A tale of two inequalities

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Abstract

The Khintchin inequality (1930) and the Grothendieck inequality (1953) are among the important and fundamental mathematical discoveries in the last century, each a milestone in the development of modern analysis. I will discuss certain upgrades of these two inequalities, and also a basic connection between them.

The setting

Start with the space of $\{-1, 1\}$-valued functions on $\mathbb{N}$,

$$\Omega = \{-1, 1\}^{\mathbb{N}},$$

endowing it with the usual product topology, i.e.,

$$\omega_n \underset{n\to\infty}{\longrightarrow} \omega \quad \text{in} \quad \Omega \quad \Leftrightarrow \quad \omega_n(j) \underset{n\to\infty}{\longrightarrow} \omega(j) \quad \text{for each} \ \ j \in \mathbb{N}.$$

Define multiplication in Ω by coordinate-wise multiplication, i.e., if $\omega, \ \omega' \in \Omega$, then

$$(\omega \cdot \omega')(j) = \omega(j)\omega'(j), \quad j \in \mathbb{N}.$$

Let $\mathbb{P}$ be the uniform probability measure on Ω, i.e., $\mathbb{P}$ is the infinite product measure $\mathbb{P}_0^{\mathbb{N}}$, where

$$\mathbb{P}_0(\{1\}) = \mathbb{P}_0(\{-1\}) = \frac{1}{2}.$$

The Rademacher system and the Walsh system

Consider the system of projections $R := \{r_j : j \in \mathbb{N}\}$,

$$r_j(\omega) = \omega(j), \quad \omega \in \Omega, \;\; j \in \mathbb{N} \quad \text{(Rademacher characters)},$$

and adjoin to it the function $r_0 \equiv 1$ on Ω.

Note. The system R is *independent*.

Next take all finite products of Rademacher characters

$$W := \left\{ w : w = \prod_{r \in F} r, \; F \subset R, \; |F| < \infty \right\} \quad \text{(Walsh characters)}.$$

Then,

$$W = \widehat{\Omega} \quad \text{(continuous characters on } \Omega).$$

I.e., W is an orthonormal basis for $L^2(\Omega, \mathbb{P})$, generated by R.

Walsh series

Recall the classical Banach spaces

$$L^p(\Omega, \mathbb{P}) \;\; (1 \leqslant p \leqslant \infty), \quad C(\Omega), \quad M(\Omega),$$

and the duality between them

$$L^p(\Omega, \mathbb{P})^* = L^q(\Omega, \mathbb{P}), \;\; 1 \leqslant p < \infty, \; q = \frac{p}{p-1}, \quad C(\Omega)^* = M(\Omega).$$

For $f \in L^1(\Omega, \mathbb{P})$ and $\mu \in M(\Omega)$, we have the Walsh transforms

$$\hat{f}(w) = \int_{\omega \in \Omega} w(\omega) f(\omega) \mathbb{P}(d\omega), \quad \hat{\mu}(w) = \int_{\omega \in \Omega} w(\omega) \mu(d\omega), \quad w \in W,$$

and then the representations of f and μ by the Walsh series

$$S[f] \sim \sum_{w \in W} \hat{f}(w) w, \quad S[\mu] \sim \sum_{w \in W} \hat{\mu}(w) w \, .$$

Issues regarding spectra

Note the proper inclusions

$$\mathcal{A}(\Omega) \subsetneqq C(\Omega) \subsetneqq L^\infty(\Omega,\mathbb{P}) \subsetneqq L^p(\Omega,\mathbb{P}) \subsetneqq L^2(\Omega,\mathbb{P}) \subsetneqq L^1(\Omega,\mathbb{P}) \subsetneqq M(\Omega),$$

where

$$\mathcal{A}(\Omega) = \{f \in C(\Omega) : \sum_{w\in W} |\hat{f}(w)| < \infty\}.$$

Let $\mathcal{D}(\Omega)$ stand for any of the spaces above, and define

$$\mathcal{D}_E(\Omega) = \{\mathbf{x} \in \mathcal{D}(\Omega) : \hat{\mathbf{x}}(w) = 0,\ w \notin E\}, \quad E \subset W.$$

Questions. $E \subset W$,

$$\mathcal{A}_E(\Omega) \stackrel{?}{=} C_E(\Omega) \stackrel{?}{=} L^\infty_E(\Omega,\mathbb{P}) \stackrel{?}{=} L^p_E(\Omega,\mathbb{P}) \stackrel{?}{=} L^2_E(\Omega,\mathbb{P}) \stackrel{?}{=} L^1_E(\Omega,\mathbb{P}) \stackrel{?}{=} M_E(\Omega).$$

Independence of the Rademacher system implies

Theorem 1

$$\mathcal{A}_R(\Omega) = C_R(\Omega) = L^\infty_R(\Omega, \mathbb{P}) \subsetneqq L^p_R(\Omega, \mathbb{P}) = L^2_R(\Omega, \mathbb{P}) = L^1_R(\Omega, \mathbb{P}) = M_R(\Omega).$$

- The first two equalities (*Sidon* property and *Rosenthal property*) are easy to verify.
- So is the *proper* inclusion $L^\infty_R \subsetneqq L^p_R,\ p < \infty$.
- The equality $L^p_R = L^2_R$ ($\Lambda(p)$-property) is a consequence of the Khintchin inequalities (Khintchin, 1924).
- The equality $L^2_R = L^1_R$ ($\Lambda(2)$-property) follows from the preceding equality (Littlewood, 1930).
- The last equality $L^1_R = M_R$ (*Riesz* property) is easy to verify.

Our focus is on the $\Lambda(2)$-property of R.

The $L^1_R \hookrightarrow L^2$ - Khintchin inequality

Theorem 2 (Littlewood, 1930)

For every $f \in L^2_R(\Omega, \mathbb{P})$*,*

$$\|f\|_{L^1(\Omega,\mathbb{P})} \leqslant \|f\|_{L^2(\Omega,\mathbb{P})} \leqslant \sqrt{3}\|f\|_{L^1(\Omega,\mathbb{P})}. \tag{1}$$

I.e.,

$$\sup\left\{\frac{\|f\|_{L^2(\Omega,\mathbb{P})}}{\|f\|_{L^1(\Omega,\mathbb{P})}} : f \in L^2_R,\ f \neq \mathbf{0}\right\} := \kappa \leqslant \sqrt{3}. \tag{2}$$

Note.

$$\kappa < \infty \iff L^1_R(\Omega, \mathbb{P}) = L^2_R(\Omega, \mathbb{P}). \tag{3}$$

The assertion in (2) with various upper estimates for the *Khintchin constant* κ was observed nearly a century ago, independently, by Littlewood, Orlicz, Steinhaus, and Zygmund; $\kappa = \sqrt{2}$ was proved by S. Szarek in his 1976 Master's thesis.

Littlewood's proof

Step 1. By independence of R and elementary counting,

$$\int_\Omega \Big|\sum_n \mathbf{x}(n) r_n\Big|^4 d\mathbb{P} \;=\; \sum_{n_1,n_2,n_3,n_4} \mathbf{x}(n_1)\mathbf{x}(n_2)\overline{\mathbf{x}(n_3)\mathbf{x}(n_4)}\; \int_\Omega r_{n_1} r_{n_2} r_{n_3} r_{n_4} d\mathbb{P}$$

$$\leqslant\; 3 \sum_{n_1,n_2} |\mathbf{x}(n_1)|^2\, |\mathbf{x}(n_2)|^2.$$

I.e.,

$$\Big\|\sum_n \mathbf{x}(n) r_n\Big\|_{L^4} \;\leqslant\; 3^{\frac{1}{4}} \Big\|\sum_n \mathbf{x}(n) r_n\Big\|_{L^2}. \tag{4}$$

Step 2. Write

$$\int_\Omega \Big|\sum_n \mathbf{x}(n) r_n\Big|^2 d\mathbb{P} \;=\; \int_\Omega \Big|\sum_n \mathbf{x}(n) r_n\Big|^{\frac{2}{3}} \Big|\sum_n \mathbf{x}(n) r_n\Big|^{\frac{4}{3}} d\mathbb{P}.$$

By Hölder, Step 1, and arithmetic,

$$\Big\|\sum_n \mathbf{x}(n) r_n\Big\|_{L^2} \;\leqslant\; \sqrt{3}\, \Big\|\sum_n \mathbf{x}(n) r_n\Big\|_{L^1}.$$

First application: Littlewood's mixed-norm inequality

Let $(a_{jk})_{j,k}$ be a finite scalar array. Then,

$$\sum_j \Big(\sum_k |a_{jk}|^2 \Big)^{\frac{1}{2}} \leqslant 2\sqrt{2} \sup_{\omega_1,\omega_2} \Big| \sum_{j,k} a_{jk} r_j(\omega_1) r_k(\omega_2) \Big|. \tag{5}$$

Proof:

$$\sup_{\omega_1,\omega_2} \Big| \sum_{j,k} a_{jk} r_j(\omega_1) r_k(\omega_2) \Big| = \sup_{\omega_1} \sup_{\omega_2} \left| \sum_j \left(\sum_k a_{jk} r_k(\omega_2) \right) r_j(\omega_1) \right|$$

$$\geqslant \frac{1}{2} \sup_{\omega} \sum_j \Big| \sum_k a_{jk} r_k(\omega) \Big|$$

$$\geqslant \frac{1}{2} \sum_j \int_\Omega \Big| \sum_k a_{jk} r_k(\omega) \Big| \mathbb{P}(d\omega) \geqslant \frac{1}{2\sqrt{2}} \sum_j \Big(\sum_k |a_{jk}|^2 \Big)^{\frac{1}{2}}$$

(by the Khintchin $(L^1_R \hookrightarrow L^2)$-inequality).

Foreshadowing the Grothendieck inequality...

Littlewood's mixed-norm inequality asserts:

for all finite scalar arrays $(a_{jk})_{j,k}$,

$$\sup\Big\{\Big|\sum_{j,k} a_{jk}\,\langle \mathbf{e}_j, \mathbf{y}_k\rangle\Big| : \mathbf{y}_k \in B_{\ell^2}\Big\} \;\leqslant\; 2\sqrt{2}\,\Big\|\sum_{j,k} a_{jk} r_j \otimes r_k\Big\|_{\infty}, \qquad (6)$$

where $\mathbf{e}_j,\ j \in \mathbb{N},$ is the standard basis in the Euclidean space $\ell^2(\mathbb{N})$, $\langle\cdot,\cdot\rangle$ is the usual dot product, and B_{ℓ^2} is the Euclidean ball in $\ell^2(\mathbb{N})$.

Whereas **Grothendieck's inequality** [Grothendieck, 1953] asserts:

there exists $K > 0$ such that for all finite scalar arrays $(a_{jk})_{j,k}$,

$$\sup\Big\{\Big|\sum_{j,k} a_{jk}\,\langle \mathbf{x}_j, \mathbf{y}_k\rangle\Big| : \mathbf{x}_j, \mathbf{y}_k \in B_{\ell^2}\Big\} \;\leqslant\; K\Big\|\sum_{j,k} a_{jk} r_j \otimes r_k\Big\|_{\infty}. \qquad (7)$$

The dual statement

The Khintchin $(L^1_R \hookrightarrow L^2)$-inequality in its dual form,

$$(L^2_R)^* = (L^1_R)^*,$$

becomes (via Hahn-Banach, F. Riesz, and Parseval)

$$\ell^2(\mathbb{N}) = \big(L^\infty(\Omega, \mathbb{P})\big)^\wedge\Big|_R \,.$$

This means: there exists a mapping

$$G:\ \ell^2(\mathbb{N}) \ \to\ L^\infty(\Omega, \mathbb{P}),$$

such that
$$G(\mathbf{x}) = \sum_n \mathbf{x}(n) r_n + g(\mathbf{x}),$$
$$g(\mathbf{x}) \in L^2_{W\setminus R} = \big(L^2_R\big)^\perp, \qquad \mathbf{x} \in \ell^2(\mathbb{N}).$$

We refer to G as an L^∞-***interpolant***, and to g as a ***perturbation***.

What can be said about the interpolants?

The Khintchin ($L^1_R \hookrightarrow L^2$)-inequality, with $\kappa = \sqrt{2}$, implies: there exist perturbations g, such that for all $\mathbf{x} \in B_{\ell^2}$,

$$\|g(\mathbf{x})\|_{L^2} \leqslant 1, \tag{8}$$

and

$$\|\sum_n \mathbf{x}(n) r_n + g(\mathbf{x})\|_{L^\infty} \leqslant \sqrt{2}. \tag{9}$$

Question 1. How "small" can g be, and still satisfy

$$\sup \left\{ \|\sum_n \mathbf{x}(n) r_n + g(\mathbf{x})\|_{L^\infty} : \mathbf{x} \in B_{\ell^2} \right\} < \infty? \tag{10}$$

To phrase **Question 1** precisely, for $\delta > 0$ and $\mathbf{x} \in B_{\ell^2}$, let

$$u_R(\mathbf{x};\delta) = \inf\left\{ \Big\| \sum_n \mathbf{x}(n) r_n + g(\mathbf{x}) \Big\|_{L^\infty} : g(\mathbf{x}) \in \left(L^2_R\right)^\perp,\ \|g(\mathbf{x})\|_{L^2} \leqslant \delta \right\},$$

and then define

$$u_R(\delta) = \sup\left\{ u_R(\mathbf{x};\delta) : \mathbf{x} \in B_{\ell^2} \right\} \quad \text{(uniformizing constants).}$$

Problem. Compute $u_R(\delta)$, $\delta > 0$.

Note. $u_R(1) = \sqrt{2}$ follows from $\kappa = \sqrt{2}$, but it is not obvious that

$$u_R(\delta) < \infty, \quad 0 < \delta < 1.$$

Another issue...

The Khintchin inequality guarantees existence of a perturbation g (via Hahn-Banach and F. Riesz), but does not guarantee its continuity.

Question 2. Can $g(\mathbf{x}),\ \mathbf{x} \in B_{\ell^2}$, be constructed continuously in $\left(L^2_R\right)^\perp$, say, with respect to the Euclidean norm in $\ell^2(\mathbb{N})$, or the weak topology in B_{ℓ^2}?

Specifically, let

$$\kappa_c = \inf\left\{\Big\|\sum_n \mathbf{x}(n) r_n + g(\mathbf{x})\Big\|_{L^\infty} : \text{norm-continuous } g : B_{\ell^2} \to \left(L^2_R\right)^\perp\right\},$$

$$\kappa_{wc} = \inf\left\{\Big\|\sum_n \mathbf{x}(n) r_n + g(\mathbf{x})\Big\|_{L^\infty} : \text{weak-continuous } g : B_{\ell^2} \to \left(L^2_R\right)^\perp\right\}.$$

Problem. $\kappa_c = ?\quad \kappa_{wc} = ?$

An L^∞-valued Riesz product

Define the product

$$\mathfrak{R}(\mathbf{x}) \sim \prod_{n\in\mathbb{N}} \big(1 + \mathbf{x}(n) r_n\big), \quad \mathbf{x} \in \mathbb{R}^{\mathbb{N}}, \tag{11}$$

to be the formal Walsh series

$$\mathfrak{R}(\mathbf{x}) \sim \sum_{k=1}^{\infty} \Bigg(\sum_{\{n_1,\ldots,n_k\}\subset\mathbb{N}} \mathbf{x}(n_1)\cdots\mathbf{x}(n_k) r_{n_1}\cdots r_{n_k} \Bigg), \tag{12}$$

and let

$$Q(\mathbf{x}) := \mathfrak{Im}\, \mathfrak{R}(\mathrm{i}\mathbf{x})$$

$$\sim \sum_{k=1}^{\infty} (-1)^{k-1} \Bigg(\sum_{\{n_1,\ldots,n_{2k-1}\}\subset\mathbb{N}} \mathbf{x}(n_1)\cdots\mathbf{x}(n_{2k-1}) r_{n_1}\cdots r_{n_{2k-1}} \Bigg)$$

where $\mathrm{i} = \sqrt{-1}$, and $\mathfrak{Im}$ denotes the imaginary part.

Lemma 3 (Salem and Zygmund, 1948)

If $\mathbf{x} \in \ell^2_{\mathbb{R}}(\mathbb{N})$ *(=* $\mathbb{R}$*eal Euclidean space), then* $Q(\mathbf{x}) \in L^\infty(\Omega, \mathbb{P})$*, and*

$$\|Q(\mathbf{x})\|_{L^\infty} \leqslant e^{\frac{\|\mathbf{x}\|_2^2}{2}}. \tag{13}$$

Proof.

For $N > 0$, $\mathbf{x} \in \ell^2_{\mathbb{R}}(\mathbb{N})$, estimate

$$\begin{aligned} \Big\| \prod_{n=1}^{N} \big(1 + \mathrm{i}\mathbf{x}(n) r_n\big) \Big\|_{L^\infty} &= \left(\prod_{n=1}^{N} \big(1 + |\mathbf{x}(n)|^2\big) \right)^{\frac{1}{2}} \\ &= e^{\frac{1}{2} \sum_{n \in F} \log(1 + |\mathbf{x}(n)|^2)} \\ &\leqslant e^{\frac{\|\mathbf{x}\|_2^2}{2}}. \end{aligned} \tag{14}$$

Now take a weak*-L^∞ limit.

$\square$

Uniformizability and continuity

Theorem 4

Let

$$G_u(\mathbf{x}) = uQ(\mathbf{x}/u), \quad u > 0, \ \ \mathbf{x} \in B_{\ell^2}. \tag{15}$$

Then,

$$\begin{aligned} G_u(\mathbf{x}) \ &= \ \sum_n \mathbf{x}(n) r_n \ + \ g_u(\mathbf{x}), \\ \|G_u(\mathbf{x})\|_{L^\infty} \ &\leqslant \ u e^{1/2u^2}, \qquad u > 0, \ \ \mathbf{x} \in B_{\ell^2}, \end{aligned} \tag{16}$$

where $g_u(\mathbf{x}) \in L^2_{W\setminus R}$, *and*

$$\|g_u(\mathbf{x})\|_{L^2} \leqslant u\sqrt{\sinh(\|\mathbf{x}/u\|_2^2) - \|\mathbf{x}/u\|_2^2}. \tag{17}$$

Moreover, $g_u : B_{\ell^2} \to L^2(\Omega, \mathbb{P})$ *is both norm- and weakly continuous.*

In particular,

$$\|g_u(\mathbf{x})\|_{L^2} \;\leqslant\; \frac{1}{u^2}, \quad u \geqslant 1, \;\; \|\mathbf{x}\|_2 \leqslant 1, \tag{18}$$

and therefore,

Corollary 5

$$\begin{gathered} u_R(\delta) = \mathcal{O}(\frac{1}{\sqrt{\delta}}), \quad \delta \in (0,1). \\ \sqrt{2} \;\leqslant \kappa_c \leqslant\; 2\sqrt{e}, \qquad \sqrt{2} \;\leqslant\; \kappa_{wc} \;\leqslant\; 2\sqrt{e}. \end{gathered} \tag{19}$$

The Grothendieck inequality

There exists $1 < K < \infty$, such that for every finite scalar array (a_{jk}),

$$\sup\left\{\big|\sum_{j,k} a_{jk}\langle \mathbf{x}_j, \mathbf{y}_k\rangle\big| : \mathbf{x}_j, \mathbf{y}_k \in B_{\ell^2}\right\} \leqslant K \sup\left\{\big|\sum_{j,k} a_{jk} s_j t_k\big| : s_j, t_k \in [-1, 1]\right\}$$

An equivalent assertion had appeared in Grothendieck's 1953 *Resumé*, and remained unnoticed until its reformulation above in [Lindenstrauss and Pelczynski,1968]. Since its reformulation, known as *the Grothendieck inequality*, it has been applied in functional, harmonic, and stochastic analysis, and recently also in theoretical physics and theoretical computer science. (See [Pisier, 2012].)

The evaluation of the "smallest" K, denoted by $\mathcal{K}_G$ and dubbed *the Grothendieck constant*, is an open problem. For the latest on it, see [Braverman et al., 2011].

The dual statement

Proposition 1

The Grothendieck inequality holds $\Leftrightarrow$ *there exists a probability space* $(\mathcal{X}, \mu)$, *and a one-one map*

$$\Phi : B_{\ell^2} \to L^{\infty}(\mathcal{X}, \mu), \tag{20}$$

such that

$$\|\Phi(\mathbf{x})\|_{L^\infty} \leqslant K, \quad \mathbf{x} \in B_{\ell^2}, \tag{21}$$

and

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{\mathcal{X}} \Phi(\mathbf{x}) \Phi(\overline{\mathbf{y}}) d\mu, \quad \mathbf{x} \in B_{\ell^2},\ \mathbf{y} \in B_{\ell^2}. \tag{22}$$

Note. The Grothendieck inequality implies $\mathcal{X} = \Omega_{B_{\ell^2}} \times \Omega_{B_{\ell^2}}$, where

$$\Omega_{B_{\ell^2}} = \{-1, 1\}^{B_{\ell^2}}.$$

Khintchin falls short

The Khintchin $(L^1_R \hookrightarrow L^2)$-inequality implies an instance of the Grothendieck inequality (the Littlewood mixed-norm inequality), but not the full statement. Specifically, it provides an L^∞-interpolant

$$G = U + g : \ell^2(\mathbb{N}) \to L^\infty(\Omega, \mathbb{P}),$$

where

$$U\mathbf{x} = \sum_n \mathbf{x}(n) r_n, \quad g(\mathbf{x}) \in \left(L^2_R\right)^\perp, \quad \mathbf{x} \in \ell^2(\mathbb{N}),$$

and

$$\int_\Omega G(\mathbf{x}) G(\overline{\mathbf{y}}) d\mathbb{P} = \langle \mathbf{x}, \mathbf{y} \rangle + \int_\Omega g(\mathbf{x}) g(\overline{\mathbf{y}}) d\mathbb{P}, \tag{23}$$

but does not assure that second term on the right side of (23) vanishes.

Idea. Let $G = G_2$ of Theorem 4, in which case

$$\|g(\mathbf{x})\|_{L^2} \leqslant \frac{1}{4}, \quad \mathbf{x} \in B_{\ell^2},$$

and then uniformizability does it...

Via Parseval,

$$\begin{aligned} \int_\Omega G(\mathbf{x})G(\overline{\mathbf{y}})\,d\mathbb{P} &= \langle \mathbf{x}, \mathbf{y}\rangle + \int_\Omega g(\mathbf{x})g(\overline{\mathbf{y}})\,d\mathbb{P} \\ &= \langle \mathbf{x}, \mathbf{y}\rangle \ + \ \text{"error"} \\ &= \langle \mathbf{x}, \mathbf{y}\rangle + \sum_{w \in W\setminus R} \widehat{g(\mathbf{x})}(w)\widehat{g(\overline{\mathbf{y}})}(w). \end{aligned} \tag{24}$$

- Note: "error" is a dot product of of two vectors in $\ell^2(W\setminus R)$.
- Apply G to each of these two vectors (after re-indexing), and apply Parseval's formula again.
- Subtract result from (24), and repeat...

Convergence of the recursion is guaranteed by uniformizability of R.

A Parseval-like formula

Theorem 6

There exists an injection

$$\Phi : B_{\ell^2} \to L^{\infty}(\Omega, \mathbb{P}),$$

which is $(\ell^2 \to L^2)$*-continuous, (weak-*$\ell^2 \to$ *weak*-*L^∞*)-continuous, and such that*

$$\|\Phi(\mathbf{x})\|_{L^\infty} \leqslant K, \quad \mathbf{x} \in B_{\ell^2}, \tag{25}$$

where $K > 1$ *is a universal constant, and*

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_\Omega \Phi(\mathbf{x})\Phi(\overline{\mathbf{y}})\,d\mathbb{P}, \quad \mathbf{x},\ \mathbf{y} \in B_{\ell^2}. \tag{26}$$

Note. The Grothendieck constant $\mathcal{K}_G \leqslant K^2$, where K is the constant in (25).

Constants?

Let

$$\|\Phi\|_{\infty,L^\infty} = \sup\left\{\|\Phi(\mathbf{x})\|_{L^\infty} : \mathbf{x} \in B_{\ell^2}\right\},$$

where Φ is the injection in Theorem 6, and let

$$\mathcal{K}_{GC} = \inf_{\Phi} \|\Phi\|^2_{\infty,L^\infty},$$

where *infimum* is taken over all continuous injections

$$\Phi : B_{\ell^2} \to L^\infty(\mathcal{X}, \mu)$$

that satisfy Theorem 6, with $(\mathcal{X}, \mu)$ in place of $(\Omega, \mathbb{P})$.
Then,

$$\mathcal{K}_G \text{ (= the Grothendieck constant)} \leqslant \mathcal{K}_{GC}.$$

Question.

$$\mathcal{K}_G < \mathcal{K}_{GC}\ ?$$

Gracias!