Grassmannian of a Hilbert space

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Intro

Definitions

$\mathcal{H}$ is a Hilbert space, $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators $\mathcal{H}$, $\mathcal{P}(\mathcal{H})$ the set of orthogonal (selfadjoint) projections in $\mathcal{H}$. We shall regard this set $\mathcal{P}(\mathcal{H})$ as the Grassmannian of $\mathcal{H}$ (i.e. the set of closed subspaces of $\mathcal{H}$), by means of the identification

$$\mathcal{S} \longleftrightarrow P_{\mathcal{S}},$$

between the closed subspace $\mathcal{S} \subset \mathcal{H}$ and the orthogonal projection $P_{\mathcal{S}}$ onto $\mathcal{S}$.

G. Corach, H. Porta y L. Recht (CPR) studied the differential geometry of $\mathcal{P}(\mathcal{H})$. They consideed a linear connection in $\mathcal{P}(\mathcal{H})$. This linear connection is based in the fact that given a projection $P \in \mathcal{P}(\mathcal{H})$, elements X of $\mathcal{B}(\mathcal{H})$ can be written as 2×2 block matices.

Any operator X decomposes as two matrices: one P-diagonal, the other P-codiagonal :

$$X = \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} + \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix}.$$

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The geometry of $\mathcal{P}(\mathcal{H})$, according to CPR, can be understood in terms of this decomposition. The tangent space of $\mathcal{P}(\mathcal{H})$ at P, consists of selfadjoint operators which are P-codiagonal. The covariant derivative of a field $X(t)$ of tangent vectors along a curve $P(t)$ in $\mathcal{P}(\mathcal{H})$, consists in differentiating $X(t)$ and taking at every t the $P(t)$-codiagonal part of $\dot{X}(t)$.

The geodesics of this connection which start at $t = 0$ in the projection P, are of the form

$$\delta(t) = e^{itZ} P e^{-itZ},$$

where Z is selfadjoint and P-codiagonal.
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The goal of this talk is to characterize when two projections P and Q can be joined by a geodesic curve. Furthermore, when is this geodesic unique.

Given two projections P and Q, P. Halmos, J. Dixmier, and C. Davis among others, suggested that to understand the geometry of the pair, one should consider the following subspaces:

$$\mathcal{H}_{11} = R(P) \cap R(Q), \ \mathcal{H}_{00} = N(P) \cap N(Q),$$

$$\mathcal{H}_{10} = R(P) \cap N(Q), \ \mathcal{H}_{01} = N(P) \cap R(Q),$$

and $\mathcal{H}_0$ the orthogonal complement to the sum of the former.

Thus

$$\mathcal{H} = \mathcal{H}_{11} \oplus \mathcal{H}_{00} \oplus \mathcal{H}_{10} \oplus \mathcal{H}_{01} \oplus \mathcal{H}_0.$$

It is not difficult to see that $\mathcal{H}_{11}$, $\mathcal{H}_{00}$ and $\mathcal{H}' = \mathcal{H}_{10} \oplus \mathcal{H}_{01}$ reduce P and Q simultaneously. Therefore so does their orthogonal complement $\mathcal{H}_0$. $\mathcal{H}_0$ is usually called **the generic part** of P and Q.

It is also useful (and straightforward to verify) that one can refer these subspaces to the difference $A = P - Q$:

$$N(A) = \mathcal{H}_{00} \oplus \mathcal{H}_{11}, \;\; N(A-1) = \mathcal{H}_{10} \text{ and } N(A+1) = \mathcal{H}_{01}.$$

In $\mathcal{H}_{11}$, P and Q act as the identity operator, in $\mathcal{H}_{00}$ they are both zero. Therefore interesting phenomena may occur in $\mathcal{H}'$ and in the generic part $\mathcal{H}_0$. Let us denote by P' and Q' the reductions of P and Q to $\mathcal{H}'$, and by P_0 and Q_0 their reductions to $\mathcal{H}_0$.

Among the results obtained by the mentioned authors (Halmos, Dixmier, Davis), which can be read for instance in 'Two subspaces' by P. Halmos (Trans. Amer. Math. Soc. 144 (1969) 381–389)) let us recall the following:
In the generic part, P_0 and Q_0 are unitarilly equivalent, and this equivalence is implemented by a specific unitary operator. There is a unitary isomorphism between $\mathcal{H}_0$ and a product space $\mathcal{K} \times \mathcal{K}$, and a positive operator $0 \leq X \leq \pi/2$ acting in $\mathcal{K}$, such that P_0 y Q_0 are transformed into (the operators of $\mathcal{K} \times \mathcal{K}$):

$$P_0 = \begin{pmatrix} 1_{\mathcal{K}} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } Q_0 = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},$$

where $C = cos(X)$ and $S = sen(X)$ has trivial nullspace.

In the subspace $\mathcal{H}'$, in terms of the decomposition $\mathcal{H}' = \mathcal{H}_{10} \oplus \mathcal{H}_{01}$, the projections P' and Q' are given by the following matrices:

$$P' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } Q' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Apparently, P' are Q' unitarilly equivalent (in $\mathcal{H}'$!) if and only if

$$\dim \mathcal{H}_{10} = \dim \mathcal{H}_{01}.$$

Returning to

$$P_0 = \begin{pmatrix} 1_{\mathcal{K}} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } Q_0 = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},$$

A simple matrix computation shows that a unitary operator which implements the equivalence between these projections is

$$U = \begin{pmatrix} C & -S \\ S & C \end{pmatrix} = exp(i \begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix}).$$

Note that the exponent

$$Z_0 = \begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix}$$

is (bringing it back to $\mathcal{H}_0$ with the isomorphism between $\mathcal{H}_0$ and $\mathcal{K} \times \mathcal{K}$) **selfadjoint** and P_0**-codiagonal**.

Summarizing: in $\mathcal{H}_0$ there is a geodesic joining P_0 and Q_0,

$$\delta_0(t) = e^{itZ_0} P_0 e^{-itZ_0},$$

which additionally satisfies that $\|Z_0\| \leq \pi/2$.
This latter fact implies that the geodesic has minimal length among all smooth curves joining P_0 and Q_0 (Porta and Recht).

In $\mathcal{H}' = \mathcal{H}_{10} \oplus \mathcal{H}_{01}$, it is easy to verify that, if both subspaces have the same dimension, then P' and Q' are unitarilly equivalent:

Let $W : \mathcal{H}_{10} \to \mathcal{H}_{01}$ be an isometric isomorphism,
put $U' : \mathcal{H}' \to \mathcal{H}'$

$$U'(\xi, \eta) = (W^*\eta, -W\xi).$$

Apparently

$$U'P'U'^* = Q'.$$

Moreover, if $Z' = -i\frac{\pi}{2}U$, it is clear that Z' is selfadjoint, P'-codiagonal, and satisfies

$$e^{iZ'} = U'.$$

That is, the geodesic $\delta'(t) = e^{itZ'}P'e^{-itZ'}$ joins P' and Q' (in $\mathcal{H}'$), and has minimal length: $\|Z'\| = \pi/2$.

Thus we have seen that if

$$\dim \mathcal{H}_{10} = \dim \mathcal{H}_{01}.$$

Then there exists a geodesic joining P and Q, which has minimal length.

Conversely, suppose that there exists a selfadjoint operator Z which is P-codiagonal (in $\mathcal{H}$), such that

$$e^{iZ}Pe^{-iZ} = Q.$$

This equation implies that,

$$e^{iZ}(\mathcal{H}_{10}) \subset \mathcal{H}_{01}.$$

and

$$e^{iZ}(\mathcal{H}_{01}) \subset \mathcal{H}_{10}.$$

In particular, $\dim \mathcal{H}_{10} = \dim \mathcal{H}_{01}$.

Thus we have proved

Theorem 1

Let P and Q be projections. There is a geodesic joining them if and only if

$$\dim \mathcal{H}_{10} = \dim \mathcal{H}_{01}.$$

Let us consider now the problem of uniquenes of geodesics joining P and Q.

We saw above the in the case when $\dim \mathcal{H}_{10} = \dim \mathcal{H}_{01}$ **any** isometric isomorphism $W : \mathcal{H}_{10} \to \mathcal{H}_{01}$ gives rise to a geodesic between P' and Q'. It follows that if

$$\dim \mathcal{H}_{10} = \dim \mathcal{H}_{01} \neq 0,$$

there are infinitely many geodesics between P and Q.

Note the fact that if Z is the exponent of a geodesic between P and Q, if we reverse the parametrization of this curve ($t \leftrightarrow 1-t$) one concludes that Z, besides P-codiagonal, is also Q-codiagonal.

In particular,

$$Z(N(P)) \subset R(P),\ Z(R(P)) \subset N(P),\ Z(N(Q)) \subset R(Q),\ Z(R(Q)) \subset N(Q).$$

Then

$$Z(\mathcal{H}_{00}) \subset \mathcal{H}_{11} \text{ and } Z(\mathcal{H}_{11}) \subset \mathcal{H}_{00}$$

as well as

$$Z(\mathcal{H}_{10}) \subset \mathcal{H}_{01} \text{ and } Z(\mathcal{H}_{01}) \subset \mathcal{H}_{10}.$$

That is, any exponent Z of a geodesic between P and Q is reduced by the (three-space) decomposition

$$(\mathcal{H}_{00} \oplus \mathcal{H}_{11}) \oplus (\mathcal{H}_{10} \oplus \mathcal{H}_{01}) \oplus \mathcal{H}_0.$$

Then any geodesic between P and Q induces three geodesics:

- Between

$$1 \oplus 0 \quad \text{and} \quad 1 \oplus 0 \ \text{ in } \ \mathcal{H}_{00} \oplus \mathcal{H}_{11}$$

  (the projection with itself!),

- between

$$P' = 1 \oplus 0 \quad \text{and} \quad Q' = 0 \oplus 1 \ \text{ in } \ \mathcal{H}' = \mathcal{H}_{10} \oplus \mathcal{H}_{01},$$

- and between

$$P_0 \text{ and } Q_0 \ \text{ in } \ \mathcal{H}_0.$$

The first geodesic is reduced to a point. This is not difficult, but we skip the proof.

In the second space, as seen before, one may construct infinitely many geodesics (always under the assumption $\dim \mathcal{H}_{10} = \dim \mathcal{H}_{01} \neq 0$).

To consider the third space, the generic part, we shall need a result by Chandler Davis (Theorem 6.1 in *Separation of two linear subspaces. Acta Sci. Math. Szeged 19 1958 172–187*):

Theorem (C. Davis)

The following conditions are necessary and sufficient in order that an operator A be the difference of two orthogonal projections:
$-1 \leq A \leq 1$, and in the generic part $\mathcal{H}_0$ (recall that $\mathcal{H}_0 = (N(A) \oplus N(A-1) \oplus N(A+1)^{\perp})$ there is a symmetry V ($V^* = V^{-1} = V$) such that

$$VA_0 = -A_0 V,$$

where A_0 is the part of A acting in $\mathcal{H}_0$.

In this case, to each V corresponds a unique pair of projections P_V, Q_V such that $A_0 = P_V - Q_V$.
Note: only A_0 needs to be decomposed as a difference of projections. In

$$\mathcal{H}_0^\perp = N(A) \oplus N(A-1) \oplus N(A+1),$$

A is trivially decomposed as a difference of projections.

Davis obtains explicit formulas for P_V and Q_V:

$$P_V = \frac{1}{2}\{1 + A_0 + V(1-A_0^2)^{1/2}\} \ , \ Q_V = \frac{1}{2}\{1 - A_0 + V(1-A_0^2)^{1/2}\}.$$

It can be shown how the projections determine V:

$$V = (1-A_0^2)^{-1/2}(P_V + Q_V - 1).$$

What this theorem of Davis says about our problem is the following:

Proposition

Let P_0 and Q_0 be projections in generic position, $A_0 = P_0 - Q_0$ and V the unique symmetry of Davis' Theorem,

$$P_V = P_0, \; Q_V = Q_0.$$

Let Z_0 be a selfadjoint operator, $\|Z_0\| \leq \pi/2$ and P_0-codiagonal, such that $Q_0 = e^{iZ_0} P_0 e^{-iZ_0}$. Then

$$V = e^{iZ_0}(2P_0 - 1).$$

Proof: straightforward matrix computation

As a consequence, e^{iZ_0} is determined by P_0 and Q_0.

Moreover, since $\|Z_0\| \leq \pi/2$, Z_0 is determined by P_0 and Q_0.

Theorem 2

Let P and Q be projections which can be joined by a geodesic δ. This geodesic determines a geodesic δ_0 between the generic parts P_0 and Q_0, which is unique (with $\|Z_0\| \leq \pi/2$).
In particular, there is a unique geodesic joining P" and Q (with exponent Z of norm less than or equal to $\pi/2$) if and only if

$$\mathcal{H}_{10} = \mathcal{H}_{01} = \{0\}.$$

If $\mathcal{H}$ is finite dimensional, or $P - Q$ is of trace class (or if the projections lie in an algebra with a finite trace) one sees that

$$Tr(P - Q) = \dim N(A - 1) - \dim N(A + 1).$$

This fact was discovered by several people (Effros, Amrein-Sinha, Avron-Seiler-Simon), and is a direct consequence of Davis' Theorem: The generic part A_0 of $A = P - Q$, has zero trace, because it anti-commutes with a symmetry:

$$Tr(A_0) = Tr(VA_0V) = -Tr(A_0).$$

Taking trace in the spectral decomposition of A, only the terms $Tr(P_{N(A-1)}) - Tr(P_{N(A+1)})$ remain, which equals the difference of ranks of these projections.

One obtains a proof that in this context (finite dimension, $P - Q$ of trace class, etc.) any pair of projections (in the same connected component of the space of projections) can be joined by a geodesic.

In finite dimensions, this is a consequence of the Hopf-Rinow Theorem, due to the fact that the space of projections is compact.