f \in \mathscr{A}(\mathbb{R})$, and $\sigma_p(C_\varphi) = \sigma(C_\varphi) = \mathbb{C} \setminus \{0\}$.

Abel equation and iteration semigroups

The motivation of Kneser for solving the Abel equation comes form the problem of finding an iteration root of the exponential map exp, i.e., of a real analytic function r such that $r^{[2]} = \exp$.

If f is an invertible solution of the Abel or Schröder equations ($\lambda > 0$), then $G(t,x) = f^{-1}(f(x) + t)$ or $G(t,x) = f^{-1}(\lambda^t f(x))$, respectively, is a so-called **real analytic iteration semigroup** in which φ embeds.

We say that φ **embeds in the real analytic iteration semigroup** G if G is real analytic satisfying the following conditions

$$G : (\mathbb{R}_+ \cup \{0\}) \times \mathbb{R} \to \mathbb{R},$$
$$G(t+s,x) = G(t, G(s,x)), \quad G(n,x) = \varphi^{[n]}(x), \text{for } n = 0, 1, \ldots .$$

Clearly, in this case $r(x) = G(1/2, x)$ is the required root of φ.

Abel equation and iteration semigroups

Theorem

A real analytic map $\varphi : \mathbb{R} \to \mathbb{R}$ embeds into a real analytic iteration semigroup whenever φ has no critical points and either φ has no fixed points or $\varphi^{[2]}$ has exactly one fixed point u and $0 < \varphi'(u) \neq 1$. In particular in that case there exist roots of the operator C_φ of arbitrary order.

It is known that there is no real analytic iteration root for $\varphi(x) = \exp(x) - 1$.

Conjecture

A real analytic map $\varphi : \mathbb{R} \to \mathbb{R}$ embeds into a real analytic iteration semigroup if and only if it has no critical point, has at most one fixed point u and in that case $0 < \varphi'(u) \neq 1$.

The case we cannot decide is when there is a fixed point u such that $\varphi'(u) = 1$. Important work related to this topic is due e.g. to I.N. Baker, M. Kuczma, G. Szekeres and P.L. Walker.

References

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