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The singularity spectrum $f(\alpha)$ for cookie-cutters

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Abstract. I use a thermodynamic formalism to study the spectrum $f(\alpha)$ which characterises the large fluctuations of pointwise dimension in a Gibbs state supported on a hyperbolic cookie-cutter. Amongst other things, it is proved that $f(\alpha)$ is the Hausdorff dimension of the set of points with pointwise dimension α, that $f(\alpha)$ is real-analytic and that its Legendre transform $\tau(q)$ is related to the Renyi dimension D_q of the Gibbs state by the formula $(1-q)D_q = \tau(q)$.

1. *Introduction*

Let ν be a measure on an interval I such as an invariant measure of a 1-dimensional map or the conditional measure on a local 1-dimensional stable or unstable manifold of an attractor. To each point x in I one can associate a range of local dimensions α given by

$$\liminf_{\varepsilon\to 0} \frac{\log \nu(B_\varepsilon(x))}{\log \varepsilon} \leq \alpha \leq \limsup_{\varepsilon\to 0} \frac{\log \nu(B_\varepsilon(x))}{\log \varepsilon},$$

where $B_\varepsilon(x)$ is the ball of radius ε about x. For many x the lim inf and lim sup will be equal so that one can talk of a local dimension $\alpha(x)$. Moreover, if the system is ergodic, $\alpha(x)$ will be independent of x on a set of full ν-measure. Nevertheless, as x ranges over I there will typically be large fluctuations in the value of α. In this paper I use a thermodynamic formalism to consider the function $f(\alpha)$ which characterizes these large fluctuations of the local pointwise dimension α (see figure 1 below). This function was first introduced by Halsey et al. [**9**] and Parisi et al. [**15**]. Using a thermodynamic formalism I will give a precise definition of f and prove a number of results for which they only gave heuristic arguments. My proofs are for a very special class of dynamical systems and invariant measures; namely hyperbolic cookie-cutters and their Gibbs states. Using the arguments of this paper it is easy to extend the results to a larger class of hyperbolic systems such as Axiom A attractors with 1-dimensional stable or unstable manifolds (see Gundlach [**8**]), but I have restricted to cookie-cutters in order to get to the heart of the problem without introducing unnecessary technical details.

These problems are all open for non-hyperbolic attractors. Indeed one of the most interesting questions in this area concerns the existence of phase transitions in the thermodynamic formalism. I prove here that for cookie-cutters f is analytic, but Bohr and Rand [**2**] gave a simple example where it is not and Arneodo et al. [**1**], Cvitanovic [**5**], Grassberger et al. [**6**] and Gunaratne and Procaccia [**7**] have conjectured that for some non-hyperbolic attractors such as the Henon attractor and certain fractal bifurcation sets f is non-analytic and hence the thermodynamic formalism has phase transitions. If this is true it would be most interesting and should have geometrical-dynamical consequences.

The thermodynamic formalism adopted here shows the generality of these ideas. In [**2**] a similar approach is used to define an entropy function for characteristic exponents and to relate it to topological entropy, Hausdorff dimension and escape rates associated with certain interesting subsets of an attractor. One can also define a spectrum which measures large fluctuations of the local metric entropy and similar ideas can be used to define entropy and dimension functions which describe the large fluctuations of rotation numbers in Birkhoff attractors.

Collet, Lebowitz and Porzio have independently obtained similar results in [**4**].

2. *Definitions and results*

2.1. *Cookie-cutters*

Let $I=[0,1]$ and $I_0, I_1 \subset I$ be two disjoint closed subintervals. A cookie-cutter is a $C^{1+\alpha}$ map $g: I_0 \cup I_1 \to \mathbb{R}$ such that (a) $|g'|>1$ and (b) $g(I_0)=I$ and $g(I_1)=I$. Let

$$\Lambda=\{x \in I : g^j x \in I \quad \text{for } j=0,1,2,\ldots\}$$

and let $h:\Lambda \to \Sigma=\{0,1\}^{\mathbb{N}}$ be defined as follows: $h(x)=a_0a_1a_2\cdots$ where $a_i \in \{0,1\}$ is such that $g^i x \in I_{a_i}$. Then it is easy to show that h is a homeomorphism from Λ to Σ (when Σ has the product topology) and

$$h \circ g = g \circ s,$$

where s is the shift: $s(a_0a_1a_2\cdots)=a_1a_2a_3\cdots$. This symbolic representation will be useful. I shall denote by Λ_n the set $\{x \in I : g^j x \in I \text{ for } 0 \le j \le n\}$ which consists of 2^n closed intervals. These intervals are called *n-cylinders* and the set of n-cylinders is denoted by $\mathscr{C}_n$. If $x \in \Lambda$ then $C_{n,x}$ denotes the n-cylinder containing x.

2.2. *Gibbs states and the Gibbsian hypothesis*

The notion of a Gibbs state is needed for what follows. Let $\varphi: I \to \mathbb{R}$ be a Hölder continuous function. The Gibbs state μ_φ of φ is the unique invariant probability measure with the following property: there exists a constant P and constants $c_1, c_2 > 0$ such that for all $C \in \mathscr{C}_n$ and all $x \in C$,

$$\mu_\varphi(C) \in [c_1, c_2]\, e^{-nP+S_n\varphi(x)},$$

where $S_n\varphi(x)=\varphi(x)+\cdots+\varphi(g^{n-1}x)$. It is easy to deduce from this that

$$P=\lim_{n\to\infty} n^{-1} \log \sum_{\mathscr{C}_n} e^{S_n\varphi(C)},$$

where $S_n\varphi(C)$ denotes the maximum value taken by $S_n\varphi(x)$ on C. $P=P(\varphi)$ is called the *pressure* of φ. The existence and uniqueness of μ_φ is due to Ruelle and Sinai and a proof is given in Bowen [3] for subshifts of finite type. To deduce the result as stated here one has simply to apply the principle of bounded variation stated in Lemma 1.

The Gibbsian condition on ν. Throughout the paper I fix a measure ν which is invariant under g. Now I introduce an important technical condition on it. I shall assume throughout that ν is the Gibbs state of a Hölder continuous function φ_ν. This is the case if and only if the ν is non-singular in the sense that if $\nu(g(A))>0$ then $\nu(A)>0$ for all measurable sets A and the Radon–Nikodyn derivative

$$J(x)=\lim_{A\searrow x}\nu(g(A))/\nu(A)$$

is ν-almost everywhere equal to a Hölder continuous function which is also denoted by J([**16**]). Then $\varphi_\nu=-\log J+u\circ g-u$ for some Hölder continuous u. For the rest of this paper I shall regard ν and J as fixed and I shall use φ_1 and φ_2 to denote the functions $-\log|g'|$ and $-\log J$ respectively.

2.3. *The singularity spectrum*

If J is an interval, $\lambda(J)$ denotes the length of J. Let A and L be open intervals and let $N_n(A, L)$ denote the number of n-cylinders C such that $l(C)=n^{-1}\log\lambda(C)\in L$ and $\alpha(C)=\log\nu(C)/\log\lambda(C)\in A$. Let

$$S(A, L)=\liminf_{n\to\infty} n^{-1}\log N_n(A, L)$$

and

$$S(\alpha, l)=\inf\{S(A, L): \alpha\in A, l\in L\}.$$

Below (see Lemma 2), it is shown that $S(\alpha, l)$ is continuous and concave in each of its arguments separately, and

$$S(A, L)=\sup\{S(\alpha, l): \alpha\in A, l\in L\}.$$

Define

$$f(\alpha, l)=-S(\alpha, l)/l$$

and

$$f(\alpha)=\sup_l f(\alpha, l).$$

Then I call $f(\alpha)$ the *singularity spectrum*. Very roughly speaking, if $\mathcal{B}$ is a typical cover of Λ by non-overlapping intervals of length ε then the number of $B\in\mathcal{B}$ with $\alpha(B)=\log\nu(B)/\log\varepsilon\in[\alpha, \alpha+d\alpha]$ grows as $\varepsilon\to 0$ as $\varepsilon^{-f(\alpha)}$ or, in terms of cylinders, the number of n-cylinders C such that $\nu(C)=\lambda(C)^\alpha$ grows, as $n\to\infty$, like $\varepsilon^{-f(\alpha)}$ where $\varepsilon=e^{nl}$ and $l=l(\alpha)$ is the value of l for which the supremum of $f(\alpha, l)$ is attained i.e. describes the dominant length scale.

2.4. *Results*

Define $\Sigma(\alpha)=\{x\in\Lambda: \alpha(C_{n,x})\to\alpha \text{ as } n\to\infty\}$ where $C_{n,x}$ denotes the n-cylinder containing x.

THEOREM 1. *The Hausdorff dimension of* $\Sigma(\alpha)$ *is* $f(\alpha)$.

Here I adopt the convention that the Hausdorff dimension of the empty set is $-\infty$ since $f(\alpha) = -\infty$ if there is no point x with local pointwise dimension α.

Throughout the paper I denote $-\log|g'|$ by φ_1, $-\log J$ by φ_2 and $\tau\varphi_1 + q\varphi_2$ by $\varphi_{q,\tau}$ where $q, \tau \in \mathbb{R}$. We will see that the pressure $P(q, \tau) = P(\varphi_{q,\tau})$ of the function $\varphi_{q,\tau}$ is the growth rate of the sums $\Sigma_n(q, \tau) = \sum_{C \in \mathscr{C}_n} \nu(C)^q \lambda(C)^\tau$. It is a concave real analytic function of q and τ. It follows (see below) that there is a real analytic function function $\tau(q)$ such that $P(q, \tau(q)) \equiv 0$.

THEOREM 2. *The singularity spectrum f is real analytic and* τ *is the Legendre transform of f i.e.* $f = q\alpha + \tau$ *where* $q = f'(\alpha)$ *and* $\alpha = -\tau'(q)$.

COROLLARY. *The maximum value of f is* $\tau(0)$ *which is the Hausdorff dimension of* Λ.

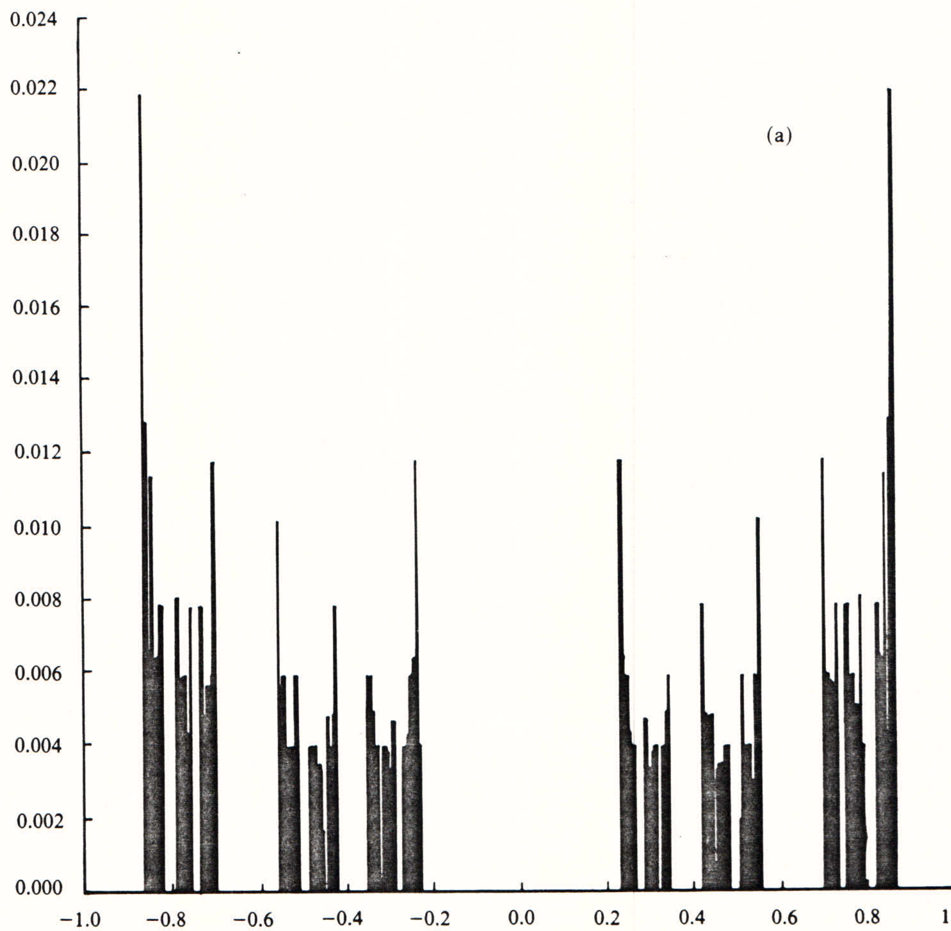

FIGURE 1. (a) A numerically computed approximation of the measure of maximal entropy for the cookie-cutter defined on the interval $[-a, a]$, $a = (1+\sqrt{11})/2$, by the map $x \to 1 - 2.5x^2$.

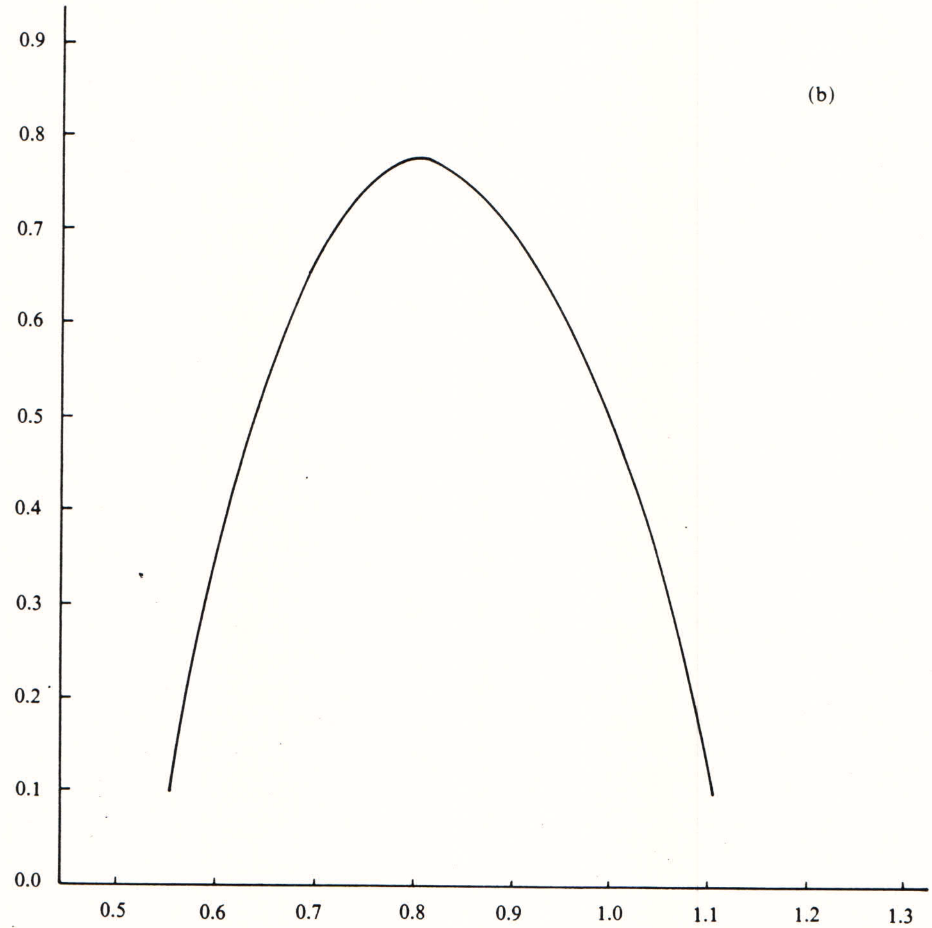

FIGURE 1. (b) A numerical approximation of σ for this measure. Note that the Hausdorff dimension and information dimension are respectively 0.779 ± 0.001 and 0.756 ± 0.001. More accuracy can easily be obtained with more computing time.

Now I introduce the Renyi dimensions D_q of a subset Δ of I. Let $\mathcal{B}$ be a cover of Δ by intervals B of length δ. Such a cover is called a δ-cover. If $q \neq 1$ define

$$D_q(\mathcal{B}) = (1-q)^{-1}\left(\log \sum_{B \in \mathcal{B}} \nu(B)^q\right) \bigg/ \log \delta^{-1}.$$

For $q = 1$ define

$$D_1(\mathcal{B}) = \left(\sum_{B \in \mathcal{B}} \nu(B) \log \nu(B)\right) \bigg/ \log \delta^{-1}.$$

Then $D_q(\mathcal{B})$ is continuous at $q = 1$ if $\nu(\Delta) = 1$. If $q \leq 1$ let $D_q(\Delta, \delta)$ denote $\inf_{\mathcal{B}} D_q(\mathcal{B})$ where the infimum is taken over all δ-covers and let $D_q(\Delta)$ denote $\liminf_{\delta \to 0} D_q(\Delta, \delta)$. Otherwise let $D_q(\Delta, \delta) = \sup_{\mathcal{B}} D_q(\mathcal{B})$ where the supremum is over all δ-covers and $D_q(\Delta) = \limsup_{\delta \to 0} D_q(\Delta, \delta)$.

THEOREM 3. $(1-q)D_q(\Lambda) = \tau(q)$.

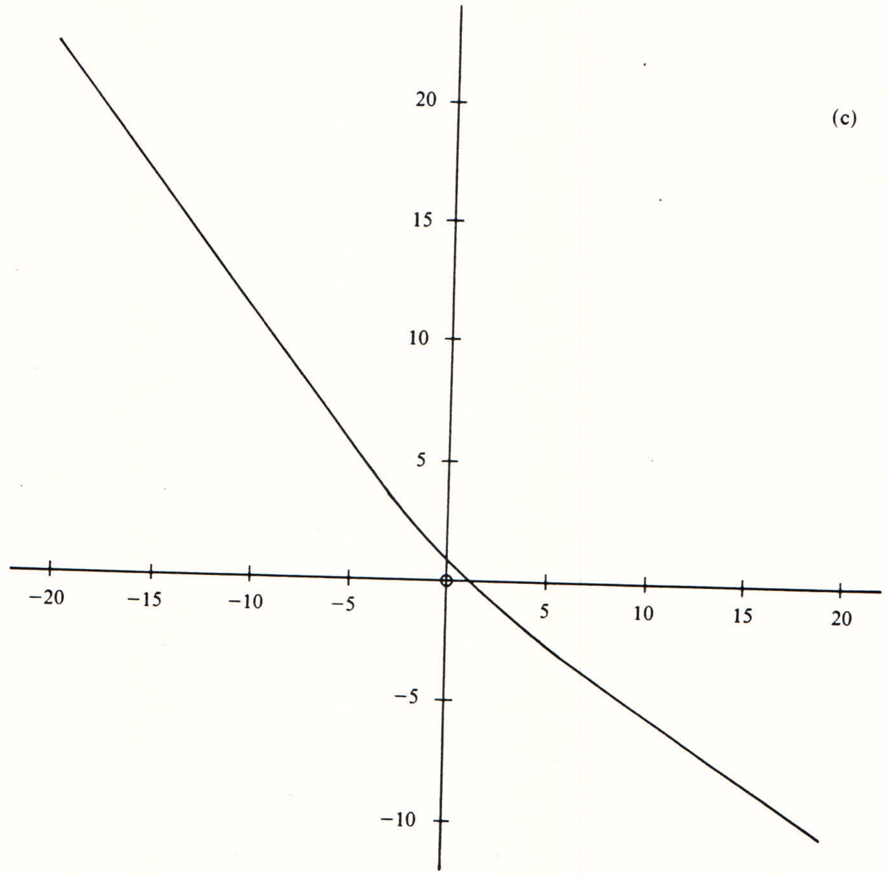

FIGURE 1. (c) A numerical approximation of τ for this measure.

Of particular interest is the *information dimenson* $ID(\nu)=D_1(\Lambda)$ of ν.

COROLLARY. *The information dimension* $ID(\nu)$ *of* ν *is* $f(\alpha)$ *where* α *is such that* $f'(\alpha)=1$.

Proof. Using Theorem 3, $\tau'(q)=D_q+(q-1)\,dD_q/dq$. Consequently, $ID(\nu)=\lim_{q\searrow 1} D_q=\tau'(1)$. But, by Theorem 3, $\tau(1)=0$ so, by Theorem 1, $f(\alpha(1))=\alpha(1)=\tau'(1)=ID(\nu)$. □

The Renyi entropies are given by

$$h_q=h_q(\nu)=(1-q)^{-1}\lim_{n\to\infty} n^{-1}\log\sum_{\mathscr{C}_n}\nu(C)^q.$$

As for the usual definitions of metric and topological entropy, a definition can be given in terms of arbitrary partitions, but it is a relatively simple exercise to show that, for cookie-cutters this boils down to the above. It is clear that h_0 is the topological entropy and that $\lim_{q\searrow 1} h_q$ is the metric entropy of ν. The following theorem follows from a proof similar to that for Theorem 3.

THEOREM 4. $(1-q)h_q=\tau(q)\chi(q)$ *where* $\chi(q)$ *is the characteristic exponent*

$-\int \varphi_1(x)\, d\mu(x)$ *of the measure* $\mu = \mu_{q,\tau(q)}$ *which is the Gibbs state of the function* $\varphi = \tau(q)\varphi_1 + q\varphi_2$. *Consequently,* $h_q/D_q = \chi(q)$.

Also, in Proposition 1 it is shown that $\tau(q)$ can be defined via a construction resembling and generalizing the definition of Hausdorff dimension so as to take the measure ν into account.

3. *Principle of bounded variation, the constants for* φ_1 *and* φ_2 *and the concavity of* S

The principle of bounded variation will play a crucial role in many arguments in the rest of the paper. In fact when this fails one gets some physically interesting deviations from our results for cookie-cutters (see **[2]** for an example).

LEMMA 1. (Principle of bounded variation.) *Suppose* $\varphi: I \to \mathbb{R}$ *is Hölder continuous. Then there exists* $d > 0$ *such that if* $C \in \mathscr{C}_n$ *and* $x, y \in C$ *then* $|S_n\varphi(x) - S_n\varphi(y)| < d$.

Proof. Since the length of the cylinders is exponentially decreasing with n, the variation of φ on an n-cylinder decreases exponentially i.e. there exists $0 < \beta < 1$ and $c > 0$ such that for all $n \geq 0$ if $C \in \mathscr{C}_n$ then $|\varphi(x) - \varphi(y)| < c\beta^n$ for all $x, y \in C$. Moreover, if x and y lie in a common n-cylinder then $g^m x$ and $g^m y$ lie in a common $(n-m)$-cylinder. Hence,

$$|S_n\varphi(x) - S_n\varphi(y)| \leq \sum_{i=0}^{n-1} |\varphi(g^i x) - \varphi(g^i y)| \leq \sum_{i=0}^{n-1} c\beta^i \leq c(1-\beta)^{-1}$$

whence the variation of $S_n\varphi$ on an n-cylinder C is bounded independently of n and C by $d = c(1-\beta)^{-1}$. □

Notation. Throughout the paper I denote by D a common bound d for the functions $\varphi_1 = -\log|g'|$ and $\varphi_2 = -\log J$ i.e. if x and y lie in a common n-cylinder then, for $i = 1, 2$,

$$|S_n\varphi_i(x) - S_n\varphi_i(y)| < D. \tag{1}$$

LEMMA 2. *S is continuous and convex in each argument separately.*

Proof. If A and L are two open intervals let $\tilde{N}_n(A, L)$ denote the number of n-cylinders C such that for $\varphi_1 = -\log|g'|$ and $\varphi_2 = -\log J$, $n^{-1}S_n\varphi_1(x) \in L$ and $S_n\varphi_2(x)/S_n\varphi_1(x) \in A$ for all $x \in C$. Then $a_n = \log \tilde{N}_n(A, L)$ is a subadditive sequence so that $\lim_{n\to\infty} a_n/n$ exists and equals $\sup_n a_n/n$. Let this limit be denoted by $\tilde{S}(A, L)$ and let $\tilde{S}(\alpha, l) = \inf\{\tilde{S}(A, L) : \alpha \in A \text{ and } l \in L\}$. Now it follows from (1) that if A' and L' denote respectively the (D/n)-neighbourhoods of A and L then

$$\tilde{N}_n(A, L) \leq N_n(A, L) \leq \tilde{N}_n(A', L').$$

It follows immediately that $\tilde{S}(\alpha, l) = S(\alpha, l)$. Now one can directly apply the arguments of Lanford **[10]** to prove that S is convex in each argument separately and is continuous. □

4. *Fundamental lemma*

Let $\mu = \mu_{q,\tau}$ be the Gibbs state of

$$\varphi = \varphi_{q,\tau} = \tau\varphi_1 + q\varphi_2 = -(\tau \log|g'| + q \log J)$$

and let $P = P(q, \tau)$ be the pressure of φ. Then $P(q, \tau)$ is convex and since $\varphi_1 = -\log|g'|$ and $\varphi_2 = -\log J$ are Hölder continuous, $P(q, \tau)$ is also real-analytic (**[17]**).

Moreover,

$$l = l(q, \tau) = \partial P/\partial\tau(q, \tau) = \mu_{q,\tau}(-\log|g'|)$$

is the $\mu_{q,\tau}$-mean decay rate of the cylinder length i.e. for all $\varepsilon > 0$, the $\mu_{q,\tau}$-measure of the n-cylinders with length in the interval $e^{n(l\pm\varepsilon)}$ tends to 1 as $n \to \infty$. Similarly, if

$$p = p(q, \tau) = \partial P/\partial q(q, \tau) = \mu_{q,\tau}(-\log J)$$

for all $\varepsilon > 0$, the $\mu_{q,\tau}$-measure of the n-cylinders with probability in the interval $e^{n(p\pm\varepsilon)}$ tends to 1 as $n \to \infty$. Let $\alpha(q, \tau) = p(q, \tau)/l(q, \tau)$.

Since $\partial P/\partial\tau(q, \tau) < 0$ it follows from the Implicit Function Theorem that there exists a real analytic function $\tau(q)$ such that $P(q, \tau(q)) \equiv 0$. Now, if $\alpha(q)$ denotes $\alpha(q, \tau(q))$, then

$$\tau'(q) = -(\partial P/\partial q)/(\partial P/\partial\tau) = -\alpha(q).$$

Let R_l (resp. R_p) denote the set of limit points of the sequences $n^{-1}S_n\varphi_1(x)$ (resp. $n^{-1}S_n\varphi_2(x)$) where x ranges over Λ and let R_α denote the set of limit points of the sequences $S_n\varphi_2(x)/S_n\varphi_1(x)$ where x ranges over Λ. Then at the end of this section I prove that R_α, R_p and R_l are (possibly trivial) closed intervals.

If R_α is a singleton then $f(\alpha)$ is clearly trivial and $\tau'(q) = \alpha(q) = \alpha_0$ is independent of q. Thus $\tau(q) = \alpha_0 q + h$ for some constant h. In fact, below I will show that $\tau(0)$ and hence h is the Hausdorff dimension of Λ. Consequently, it is henceforth assumed that R_α is a non-trivial interval.

If R_l is a singleton then $\varphi_1 = -\log|g'|$ is of the form $l + u - u \circ g$ where u is a Hölder continuous function and l is a constant ([**3**, Theorem 1.28 and Proposition 4.5]). Consequently, I henceforth assume that R_l is a non-trivial interval. It follows from this that $P(q, \tau)$ is a strictly convex function in the argument τ i.e. $\partial^2 P/\partial^2\tau(q, \tau) > 0$ ([**17**]). But

$$\tau''(q) = ((\partial P/\partial\tau)\cdot(\partial^2 P/\partial q^2) - (\partial P/\partial q)^2 \cdot (\partial^2 P/\partial\tau^2))/(\partial P/\partial\tau)^3$$

with each of the functions in the righthand side evaluated at $(q, \tau(q))$. Therefore, $\tau''(q) > 0$ because $\partial P/\partial\tau < 0$, $\partial P/\partial q < 0$, $\partial^2 P/\partial q^2 \geq 0$ and $\partial^2 P/\partial\tau^2 > 0$. Thus $\tau(q)$ is strictly convex.

Now consider the function $\alpha : \mathbb{R} \to R_\alpha$ given above by $\alpha(q) = -\tau'(q)$.

LEMMA 3. *$\alpha(q)$ is invertible on* int R_α *and its inverse $q(\alpha)$ is real analytic.*

Proof. Let $\alpha \in \text{int } R_\alpha$. Consider the sum

$$\sum_{C\in\mathscr{C}_n} \exp(S_n\varphi_{q,\tau}(C)) = \sum_{C\in\mathscr{C}_n} \exp(\tau S_n\varphi_1(C)) \cdot \exp(qS_n\varphi_2(C)).$$

(Recall that $S_n\psi(C)$ denotes $\max_{x\in C} S_n\psi(x)$.) Then by the principle of bounded variation and the definition of D (see (1)), if $\kappa = (|q| + |\tau|)D$, for any choice of $x_C \in C$, this sum is not less than

$$\begin{aligned}\sum_{C\in\mathscr{C}_n} \exp(S_n\varphi_{q,\tau}(x_C) - \kappa) &= \sum_{C\in\mathscr{C}_n} \exp(\tau S_n\varphi_1(x_C) + qS_n\varphi_2(x_C) - \kappa)\\ &= \sum_{C\in\mathscr{C}_n} \exp((\tau S_n\varphi_1(x_C)\\ &\quad + q(S_n\varphi_2(x_C)/S_n\varphi_1(x_C)))S_n\varphi_1(x_C) - \kappa). \qquad (2)\end{aligned}$$

If $\alpha' \in R_\alpha$ there exists $x \in \Lambda$ and a sequence $n_i \to \infty$ such that $S_{n_i}\varphi_2(x)/S_{n_i}\varphi_1(x) \to \alpha'$ as $i \to \infty$.

Firstly consider the case $q > 0$. Choose $\alpha' = \alpha - 2\varepsilon \in R_\alpha$, where $\varepsilon > 0$. Each term in the sum (2) can be written as

$$\exp\left((\tau + \alpha q + q(S_n\varphi_2(x_C)/S_n\varphi_1(x_C) - \alpha))S_n\varphi_1(x_C) - \kappa\right).$$

If $x = x_C$ and $n = n_i$, then for large i, $S_n\varphi_2(x)/S_n\varphi_1(x) - \alpha < -\varepsilon$ and, since $S_n\varphi_1(x_C) < 0$, the corresponding term in the sum is not less than

$$\exp\left((\tau + \alpha q - q\varepsilon)S_n\varphi_1(x_C)\right) > \exp\left((\tau + \alpha q - q\varepsilon)nl\right),$$

where $l = \inf\{\varphi_1(x) : x \in \Lambda\}$ if $\tau + \alpha q - q\varepsilon > 0$ and $l = \sup\{\varphi_1(x) : x \in \Lambda\}$ otherwise. Thus $P(q, \tau) \geq (\tau + \alpha q + q\varepsilon)l$. But $P(q, \tau) = 0$ if $\tau = \tau(q)$. Consequently, if $\tau = \tau(q)$, since $l < 0$, $\tau + \alpha q - \varepsilon q \geq 0$ or $\tau + \alpha q \geq \varepsilon q$. This implies that $\tau + \alpha q \to \infty$ as $q \to \infty$. If $q < 0$ then a similar argument with $\alpha' = \alpha + 2\varepsilon$ proves that $\tau + \alpha q \to \infty$ as $q \to -\infty$. Thus $\tau + \alpha q$ has a minimum at $q(\alpha)$ say and $(\tau + \alpha q)'(q(\alpha)) = 0$ or $\tau'(q(\alpha)) = \alpha$. But since τ is strictly convex then so is $\tau + \alpha q$ and this minimum is unique. This proves the existence of the function $q(\alpha)$. It is real-analytic by the inverse function theorem because $\tau'(q(\alpha)) = \alpha$ and $\tau''(q) > 0$. □

Now fix $q \in \mathbb{R}$ and consider the map $l_q : \mathbb{R} \to R_l$ given by $l_q(\tau) = l(q, \tau) = \partial P/\partial\tau(q, \tau)$.

LEMMA 4. *l_q is invertible on* int R_l *and its inverse $\tau_q(l)$ is real-analytic.*

Proof. Let $l \in \operatorname{int} R_l$. Let $\tilde{\varphi}_{q,\tau} = \varphi_{q,\tau} - l\tau = \tau\tilde{\varphi}_1 + q\varphi_2$ where $\tilde{\varphi}_1 = \varphi_1 - l$. Let $\tilde{P}(q, \tau) = P(\tilde{\varphi}_{q,\tau})$ be the pressure of $\tilde{\varphi}_{q,\tau}$. Then $\tilde{P}(q, \varphi) = P(q, \tau) - l\tau$. Thus solving $l_q(\tau) = \partial P/\partial\tau(q, \tau) = l$ is equivalent to solving $\partial\tilde{P}/\partial\tau(q, \tau) = 0$.

If $l' \in R_l$ there exists $x \in \Lambda$ and a sequence $n_i \to \infty$ such that $n_i^{-1}S_{n_i}\varphi_1(x) \to l'$ as $i \to \infty$. But then if $y \in C_{n_i,x}$, the n_i-cylinder containing x, $|n_i^{-1}S_{n_i}\varphi_1(x) - n_i^{-1}S_{n_i}\varphi_1(y)| \leq D/n_i$ by (1). Thus $n_i^{-1}S_{n_i}\varphi_1(C_{n_i,x}) \to l'$ as $i \to \infty$. (Recall $S_n\varphi(C) = \max_{x \in C} S_n\varphi(x)$.) Consequently, $n_i^{-1}S_{n_i}\tilde{\varphi}_1(C_{n_i,x}) \to l' - l$.

Fix $\varepsilon > 0$ so that $[l - 2\varepsilon, l + 2\varepsilon] \subset R_l$. Then taking $l' = l + 2\varepsilon$, there exists a sequence $n_i \to \infty$ such that for each $i \geq 0$ there is a n_i-cylinder C_i such that $n_i^{-1}S_{n_i}\tilde{\varphi}_1(C_i) > \varepsilon$. Now consider the sum

$$\sum_{C \in \mathscr{C}_n} \exp\left(S_n\tilde{\varphi}_{q,\tau}(C)\right) = \sum_{C \in \mathscr{C}_n} \exp\left(\tau S_n\tilde{\varphi}_1(C)\right) \cdot \exp\left(qS_n\varphi_2(C)\right).$$

If $\tau < 0$ and $n = n_i$ one of the terms in this sum is $\exp(\tau S_{n_i}\varphi_1(C_i)) \cdot \exp(qS_{n_i}\varphi_2(C_i))$ which is greater than $\exp(\tau n_i\varepsilon) \cdot \exp(qS_{n_i}\varphi_2(C_i))$. Thus $\tilde{P}(q, \tau) > \tau\varepsilon + c$ where $c = q \cdot \min(\varphi_2)$ if $q \geq 0$ and $c = q \cdot \max(\varphi_2)$ if $q < 0$. If $\tau < 0$ then using a sequence C_i so that $n_i^{-1}S_{n_i}\varphi_1(C_i) < -\varepsilon$, one proves in a similar fashion that $\tilde{P}(q, \tau) > |\tau|\varepsilon + c$. Thus in any case, $\tilde{P}(q, \tau) \to \infty$ as $|\tau| \to \infty$. As has already been noted, since R_l is a nontrivial closed interval, $\tilde{P}(q, \tau)$ is a strictly convex function of τ. Therefore since $\tilde{P}(q, \tau) \to \infty$ as $|\tau| \to \infty$ it has a unique minimum, say at τ_0. Consequently, $\partial\tilde{P}/\partial\tau(q, \tau_0) = 0$ as required and we set $\tau_q(l) = \tau_0$.

Since $l_q'(\tau) = \partial^2\tilde{P}/\partial^2\tau(q, \tau) > 0$ it follows immediately from the Inverse Function Theorem that the inverse function $\tau_q(l)$ is a real-analytic. □

FUNDAMENTAL LEMMA. *$f(\alpha) = \tau(q) + \alpha q$ where $q = q(\alpha)$ and $l(\alpha) = l_{q(\alpha)}(\tau(q(\alpha))) = l(q(\alpha), \tau(q(\alpha)))$ is the unique value of l at which the supremum of $f(\alpha, l)$ is achieved.*

Proof. Given α and l let A (resp. L) be a small interval containing α (resp. l). Let $q=q(\alpha)$ and $\tau=\tau_q(l)$. Then if C is an n-cylinder such that $l(C)\in L$ and $\alpha(C)\in A$, for some $x\in C$,

$$\mu_{q,\tau}(C)\in[c_1, c_2]\exp[n(-P(q,\tau)+\tau S_n\varphi_1(x)+qS_n\varphi_2(x))]$$
$$\subset[e^{-2d}c_1, e^{2d}c_2]\exp[n(-P(q,\tau)+\tau L+qAL)],$$

where $d=(|q|+|\tau|)D$. Consequently, if $C_n(A, L)$ denotes the union of such n-cylinders,

$$\mu_{q,\tau}(C_n(A,L))\in[e^{-2d}c_1, e^{2d}c_2]N_n(A,L)\exp[n(-P(q,\tau)+\tau L+qAL)].$$

But by the choice of q and τ, $\mu_{q,\tau}(C_n(A,L))\to 1$ as $n\to\infty$ so

$$S(A,L)\in P(q,\tau)-qAL-\tau L.$$

Taking the limits $A\searrow\alpha$ and $L\searrow l$ it follows that

$$f(\alpha, l)=-l^{-1}P(q,\tau)+q\alpha+\tau.$$

But then $\partial f/\partial l=l^{-2}P(q,\tau)$ so the maximum value is achieved when $P(q,\tau)=0$ and then $f(\alpha)=q\alpha+\tau$ which is the first part of the required result. Moreover, $P(q,\tau)=0$ when $\tau=\tau(q)$ so that the maximum occurs when $l=l_q(\tau(q))$. This is the second part of the result as $q=q(\alpha)$. □

Proof of Theorem 2. By the Fundamental Lemma, $f(\alpha)=\tau(q)+\alpha q$ where $q=q(\alpha)$. But, $q=q(\alpha)$ implies that $\alpha=\alpha(q)=-\tau'(q)$ and, moreover,

$$f'(\alpha)=\tau'(q)q'(\alpha)+q+\alpha q'(\alpha)=q$$

since $\alpha=-\tau'(q)$. Thus f is the Legendre transform of τ. Since $\tau(q)$ and $q(\alpha)$ are real-analytic then so is f. □

Proof of Corollary to Theorem 2. One immediate consequence of the relation $f(\alpha)=\tau(q)+\alpha q$ is that the maximum value of $f(\alpha)$ is the Hausdorff dimension of Λ. This is the case because at the maximum, $q=f'=0$. Thus for this value of α, $P(0,\tau(0))=0$ i.e. the pressure of $-d\log|g'|$ is zero when $d=\tau(0)$. By standard arguments (e.g. **[14]**; see also **[2]** § 7) this means that $d=\tau(0)=f(\alpha(0))$ is the Hausdorff dimension. This will also be proved via a direct argument below. □

Proof that R_α and R_l are intervals. This is quite general and follows from the fact that $\varphi_1=-\log|g'|$ and $\varphi_2=-\log J$ are Hölder continuous and that φ_1 is bounded away from zero, say $0<\delta<|\varphi_1|$. In particular, φ_1 and φ_2 are bounded, say $|\varphi_1|, |\varphi_2|<c$.

To prove the result for R_α, firstly note that if $B_{n,i}(x)=n^{-1}S_n\varphi_i(x)$ then the set $L(x)$ of limit points of the sequence $s_n(x)=B_{n,2}(x)/B_{n,1}(x)$ is either a single point or a closed interval $[l, r]$ with, of course, $l=\liminf s_n$ and $r=\limsup s_n$. To see this cover $[l, r]$ by intervals of length ε and choose $n>2c^2/\varepsilon\delta$ such that s_n lies in the leftmost interval. Then a simple calculation shows that $|s_{n+i}-s_n|<i\varepsilon/2$ for all $i\geq 0$, so every interval of the cover contains a point of the sequence s_i. Consequently, the sequence is dense in $[l, r]$ which proves that $[l, r]$ is the set of limit points.

I must now prove that $R_\alpha=\bigcup\{L(x):x\in\Lambda\}$ is an interval. To see this suppose that $x<y$ are in Λ, $a\in L(x)$ and $b\in L(y)$. Fix $\varepsilon>0$ small. By the principle of bounded variation (1), if $x, y\in C\in\mathscr{C}_n$ then $|S_n\varphi_i(x)-S_n\varphi_i(y)|<D$ for $i=1, 2$. Then, using the fact that the variation of $B_{n,j}$ on $C_{n(i)}$ is bounded by D/n_i, one easily

sees that there exist a sequence $n_i \to \infty$ and n_i-cylinders C_{n_i} such that for all $t \in C_{n_i}$, $|s_{n_i}(t) - a| < \varepsilon$. Similarly there exists another sequence so that $|s_{n_i}(t) - b| < \varepsilon$. Thus there exists a sequence $n_i \to \infty$ such tht $n_i/n_{i+1} \to 0$ and such that for each i there is a $(n_{i+1} - n_i)$-cylinder C_i such that for i odd,

$$|s_{n_{i+1}-n_i}(t) - a| < \varepsilon \quad \text{for all } t \in C_i$$

and for all even i,

$$|s_{n_{i+1}-n_i}(t) - b| < \varepsilon \quad \text{for all } t \in C_i.$$

Also, since the image of any n-cylinder under g^n contains I, there exists $x \in \Lambda$ such that $g^{n_i}x \in C_i$ for all $i > 0$. Now, since $|\varphi| < c$, it is clear that

$$|s_{n_{i+1}}(x) - s_{n_{i+1}-n_i}(g^{n_i}x)| \leq 4n_i c^2/n_{i+1}\delta^2.$$

Consequently, since $g^{n_i}x \in C_i$, for large i, the sequence $s_{n_i}(x)$ oscillates between the intervals $[a - 2\varepsilon, a + 2\varepsilon]$ and $[b - 2\varepsilon, b + 2\varepsilon]$. Thus $L(x)$ contains $[a, b]$ since $L(x)$ is an interval. This proves that R_α is an interval.

This same argument, but taking $\varphi_1 \equiv 1$ and $\varphi_2 = -\log|g'|$ also proves that R_l is an interval. □

5. *The meaning of the function $\tau(q)$*

Since $P(q, \tau(q)) = 0$, there are constants $c_1, c_2 > 0$ such that

$$1 = \sum_{\mathscr{C}_n} \mu_{q,\tau(q)}(C) \in [c_1, c_2] \sum_{\mathscr{C}_n} \exp(S_n\varphi(C)),$$

where $\varphi = \varphi_{q,\tau(q)} = -(\tau(q)\log|g'| + q \log J)$, $S_n\varphi(C)$ is the supremum of $\varphi(x) + \cdots + \varphi(g^{n-1}x)$ over the n-cylinder C and the sum is over the set of n-cylinders $\mathscr{C}_n$. This used the Gibbs state property of $\mu_{q,\tau}$. But, using the principle of bounded variation, if $d = (|q| + |\tau|)D$ where D is given by (1) then

$$\exp S_n\varphi(C) \in [e^{-d}, e^d]\nu(C)^q\lambda(C)^{\tau(q)}.$$

Thus there exists constants $d_1, d_2 > 0$ such that

$$1 \in [d_1, d_2] \sum_{\mathscr{C}_n} \nu(C)^q\lambda(C)^{\tau(q)}.$$

Proposition 1 follows from this.

PROPOSITION 1. *$\tau(q)$ is defined by the following property: as $n \to \infty$,*

$$\sum_{\mathscr{C}_n} \nu(C)^q\lambda(C)^\tau \to \begin{matrix} \infty & \text{if } \tau < \tau(q) \\ 0 & \text{if } \tau > \tau(q). \end{matrix}$$

From this one sees that it corresponds precisely to the corresponding function introduced by Halsey et al. [**9**]. There the definition is given in terms of arbitrary covers of bounded diameter and not just in terms of cylinders, but you will see from the arguments of the next section that this makes no difference.

I now prove that the Hausdorff dimension of the set $\Sigma(\alpha)$ of points with exponent α is $f(\alpha)$ (see § 2.4).

Proof of Theorem 1. In this proof and what follows I denote by $\mathscr{H}^d_\delta(E)$ the quantity $\inf_{\mathscr{B}} \sum_{B\in\mathscr{B}} \lambda(B)^q$ where the infimum is over all covers $\mathscr{B}$ of E by intervals B with $\lambda(B) \leq \delta$. Let $\mathscr{H}^d(E)$ denote $\lim_{\delta\to 0} \mathscr{H}^d_\delta(E) = \sup_{\delta>0} \mathscr{H}^d_\delta(E)$. Then the Hausdorff dimension $HD(E)$ of E is given by $\inf\{d: \mathscr{H}^d(E) = 0\}$.

Let $\mu_q = \mu_{q,\tau(q)}$. Since $\mu_q(\Sigma(\alpha)) = 1$ if $\alpha = \alpha(q)$, then it follows from Manning [13] or Young [18] that

$$HD(\Sigma(\alpha)) \geq h(\mu_q)/\chi(\mu_q),$$

where $h(\mu_q)$ are $\chi(\mu_q)$ respectively the metric entropy and characteristic exponent associated with μ_q. But by the variational principle [3],

$$h(\mu_q) = P(q, \tau(q)) + \int (-\tau(q) \log |g'| - q \log J) \, d\mu_q.$$

Thus, since $P(q, \tau(q)) = 0$, $h(\mu_q) = \tau(q)\chi(\mu_q) + q\alpha\chi(\mu_q)$; from which it follows that $HD(\Sigma(\alpha)) \geq \tau(q) + q\alpha$.

Now I prove the reverse inequality. Let A be an open interval containing $\alpha = \alpha(q)$ and $A_n = \{x \in \Lambda : \alpha(C_{m,x}) = \log \nu(C_{m,x})/\log \lambda(C_{m,x}) \in A \text{ for all } m \geq n\}$. Let $\mathscr{C}_m(A)$ denote the set of those $C \in \mathscr{C}_m$ such that $\alpha(C) = \log \nu(C)/\log \lambda(C) \in A$. Then there exists $d_1, d_2 > 0$ and independent of n and C such that

$$\sum_{\mathscr{C}_n(A)} \mu_q(C) \in [d_1, d_2] \sum_{\mathscr{C}_n(A)} \lambda(C)^{\tau(q)} \nu(C)^q \subset [d_1, d_2] \sum_{\mathscr{C}_n(A)} \lambda(C)^{\tau(q)+Aq}.$$

But the lefthand side converges to 1 as $n \to \infty$ and $\mathscr{C}_m(A)$ is a cover of A_n for all $m \geq n$. Thus choosing m so large that each cylinder in $\mathscr{C}_m(A)$ has length less than δ one sees that if $d > \tau(q) + Aq$ then $\mathscr{H}^d(A_n) = 0$. Hence, since $\mathscr{H}^d$ defines an outer measure $\mathscr{H}^d(\bigcup_{n>0} A_n) = 0$. But $\Sigma(\alpha) \subset \bigcup_{n>0} A_n$ so $\mathscr{H}^d(\Sigma(\alpha)) = 0$. Letting $A \searrow \alpha$ one deduces that if $d > \tau(q) + \alpha q$ then $\mathscr{H}^d(\Sigma(\alpha)) = 0$ i.e. $HD(\Sigma(\alpha)) \leq \tau(q) + \alpha q$. □

6. *The Renyi dimensions*

Recall the definition of the Renyi dimensions and the statement of Theorem 3 from § 2.4.

Proof of Theorem 3. If $q \in \mathbb{R}$ is fixed, for $\varepsilon > 0$ and $N > 0$ let

$$\Lambda_N(\varepsilon) = \{x \in \Lambda : \lambda(C_{n,x}) \in \exp[n(l \pm \varepsilon)]$$

and

$$\nu(C_{n,x}) \in \exp[n(p \pm \varepsilon)] \quad \text{for all } n \geq N\},$$

where $l = l(q) = \partial P/\partial\tau(q, \tau(q))$ and $p = p(q) = \partial P/\partial q(q, \tau(q))$. For N sufficiently large $\mu_q(\Lambda_N(\varepsilon)) > 1 - \varepsilon$. Then, if $\mathscr{C}_n(\varepsilon)$ denotes the set of n-cylinders C such that $\lambda(C) \in \mathrm{e}^{n(l\pm\varepsilon)}$ and $n \geq N$,

$$1 - \varepsilon < \mu_q(\Lambda_N(\varepsilon)) \leq \sum_{\mathscr{C}_n(\varepsilon)} \mu_q(C) \in [d_1, d_2] \sum_{\mathscr{C}_n(\varepsilon)} \nu(C)^q \lambda(C)^{\tau(q)}$$

for some $d_1, d_2 > 0$ independent of n and C. This uses the Gibbs state property and the principle of bounded variation. But, since $\lambda(C) \in \mathrm{e}^{n(l\pm\varepsilon)}$, this implies that there exist $c_1, c_2 > 0$ such that

$$\mathrm{e}^{nl\tau(q)} \sum_{\mathscr{C}_n(\varepsilon)} \nu(C)^q \in [c_1 \exp(-n\tau(q)\varepsilon), c_2 \exp(n\tau(q)\varepsilon)]. \tag{3}$$

This inequality will be used several times in the proof. Also assume henceforth that $\varepsilon < \min(|l|, |p|)$.

Firstly, I consider the case $q \geq 0$, $q \neq 1$ and show that $(1-q)D_q(\Lambda) \geq \tau(q)$. (The result for $q = 1$ follows by continuity.) If B is an open interval let $n = n(B)$ be the smallest non-negative integer such that B contains a cylinder in $\mathscr{C}_n(\varepsilon)$. If $B \cap \Lambda_N(\varepsilon) \neq \varnothing$ then $n(B) < \infty$. Now, by the definition of $n(B)$, B can meet at most two elements of $\mathscr{C}_{n(B)-1}(\varepsilon)$. Let B_1 and B_2' denote the intersection of these with B, with B_1 containing an element of $\mathscr{C}_{n(B)}(\varepsilon)$ and with $B_2' = \varnothing$ if B only meets one element of $\mathscr{C}_{n(B)-1}(\varepsilon)$. If B_2' does not intersect $\Lambda_N(\varepsilon)$ then set $B_2' = \varnothing$. If $B_2' \neq \varnothing$ then B_2' only meets one element C of $\mathscr{C}_n(\varepsilon)$ with $n = n(B_2') - 1$. Let $B_2 = B_2' \cap C$. Then $2\nu(B)^q \geq \nu(B_1)^q + \nu(B_2)^q$.

If $\mathscr{B}$ is a δ-cover of $\Lambda_N(\varepsilon)$ let $\mathscr{B}'$ denote the set of intervals of the form B_1 and B_2 where B ranges over $\mathscr{B}$. Then $\mathscr{B}'$ is a cover of $\Lambda_N(\varepsilon)$ with diameter $\leq \delta$. Moreover, if $B \in \mathscr{B}'$ then since B contains an element of $\mathscr{C}_{n(B)}(\varepsilon)$, $\nu(B) \geq e^{n(B)(p-\varepsilon)}$ and since B is contained in an element of $\mathscr{C}_{n(B)-1}(\varepsilon)$, $\lambda(B) \leq \exp[(n(B)-1)(l+\varepsilon)]$ or $n(B) \leq 1 + \log \lambda(B)/(l+\varepsilon)$. Thus

$$\begin{aligned} 2\sum_{\mathscr{B}} \nu(B)^q &\geq \sum_{\mathscr{B}'} \nu(B)^q \geq \sum_{\mathscr{B}'} e^{n(B)(p-\varepsilon)q} \\ &\geq c \sum_{\mathscr{B}'} \lambda(B)^{q(p-\varepsilon)/(l+\varepsilon)} \\ &= c\delta^{-\tau(q)} \sum_{\mathscr{B}'} \delta^{\tau(q)} \lambda(B)^{q(p-\varepsilon)/(l+\varepsilon)} \\ &\geq c\delta^{-\tau(q)} \sum_{\mathscr{B}'} \lambda(B)^{\tau(q)+q(p-\varepsilon)/(l+\varepsilon)} \\ &\geq c\delta^{-\tau(q)} \mathscr{H}_\delta^d(\Lambda_N(\varepsilon)), \end{aligned}$$

where $d = \tau(q) + q(p-\varepsilon)/(l+\varepsilon)$ and $c = e^{(p-\varepsilon)q}$. This used the fact that $\tau(q) \geq 0$ for $q \leq 1$. If $q > 1$, $\tau(q) \leq 0$ and the last two inequalities are replaced by

$$\begin{aligned} &\geq c\delta^{-\tau(q)} \sum_{\mathscr{B}'} \lambda(B)^{q(p-\varepsilon)/(l+\varepsilon)} \\ &\geq c\delta^{-\tau(q)} \mathscr{H}_\delta^d(\Lambda_N(\varepsilon)), \end{aligned}$$

where $d = q(p-\varepsilon)/(l+\varepsilon)$. Thus,

$$(\log \sum \nu(B)^q)/(\log \delta^{-1}) \geq \log(2c^{-1})/\log \delta + \tau(q) + (\log \mathscr{H}_\delta^d(\Lambda_N(\varepsilon)))/(\log \delta).$$

This proves that

$$(1-q)D_q(\Lambda(\varepsilon)) \geq \log 2/\log \delta + \tau(q) + (\log \mathscr{H}_\delta^d(\Lambda_N(\varepsilon)))/(\log \delta).$$

But, it follows exactly as in the proof of Theorem 3 that the Hausdorff dimension of $\Lambda_N(\varepsilon)$ is $f(\alpha)$. Thus, since in both cases $d > f(\alpha)$, $\mathscr{H}_\delta^d(\Lambda_N(\varepsilon)) \to 0$ as $\delta \to 0$ and one deduces that

$$(1-q)D_q(\Lambda(\varepsilon)) \geq \tau(q).$$

Since $\Lambda(\varepsilon) \subset \Lambda$, this implies that $(1-q)D_q \geq \tau(q)$ as required. Now I prove the reverse inequality for the case $q \geq 0$.

Each $C \in \mathscr{C}_n(\varepsilon)$ can be covered by $\leq N = e^{2n\varepsilon}$ intervals B of length $e^{n(l-\varepsilon)}$. Let $\mathscr{B}$ denote such a cover. Then

$$\sum_{\mathscr{C}_n(\varepsilon)} \nu(C)^q \geq N^{-1} \sum_{\mathscr{B}} \nu(B)^q.$$

But by (3) the left-hand side is $\leq c_2 e^{-n\tau(q)(l-\varepsilon)}$ for n sufficiently large. Thus

$$\begin{aligned} (1-q)D_q(\Lambda_N(\varepsilon), e^{n(l-\varepsilon)}) &\leq -\left(\log \sum_{\mathscr{B}} \nu(B)^q\right) \Big/ nl \\ &\leq (-nl)^{-1} \log Nc_2 + \tau(q)(1-\varepsilon l^{-1}). \end{aligned}$$

Thus $(1-q)D_q(\Lambda'_\varepsilon) \leq \tau(q)(1-\varepsilon l^{-1})$ where $\Lambda'_\varepsilon = \bigcup_{n\geq 0} \Lambda_n(\varepsilon)$. Since Λ'_ε is dense in Λ, it follows that

$$(1-q)D_q(\Lambda) = (1-q)D_q(\Lambda'_\varepsilon) \leq \tau(q)(1-\varepsilon l^{-1})$$

for all $\varepsilon > 0$ sufficiently small, whence $(1-q)D_q(\Lambda) \leq \tau(q)$. This completes the case $q \geq 0$.

Finally, I consider the case $q < 0$. Let $\mathcal{B}$ be a $e^{n(l-\varepsilon)}$-cover of $\Lambda_N(\varepsilon)$. Then for each $C \in \mathcal{C}_n(\varepsilon)$, $\sum \nu(B) \leq \nu(C)$ where the sum is over all $B \in \mathcal{B}$ which meet C. Thus, since $q < 0$, $\sum \nu(B)^q \geq \nu(C)^q$ and summing over all C it follows that

$$\sum_{\mathcal{B}} \nu(B)^q \geq 2 \sum_{\mathcal{C}_n(\varepsilon)} \nu(C)^q$$

since each B can meet at most two Cs. It follows that $(1-q)D_q(\Lambda) \geq (1-q)D_q(\Lambda_n(\varepsilon)) \geq \tau(q)$. Finally, for the reverse inequality, take for $\mathcal{B}$ any cover whose intervals B are in one-to-one correspondence with the cylinders C in $\mathcal{C}_n$ and contain the corresponding cylinder. Then $\sum_{\mathcal{C}} \nu(C)^q \geq \sum_{\mathcal{B}} \nu(B)^q$ and as above one deduces that $\tau(q) \geq (1-q)D_q(\Lambda)$. □

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