Exercises for Deformation Theory (Roughly two per day)

1. Twisted cubic curves.
   a) Show that the twisted cubic curve in $\mathbb{P}^3$ form a family of dimension 12.
   b) If $Y$ is a twisted cubic curve, compute the normal bundle $\mathcal{N}_Y/\mathbb{P}^3$ and show that $h^0(Y, \mathcal{N}_Y/\mathbb{P}^3) = 12$. Hence the Hilbert scheme is smooth at that point.

2. Zero-schemes of length $k$. Y.
   a) Show that sets of four distinct points in $\mathbb{P}^3$ form an open subset $U$ of dimension 12 of the Hilbert scheme $X$ of zero schemes of length $k$ in $\mathbb{P}^3$.
   b) Let $Y$ be the length 4 zero-scheme concentrated at a point whose local form is $\text{Spec } k[x,y,z]/m^2$, where $m = (x,y,z)$. Show that $h^0(Y, \mathcal{N}_Y/\mathbb{P}^3) = 18$, so $Y$ corresponds to a singular point of the Hilbert scheme. To prove this, you need to show also
   c) $Y$ is in the closure of the open set $U$ of a).

3. A rigid scheme.
   Let $X$ be the affine scheme defined as $X = \text{Spec } k[x,y,z,w]/(x,y,z,w)$. Show that $X$ is a rigid scheme, even though $X$ is singular. (Geometrically, it is the union of two planes meeting at a simple point.)
4. A node.

Let $B = k[[x,y]]/(xy)$. Compute $T^2(B/k, B)$. Show that the family defined by $k[[x,y,t]]/(xy-t)$ gives a first-order deformation that corresponds to a non-zero element of $T^2$.

5. Twisted cubic curve.

If $Y$ is a twisted cubic curve in $P^3$, verify that $H^1(Y, N_{Y/P^3}) = 0$, giving another proof (cf. #1 above) that the Hilbert scheme is smooth at that point.


a) Show that the Hilbert scheme of quartic (degree 4) surfaces in $P^3$ is smooth of dimension 34.

b) If $X_1, X_2 \subseteq P^3$ are two quartic surfaces, and if $\sigma \in Aut P^3$ is an automorphism of $P^3$, then $X_1 \cong X_2$ are isomorphic as abstract surfaces. Since the group $Aut P^3$ has dimension 15, we expect the family of abstract nonsingular quartic surface to have dimension $\leq 34 - 15 = 19$.

c) Show however that for $X$ a nonsingular quartic surface in $P^3$, $H^0(X, \Omega_X) = 0$, $H^1(X, \Omega_X) = 20$, $H^2(X, \Omega_X) = 0$, so we expect a family of abstract nonsingular surface of dimension 20.

d) Prove that if $X_0$ is a nonsingular quartic surface in $P^3$, then there exists a deformation of $X_0$, as an abstract surface, over the dual numbers $D$, that does not arise from any embedding $X \rightarrow \mathbb{P}^3$ but family $X \subseteq \mathbb{P}_D^3$, flat over $D$. 
7. **Elliptic curves.**

We define an elliptic curve to be a nonsingular projective curve of genus 1 over an algebraically closed field \(k\), together with a fixed point \(P\).

a) Show that to each elliptic curve \((E, P)\) we associate its \(j\)-invariant \([AG, IV, §9]\), then \(k\)-line \(Spec \mathbb{C}[j]\) acts as a coarse moduli space for elliptic curves.

b) However, show that there does not exist a flat, topological family \(\mathcal{X}\) of elliptic curves over \(Spec \mathbb{C}[j]\), with the property that for each \(j\), the fiber \(\mathcal{X}_j\) is an elliptic curve with invariant \(j\).

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8. **Invertible sheaves on an integral curve.**

Let \(X\) be an integral projective curve on \(k\) algebraically closed.

a) Show that the family of all invertible sheaves \(\mathcal{L}\) on \(X\) of fixed degree \(d\) is a bounded family, i.e., there exists a scheme of finite type \(\mathcal{T}/k\), such that \(\mathcal{T}\) is a family of invertible sheaves on \(X \times \mathcal{T}\), whose fibers \(\mathcal{L}_t\) for \(t \in \mathcal{T}\) include all possible invertible sheaves on \(X\).

b) Show that the family of invertible sheaves is separated, i.e., if \(\mathcal{L}_1\) and \(\mathcal{L}_2\) are two families on \(X \times \mathcal{T}\), where \(\mathcal{T}\) is a nonsingular curve, \(t \in \mathcal{T}\) is a point, and \(t_0 \in \mathcal{T}\), for all \(t \neq t_0\), then also \((\mathcal{L}_1)_t \cong (\mathcal{L}_2)_t\).

c) Show however that the family of all \(\mathcal{L}'s\) may not be complete, i.e., it is possible to have a family \(\mathcal{L}\) defined on \(X \times (\mathcal{T}, \mathcal{O})\), where \(\mathcal{T}\) is a nonsingular curve, \(\mathcal{T}/k\), but there is no family \(\mathcal{L}\) on \(X \times T\) extending \(\mathcal{L}\). (If \(X\) is nonsingular, then the family is complete.)
9. **Rational curves of degree 5 in \( \mathbb{P}^3 \)**

* Show that the Hilbert scheme of smooth, rational curves in \( \mathbb{P}^3 \) of degree 5 is smooth and irreducible of dimension 20.

* Show that a general such curve (meaning a general point in that Hilbert scheme) is contained in some cubic surface, but is not contained in any quadric surface.

10. **Stable curves.**

A stable curve is a reduced, connected, projective curve having at most nodes as singularities, of arithmetic genus \( g \geq 2 \) with the additional property that if any irreducible component has \( g = 0 \), then that component meets the rest of the curve in at least three points. (Example: two rational curves meeting each other in three points.)

* If \( X_0 \) is a stable curve, show that there are no obstructions to deformation of \( X_0 \) (as an abstract curve).

* Show that deformations of \( X_0 \) over the dual numbers \( \mathbb{D} \) are classified by a vector space of dimension \( 3g - 3 \).

* Conclude that the modular family of stable curves (assuming it exists) is smooth of dimension \( 3g - 3 \).