Maximal spaceability and optimal estimates for summing multilinear operators

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Joint work with Daniel Pellegrino

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Introduction

Definition (M. Matos, 2003 – D. Pérez García, 2004)

Let $1 \leq s \leq r < \infty$. A multilinear operator $T \in \mathcal{L}(E_1, \dots, E_m; F)$ is multiple $(r; s)$–summing if there exists a $C > 0$ such that

$$\left(\sum_{j_1,\dots,j_m=1}^{\infty} \left\| T\left(x_{j_1}^{(1)}, \dots, x_{j_m}^{(m)}\right) \right\|^r \right)^{\frac{1}{r}} \leq C \prod_{k=1}^{m} \left\| \left(x_{j_k}^{(k)}\right)_{j_k=1}^{\infty} \right\|_{w,s}$$

for all $\left(x_{j_k}^{(k)}\right)_{j_k=1}^{\infty} \in \ell_s^w(E_k)$, $k \in \{1, \dots, m\}$. We represent the class of all multiple $(r; s)$–summing operators from $E_1, \dots, E_m$ to F by $\prod_{\mathrm{mult}(r,s)}(E_1, \dots, E_m; F)$ and $\pi_{\mathrm{mult}(r,s)}(T)$ denotes the infimum over all C as above.

Theorem (H. F. Bohnenblust and E. Hille, 1931)

There exist $C = C(m) \geq 1$ *such that*

$$\left(\sum_{j_1, \ldots, j_m = 1}^{\infty} \left| T\left(e_{j_1}, \ldots, e_{j_m}\right) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C \|T\|$$

for all continuous m–linear operators $T : c_0 \times \cdots \times c_0 \to \mathbb{K}$ *and the exponent* $\frac{2m}{m+1}$ *is optimal.*

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for all continuous m–linear operators $T : c_0 \times \cdots \times c_0 \to \mathbb{K}$ and the exponent $\frac{2m}{m+1}$ is optimal.

Using a standard argument we can lift the result from c_0 to arbitrary Banach spaces:

Theorem (H. F. Bohnenblust and E. Hille, 1931 – D. Pérez García, 2004)

Let $E_1, ..., E_m$ Banach spaces. Then

$$\mathcal{L}\left(E_1, ..., E_m; \mathbb{K}\right) = \prod\nolimits_{\mathrm{mult}\left(\frac{2m}{m+1};1\right)} \left(E_1, ..., E_m; F\right)$$

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and $\frac{2m}{m+1}$ is optimal.

i) For which values $(r; s) \in [1, \infty) \times [1, r]$ do we have coincidence, i.e., when does

$$\mathcal{L}\left(E_1, \ldots, E_m; \mathbb{K}\right) = \prod\nolimits_{\mathrm{mult}(r;s)}\left(E_1, \ldots, E_m; F\right) \quad \forall\ E_1, \ldots, E_m?$$

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ii) Let $(r; s) \in [1, \infty) \times [1, r]$ and let $E_1, \ldots, E_m$ such that $\mathcal{L}\left(E_1, \ldots, E_m; \mathbb{K}\right) \neq \prod_{\mathrm{mult}(r,s)} \left(E_1, \ldots, E_m; \mathbb{K}\right)$. How "big" is the set

$$\mathcal{L}\left(E_1, \ldots, E_m; \mathbb{K}\right) \setminus \prod\nolimits_{\mathrm{mult}(r,s)} \left(E_1, \ldots, E_m; \mathbb{K}\right)?$$

Maximal spaceability and multiple summability

Definition (Aron, Gurariy, Seoane–Sepúlveda, 2004)

For a given Banach space E, a subset $A \subset E$ is *spaceable* if $A \cup \{0\}$ contains a closed infinite–dimensional subspace V of E. When $\dim V = \dim E$, A is called *maximal spaceable*.

From now on $\mathfrak{c}$ denotes the cardinality of the continuum.

Theorem

Let $m \geq 1,\ p \in [2, \infty)$. *If* $1 \leq s < p^*$ *and* $r < \frac{2ms}{s+2m-ms}$ *then* $\mathcal{L}\left({}^m\ell_p; \mathbb{K}\right) \smallsetminus \prod_{\mathrm{mult}(r,s)}\left({}^m\ell_p; \mathbb{K}\right)$ *is maximal spaceable.*

Sketch of the proof:

Theorem

Let $m \geq 1$, $p \in [2, \infty)$. If $1 \leq s < p^*$ and $r < \frac{2ms}{s+2m-ms}$ then $\mathcal{L}\left({}^m\ell_p; \mathbb{K}\right) \smallsetminus \prod_{\mathrm{mult}(r,s)}\left({}^m\ell_p; \mathbb{K}\right)$ is maximal spaceable.

Sketch of the proof:

• An extended version of the Kahane–Salem–Zygmund inequality (see [2, N. Albuquerque, F. Bayart, D. Pellegrino and J. Seoane–Sepúlveda, 2014]) asserts that if $m, n \geq 1$ and $p \in [2, \infty]$, then there exists a m–linear map $A_n : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ of the form

$$A_n(z^{(1)}, \ldots, z^{(m)}) = \sum_{j_1, \ldots, j_m = 1}^{n} \pm z_{j_1}^{(1)} \cdots z_{j_m}^{(m)} \tag{1}$$

and a constant $C_m > 0$ such that

$$\|A_n\| \leq C_m n^{\frac{m+1}{2} - \frac{m}{p}}.$$

- Let $\beta := \frac{p+s-ps}{ps}$. We have, for $n \geq 2$,

$$\left(n^{1-r\beta}\right)^{\frac{m}{r}} < (1+\log n)^{m\beta}\, \pi_{\mathrm{mult}(r,s)}\left(A_n\right).$$

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- Since $\|A_n\| \leq C_m n^{\frac{m+1}{2}-\frac{m}{p}}$ we have

$$\frac{\pi_{\mathrm{mult}(r,s)}\left(A_n\right)}{\|A_n\|} > \frac{n^{\frac{m}{r}-\left(\frac{p+s-ps}{ps}\right)m}}{(1+\log n)^{m\beta}\, C_m n^{\frac{m+1}{2}-\frac{m}{p}}} = \frac{n^{\frac{m}{r}+\frac{m}{2}-\frac{m}{s}-\frac{1}{2}}}{C_m\,(1+\log n)^{m\beta}}$$

and consequently, by making $n \to \infty$ and using that $r < \frac{2ms}{s+2m-ms}$ we get

$$\lim_{n\to\infty} \frac{\pi_{\mathrm{mult}(r,s)}\left(A_n\right)}{\|A_n\|} = \infty. \tag{2}$$

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Theorem (Drewnowski, 1984)

Let X and Z be Banach spaces and $T : Z \to X$ a continuous linear operator with range $Y = T(Z)$ not closed. Then the complement $X \setminus Y$ is spaceable.

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It remains to prove the maximal spaceability.

It is not difficult to prove that

$$\dim\left(\mathcal{L}({}^m\ell_p;\mathbb{K})\right)=\mathfrak{c}.$$

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Therefore, if

$$V\subseteq\left(\mathcal{L}(^m\ell_p;\mathbb{K})\smallsetminus\prod\nolimits_{\mathrm{mult}(r;s)}(^m\ell_p;\mathbb{K})\right)\cup\{0\}$$

is a closed infinite–dimensional subspace of $\mathcal{L}(^m\ell_p;\mathbb{K})$, we have $\dim(V)\leq\mathfrak{c}$. Since V is Banach, we know that $\dim(V)\geq\mathfrak{c}$ (see for instance [3, G. Botelho, D. Cariello, V. Favaro, D. Pellegrino and J. B. Seoane–Sepúlveda, 2013]). Thus $\dim(V)=\mathfrak{c}$ and the proof is done. $\square$

Corollary

Let $m \geq 2$ and let $E_1, \ldots, E_m$ be Banach spaces.

(i) *If* $r \in \left[\frac{2m}{m+1}, 2\right]$, *then*

$$\mathcal{L}\left(E_1, \ldots, E_m; \mathbb{K}\right) = \prod\nolimits_{\mathrm{mult}\left(r, \frac{2mr}{mr+2m-r}\right)} \left(E_1, \ldots, E_m; \mathbb{K}\right).$$

and the value $\frac{2mr}{mr+2m-r}$ *is optimal.*

(ii) *If* $r \in (2, \infty)$, *the optimal value of* s *such that*

$$\mathcal{L}\left(E_1, \ldots, E_m; \mathbb{K}\right) = \prod\nolimits_{\mathrm{mult}(r,s)} \left(E_1, \ldots, E_m; \mathbb{K}\right)$$

belongs to $\left[\frac{2m}{2m-1}, \frac{m}{m-1}\right)$.

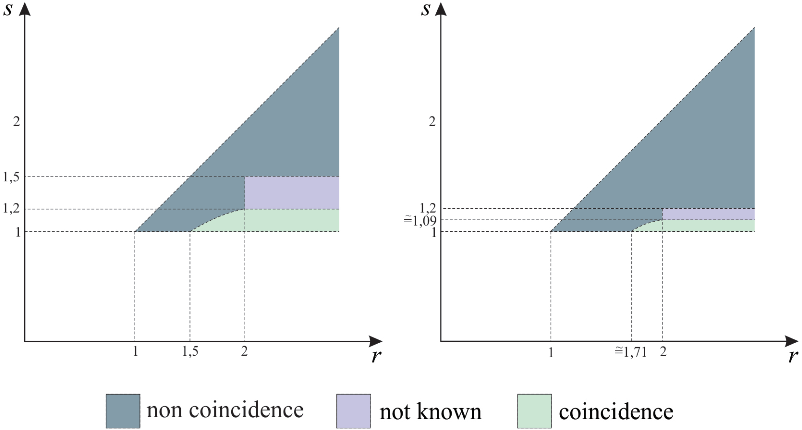

Figure: Coincidence zone for $m = 3$ and $m = 6$, respectively.

The table below details the results of coincidence and non coincidence in the "boundaries" of Figure 1.

$r \geq 1$	$s = r$	non coincidence
$1 \leq r < \frac{2m}{m+1}$	$s = 1$	non coincidence
$\frac{2m}{m+1} \leq r \leq 2$	$s = \frac{2mr}{mr+2m-r}$	coincidence
$r \geq \frac{2m}{m+1}$	$s = 1$	coincidence
$r = 2$	$\frac{2m}{2m-1} < s \leq \frac{m}{m-1}$	non coincidence
$r \geq 2$	$s = \frac{2m}{2m-1}$	coincidence
$r \geq 2$	$s = \frac{m}{m-1}$	non coincidence

Absolutely summing polynomials and multilinear operators

Definition (R. Alencar and M. C. Matos, 1989)

For $1 \leq s < \infty$ and $r \geq \frac{s}{m}$ a continuous m–linear operator $A : E_1 \times \cdots \times E_m \to F$ is absolutely $(r; s)$–summing if there is a $C > 0$ such that

$$\left(\sum_{j=1}^{n} \left\| A\left(x_j^{(1)}, \ldots, x_j^{(m)}\right) \right\|^r \right)^{\frac{1}{r}} \leq C \prod_{k=1}^{m} \left\| \left(x_j^{(k)}\right)_{j=1}^{\infty} \right\|_{w,s}$$

for all positive integers n and all $\left(x_j^{(k)}\right)_{j=1}^{n} \in \ell_s^w\left(E_k\right)$, $k = 1, \ldots, m$.
We represent the class of all absolutely $(r; s)$–summing operators from $E_1, \ldots., E_m$ to F by $\prod_{\mathrm{as}(r,s)}\left(E_1, \ldots, E_m; F\right)$ and $\pi_{\mathrm{as}(r,s)}\left(T\right)$ denotes the infimum over all C as above.

Combining

Theorem (M. C. Matos, 1993)

Let $p \leq q$ *and* $p_j \leq q_j$, $j = 1, \ldots, m$ *be such that*

$$0 \leq \sum_{j=1}^{m} \frac{1}{p_i} - \frac{1}{p} \leq \sum_{j=1}^{m} \frac{1}{q_i} - \frac{1}{q}.$$

Then $\prod_{\mathrm{as}(p;p_1,\ldots,p_m)} \subseteq \prod_{\mathrm{as}(q;q_1,\ldots,q_m)}$.

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and

Theorem (A. Defant and D. Vogt)

$$\prod_{\mathrm{as}(1;1)}(E_1, \ldots, E_m; \mathbb{K}) = \mathcal{L}(E_1, \ldots, E_m; \mathbb{K}).$$

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$$\prod_{\mathrm{as}(1;1)}(E_1, \ldots, E_m; \mathbb{K}) = \mathcal{L}(E_1, \ldots, E_m; \mathbb{K}).$$

we obtain that, for $r, s \geq 1$ and $s \leq \frac{mr}{mr+1-r}$,

$$\prod_{\mathrm{as}(r;s)}(E_1, \ldots, E_m; \mathbb{K}) = \mathcal{L}(E_1, \ldots, E_m; \mathbb{K}).$$

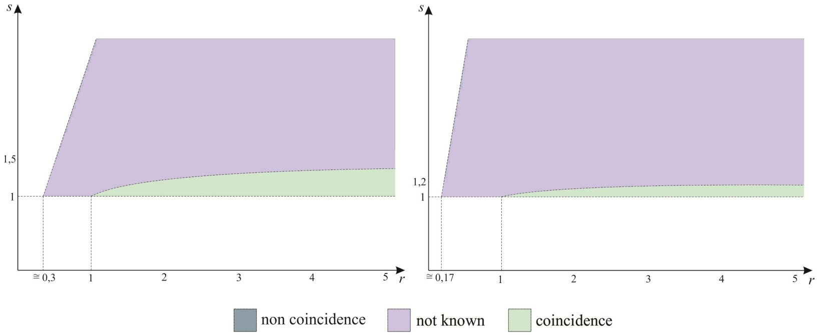
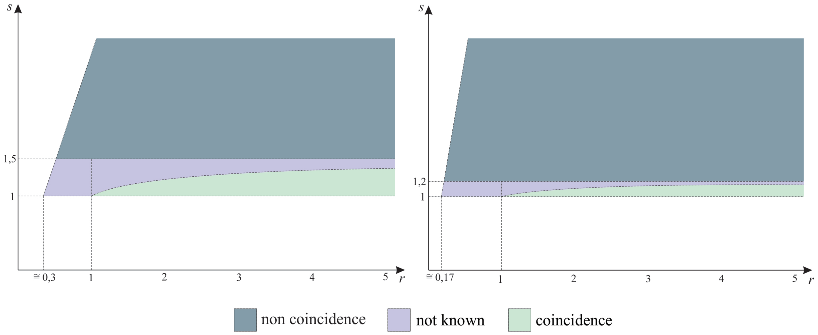


Figure: Coincidence zone for $m = 3$ and $m = 6$, respectively.



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As a particular case of our results of this section we will improve the information contained in the graphic above.

Definition

For $1 \leq s < \infty$ and $r \geq \frac{s}{m}$, a continuous m–homogeneous polynomial $P : E \to F$ is absolutely $(r; s)$–summing if there is a $C > 0$ such that

$$\left(\sum_{j=1}^{n} \| P(x_j) \|^r \right)^{\frac{1}{r}} \leq C \left\| (x_j)_{j=1}^{n} \right\|_{w,s}^{m}$$

for all positive integers n and all $(x_j)_{j=1}^{n} \in \ell_s^w(E)$. We will represent the class of all absolutely $(r; s)$–summing polynomials from E to F by $\mathcal{P}_{\mathrm{as}(r;s)}({}^m E; F)$.

It is well–known that every continuous scalar–valued m–homogeneous polynomial is absolutely $(1;1)$–summing. More precisely

$$\mathcal{P}(^m E; \mathbb{K}) = \mathcal{P}_{\mathrm{as}(1;1)}(^m E; \mathbb{K})$$

regardless of the Banach space E (Defant–Voigt Theorem for polynomials).

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For a given $m \geq 2$, what is the infimum of the r such that $\mathcal{P}(^m E;\mathbb{K}) = \mathcal{P}_{\mathrm{as}(r;1)}(^m E;\mathbb{K})$ for all infinite–dimensional Banach spaces E?

From the previous results we know that, for $m \geq 3$,

$$V_m := \inf \left\{ r : \begin{array}{l} \mathcal{P}(^m E; \mathbb{K}) = \mathcal{P}_{\mathrm{as}(r;1)}(^m E; \mathbb{K}) \text{ for all} \\ \text{infinite–dimensional Banach spaces } E \end{array} \right\} \in \left[\frac{1}{m}, 1 \right].$$

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Proposition (G. Botelho, D. Pellegrino and P. Rueda [5], 2010)

If m is even, then

$$\inf \left\{ r : \mathcal{P}(^m E; \mathbb{K}) = \mathcal{P}_{\mathrm{as}(r;1)}(^m E; \mathbb{K}) \right\} \geq \frac{\cot E}{m + \cot E}.$$

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Therefore, if m is even, we can prove the following result:

Theorem

If m is even, then $V_m = 1$.

Lemma

If every continuous m-homogeneous polynomial from E to F is absolutely $(r; s)$–summing, then every continuous symmetric $(m+1)$–linear forms from E^{m+1} to F is absolutely $(r; s, \ldots, s, 1)$–summing.

This lemma is known in the framework of multilinear operators; the essence of its proof goes back to [1, R. M. Aron, M. Lacruz, R. A. Ryan and A. M. Tonge, 1992]. Our result is slightly different, although of the proof is standard.

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Theorem

If $r<1$, $\dim E=\infty$, $\mathcal{P}(^mE;\mathbb{K})=\mathcal{P}_{\mathrm{as}(r;1)}(^mE;\mathbb{K})$ *and m is odd, then* $\cot E<\infty$ *and*

$$r \geq \frac{\cot E}{m+1+\cot E}.$$

Corollary

$V_m = 1$ for all positive integers m.

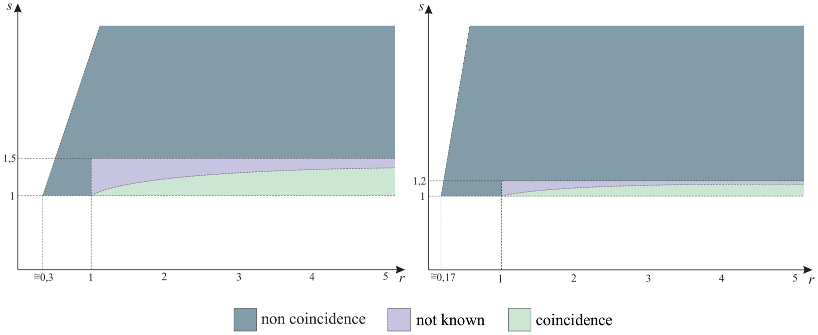

Figure: Coincidence zone for $m = 3$ and $m = 6$, respectively.

The table below details the results of coincidence and non coincidence in the "boundaries" of Figure 4.

$\frac{1}{m} \leq r < 1$	$s = 1$	non coincidence
$r \geq \frac{1}{m}$	$s = mr$	non coincidence
$r = 1$	$1 < s \leq \frac{m}{m-1}$	non coincidence
$r \geq 1$	$s = 1$	coincidence
$r \geq 1$	$s = \frac{mr}{mr+1-r}$	coincidence
$r \geq 1$	$s = \frac{m}{m-1}$	non coincidence

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Thank you very much for your attention!