Some old friends revisited

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1 Introduction

In his famous memoir on trigonometric series published in 1867, Riemann introduces the definition of integral that now bears his name. He considers in particular some functions that are very irregular, but nonetheless "Riemann integrable". One of these examples is

$$R(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2}, \tag{1}$$

where (x) represents x minus the integer closest to x (and $(x) = 0$ if $x \in \mathbf{Z} + 1/2$).

Riemann remarks that R is continuous except at the rationals $p/2q$ (p and $2q$ having no common factor), with the following right and left limits at these rationals

$$\left.\begin{aligned} R(\frac{p}{2q}^-) &= R(\frac{p}{2q}) - \frac{\pi^2}{16q^2} \\ R(\frac{p}{2q}^+) &= R(\frac{p}{2q}) + \frac{\pi^2}{16q^2}. \end{aligned}\right\} \tag{2}$$

Thus R is discontinuous on a dense set; but in contrast with the characteristic function of rationals, the following property holds: For all $\epsilon > 0$, the set of points where R has a discontinuity of amplitude larger than ϵ is finite; thus R is Riemann integrable; for this reason, we will call R the "Integrable Riemann function". Can we study more precisely the regularity of R?

Consider for instance the neighbourhood of a rational $\frac{p}{2q+1}$ (with an odd denominator). The function (nx) has its discontinuities at the points $\frac{2k+1}{2n}$ ($k \in \mathbf{Z}$), so that the distance between $\frac{p}{2q+1}$ and a discontinuity of (nx) is at least $1/2n(2q+1)$; thus (nx) is linear on the interval $[\frac{p}{2q+1} - h, \frac{p}{2q+1} + h]$ provided that $h \leq \frac{1}{2n(2q+1)}$. Let h be fixed and $N = [\frac{1}{2(2q+1)h}]$. Then

$$\begin{aligned} f(\frac{p}{2q+1} + h) - f(\frac{p}{2q+1}) &= \sum_{n=1}^{N} \frac{h}{n} + \sum_{n=N+1}^{\infty} \frac{(n(\frac{p}{2q+1} + h)) - (n\frac{p}{2q+1})}{n^2} \\ &= h \log(\frac{1}{2qh}) + O(h); \end{aligned}$$

thus R is quite smooth at rationals with an odd denominator: its modulus of continuity is $h \log(1/h)$ at these points. So we see that the regularity of R changes completely from point to point (in sharp contrast for instance with the Weierstrass functions

$$W_{a,b}(x) = \sum a^n \sin(b^n x)$$

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which have a "very regular irregularity"). Let us make this point more precise. The regularity of a function f at a point x_0 is measured by the $C^\alpha(x_0)$ conditions:
F is $C^\alpha(x_0)$ $(\alpha \geq 0)$ if there exists a polynomial P of degree at most $[\alpha]$ such that

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha.$$

In practice one often uses the *Hölder exponent* of f at x_0 defined by

$$h_f(x_0) = \sup\{\alpha : f \quad \text{is} \quad C^\alpha(x_0)\}.$$

The Hölder exponent of $W_{a,b}$ is constant (and equal to $-\log a / \log b$), whereas we saw that R's can jump from point to point. Such a behavior of the Hölder exponent is characteristic of *multifractal functions*; these are functions whose Hölder exponent jumps from one point to another in such an erratic way that the set of points E_α where the function has a given exponent α is a fractal set. The relevant parameters one tries to determine are contained in the *spectrum of singularities*

$$d(\alpha) = dim\{x_0 : h(x_0) = \alpha\} \tag{3}$$

where dim stands for the Hausdorff dimension (and by convention $dim(\emptyset) = -\infty$).

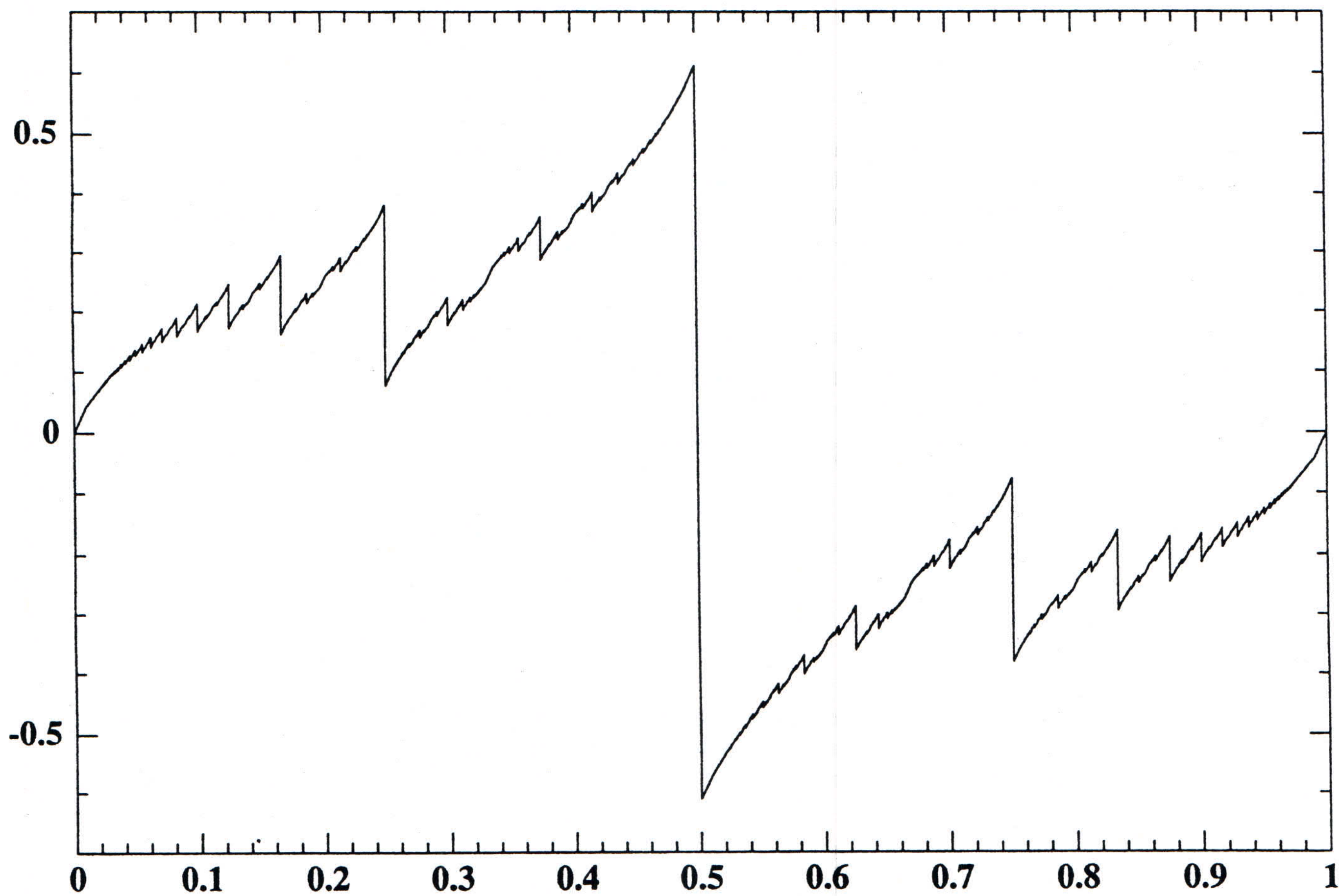

Fig.1: Riemann's integrable function R.

We will show that R is a multifractal function. More precisely, let $\frac{p_n}{q_n}$ be the sequence of convergents of the continued fraction expansion of x_0. Let us define (as in [4]) τ_n by

$$|x_0 - \frac{p_n}{q_n}| = \frac{1}{q_n^{\tau_n}}$$

(thus $\tau_n \geq 2$). Let

$$\tau(x_0) = \limsup_{n \in A} \tau_n \tag{4}$$

where A is the set of integers n such that q_n is even (if this set contains only a finite number of elements, we set $\tau(x_0) = 2$). We will prove the following theorem in Section 2.

Theorem 1 *The Hölder exponent of the integrable Riemann function R at x_0 is*

$$h_R(x_0) = \frac{2}{\tau(x_0)},$$

and its spectrum of singularities is given by

$$\begin{aligned} d(\alpha) \quad &= \alpha \quad \textit{for} \quad \alpha \in [0,1] \\ &= -\infty \quad \textit{elsewhere.} \end{aligned}$$

Note that this Hölder exponent has a striking similarity with the Hölder exponent of another function introduced by Riemann:

$$\mathcal{R}(x) = \sum \frac{1}{n^2} \sin(\pi n^2 x).$$

Indeed, let

$$\tau'(x_0) = \limsup_{n \in B} \tau_n$$

where B is the set of integers n such that p_n and q_n are not both odd. The Hölder exponent of $\mathcal{R}$ is

$$h_{\mathcal{R}}(x_0) = \frac{1}{2} + \frac{1}{2\tau'(x_0)},$$

(see [4]) and $\mathcal{R}$ is also a multifractal function of spectrum

$$\left.\begin{aligned} d_{\mathcal{R}}(\alpha) \quad &= 4\alpha - 2 \quad \text{if} \quad \alpha \in [\frac{1}{2}, \frac{3}{4}] \\ &= 0 \qquad\quad \text{if} \quad \alpha = \frac{3}{2} \\ &= -\infty \qquad \text{elsewhere.} \end{aligned}\right\} \tag{5}$$

Let us quote [8]; the function R "is exactly what Paul Lévy called a compensated jump function: all jumps are negative and their sum is infinite but the continuous parts of (nx) provide a shift such that the series converges. Paul Lévy considerd the simpler function

$$\mathcal{L}(x) = \sum_{n=0}^{\infty} \frac{(2^n x)}{2^n}.$$

as an illustration of what occurs frequently in the theory of stochastic processes with independant increments."

We will study the pointwise regularity of Lévy's function $\mathcal{L}$ and prove that it is another example of multifractal function in Section 5. Can we infer from Lévy's intuition that there are natural examples of stochastic processes with independant increments which are multifractal? We are not aware of such existing results; nonetheless, there exists encouraging hints: in [5] it is proved that the most simple model of (random) lacunary wavelet series yields almost surely multifractal functions.

A slight modification of R yields

$$J(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^3}.$$

In a Note au Compte-Rendu [7] published in 1881, Jordan introduced the notion of bounded variation function, and proposed J as an example of a bounded variation function which is nonetheless discontinuous on a dense set.

Let us compute the amplitude of the jump of J at $(2k+1)/2n$ ($2k+1$ and $2n$ having no common factor). All the functions (mx) that have a jump at $(2k+1)/2n$ satisfy:

$$\exists l \in \mathbf{Z}: \ \frac{l+\frac{1}{2}}{m} = \frac{2k+1}{2n},$$

so that $m = An$ and $2l+1 = A(2k+1)$. All possible values of A are the odd integers, so that the total jump at $(2k+1)/2n$ is

$$\sum_{A \ odd} \frac{1}{(An)^3} = \frac{1}{n^3} \sum_{p=0}^{\infty} \frac{1}{(2p+1)^3} = \frac{1}{n^3} \frac{7\zeta(3)}{8}.$$

The study of Jordan's function is quite similar to Riemann's, and J is another example of multifractal function. We will prove the following theorem in Section 3.

Theorem 2 *The Hölder exponent of Jordan's function J at x_0 is*

$$h_J(x_0) = \frac{3}{\tau(x_0)}$$

except if x_0 is a rational with an odd denominator, in which case

$$h_J(x_0) = 3;$$

the spectrum of singularities of J is given by

$$\left.\begin{array}{lll} d_J(\alpha) & = 2\alpha/3 & \text{if} \quad \alpha \in [0, 3/2] \\ & = 0 & \text{if} \quad \alpha = 3 \\ & = -\infty & \text{elsewhere.} \end{array}\right\} \tag{6}$$

The only qualitative difference with R is the regularity at rationals with an odd denominator (note that the resemblance with $\mathcal{R}$ is even more striking here).

One can see Jordan's function as a modification of Riemann's function R. But an important property that is not shared with Riemann's function is that it is the primitive of a singular measure (up to a linear term). Indeed, the derivative of Jordan's function is

$$\frac{\pi^2}{6}x - \sum_{2k+1 \wedge 2n = 1} \frac{7\zeta(3)}{8n^3} \delta_{\frac{2k+1}{2n}}$$

We can thus reinterpret Theorem 2 as follows: The measure

$$\mu_J = \sum_{2k+1 \wedge 2n = 1} \frac{1}{n^3} \delta_{\frac{2k+1}{2n}} \tag{7}$$

is a multifractal positive measure whose spectrum is given by (6). Let us explain this assertion.

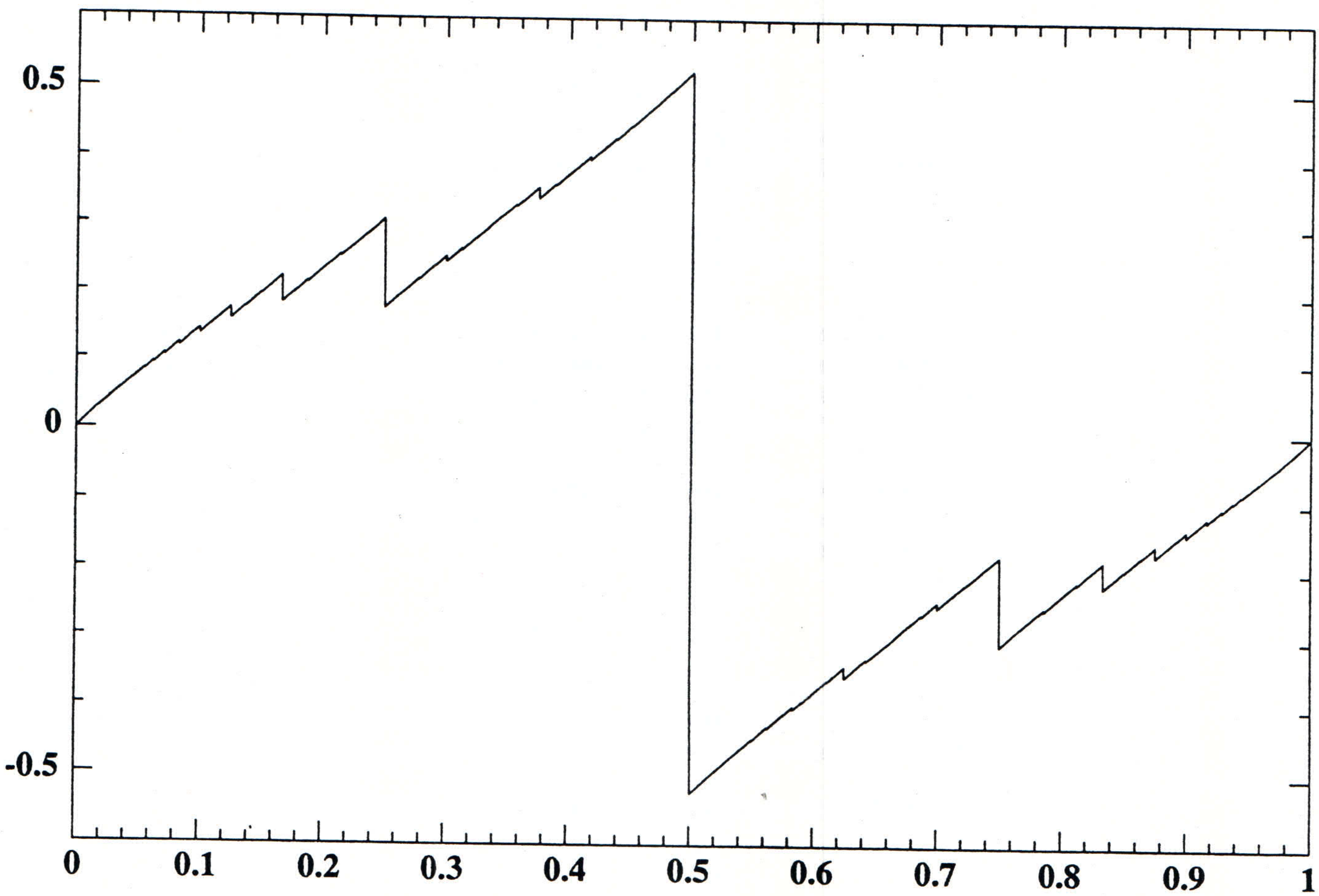

Fig.2: Jordan's function J.

The Hölder exponent of a measure μ at x_0 is defined by

$$h_\mu(x_0) = \sup\{\alpha : \mu([x_0 - \epsilon, x_0 + \epsilon]) \leq C\epsilon^\alpha\}. \tag{8}$$

The spectrum of singularities of μ is then defined as in (3). If f is the primitive of the measure μ,

$$\mu([x_0 - \epsilon, x_0 + \epsilon]) = f(x_0 + \epsilon) - f(x_0 - \epsilon)$$

and the exponent of μ and f at x_0 are the same as long as one can choose $P(x - x_0) = f(x_0)$ in (2). It is always the case if $0 \leq h(x_0) \leq 1$. If $1 < h(x_0) \leq 2$, the two exponents will coincide for an important class of functions: The functions called (by Lebesgue and later by Salem) *purely singular increasing functions*. By definition, these functions are differentiable almost everywhere with a vanishing derivative at every point of differentiablility.(Actually, these authors require f to be continuous; we do not make this asumption here). The Devil's staircase is an example of purely singular increasing functions; we will see that $\frac{\pi^2 x}{6} - J(x)$ is another example.

Properties of such functions, or equivalently of their derivative have been extensively studied, starting with Jordan and Lebesgue, and including Denjoy, Rajchman, Salem,... In many cases, these authors have considered specific examples that we can now interpret as multifractal functions. We will study some of them.

We can try to differentiate also Riemann's function R. If we are not careful, we obtain the difference between an infinite linear functions $(\sum \frac{1}{n})x$ and an infinite measure

$$\sum_{2k+1\wedge 2n=1} \frac{1}{n^2}\delta_{\frac{2k+1}{2n}}. \tag{9}$$

Of course this calculationshould be given a sense by differentiating R in the sense of distributions; thus if ψ is a C^∞, one-periodic function, we obtain

$$\langle R'|\psi\rangle = \lim_{N\to\infty} \int \left(\sum_{n=1}^{N} \frac{1}{n}\right) \psi(x)dx - \langle \left(\sum_{n=1}^{N} \frac{1}{n^2} \sum_{2k+1\wedge 2n=1} \delta_{\frac{2k+1}{2n}}\right) |\psi\rangle. \tag{10}$$

The two terms in the limit usually do not have a limit independantly, except if $\int \psi = 0$. Thus (9) makes sense when integrated against functions with a vanishing integral; and we can interpret (10) as the correct way to "renormalize" the infinite measure (9) by substracting the right "floating constant".

We will study measures (finite or infinite) similar to (9) in Section 4. We can actually slightly simplify the model given by (9) or by μ_J, without changing the specific properties of these measures as follows. Consider

$$\mu_\beta = \sum_{m\wedge n=1} \frac{1}{n^\beta}\delta_{\frac{m}{n}} \tag{11}$$

which are also "real measures" if $\beta > 2$ and must be renormalized if $\beta \leq 2$. We will explain how one can define an Hölder exponent for the "measure" μ_β when $1 < \beta \leq 2$. When using this generalization, all measures μ_β will be multifractal $(1 < \beta < \infty)$.

Theorem 3 *If $\beta \geq 2$, the spectrum of singularities of (11) is*

$$\begin{aligned} d_\beta(\alpha) &= \frac{2\alpha}{\beta} \quad \text{for} \quad \alpha \in [0, \beta/2] \\ &= -\infty \quad \text{elsewhere.} \end{aligned}$$

If $1 < \beta < 2$ the spectrum of singularities of (11) satisfies

$$\begin{aligned} d_\beta(\alpha) &= \frac{2\alpha}{\beta} \quad \text{for} \quad \alpha \in [0, \beta - 1) \\ &= -\infty \quad \text{for} \quad \alpha > \frac{\beta}{2}. \end{aligned}$$

Note that, in the case $\beta = 3$, (11) is the derivative of

$$R_a(x) = \sum_{n=1}^{\infty} \frac{E(nx)}{n^3}$$

(up to the numerical factor $\zeta(3)$). This function appears in a paper of Rajchman [12], where he studied purely singular increasing functions, and proved that a convergent series of such functions is still purely singular. As an example he considers R_a and thus obtains directly that it is almost everywhere differentiable. The proof of Theorem 3 will actually yield a more precise result: Rajchman's function R_a is differentiable except on a set of dimension $2/3$.

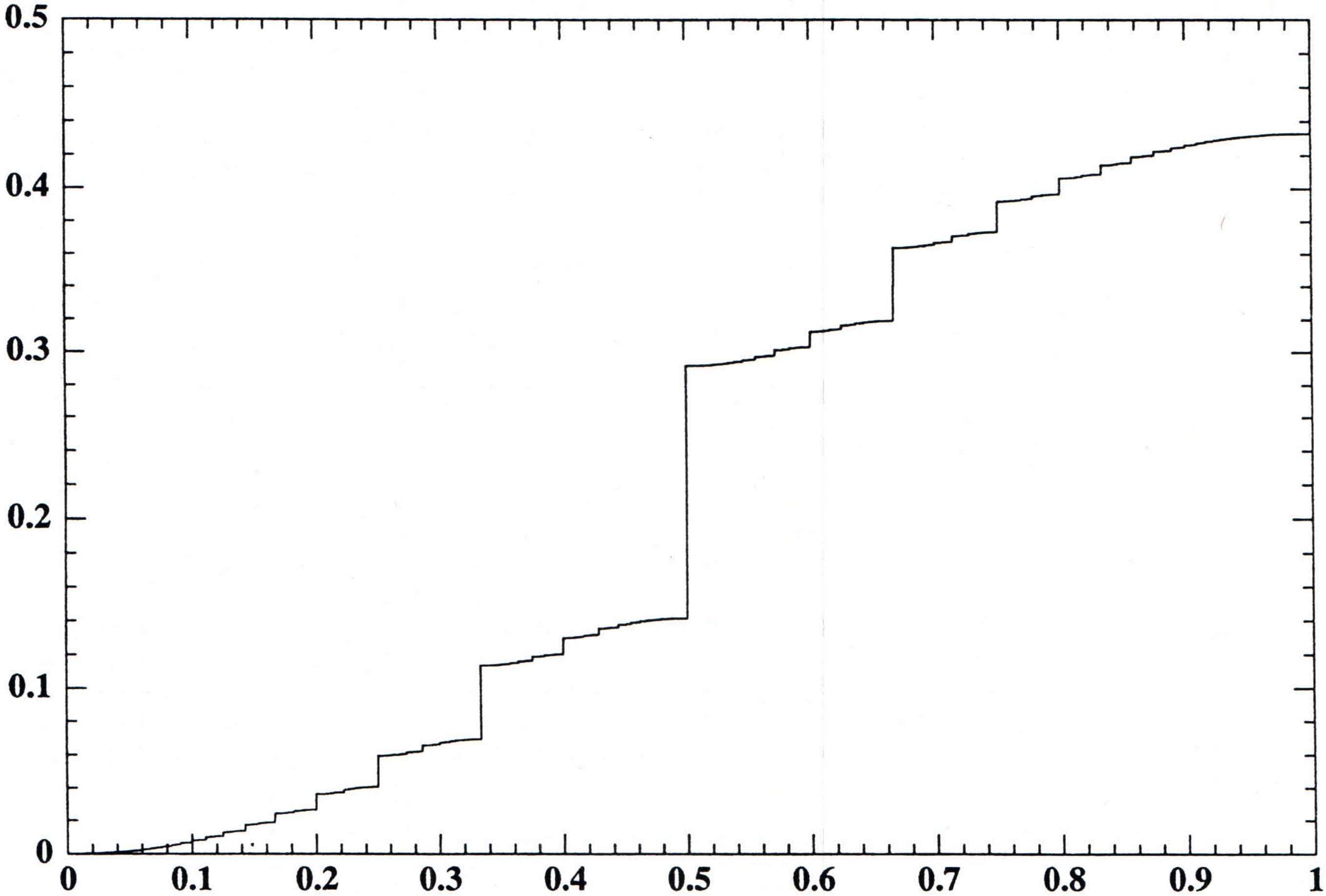

Fig.3: Rajchman's function.

Other functions have been introduced in the past as examples or counterexamples of functions with unexpected properties and turn out to be multifractal (for instance Polya's "triangle-filling" function, studied by B.Mandelbrot and the author in [6], or de Rham's function, studied by Y.Meyer in [11]).

The motivation to consider multifractal functions came from physical problems ([2],[10]). Within mathematics, it leads to reconsider all the historical examples we mentioned with a new eye; and, by deepening our understanding of these functions, it allows us perceive some similarities and recurrent structures, in what was before a collection of unrelated curiosities.

Acknowledgement: I wish to thank Jean-Pierre Kahane who suggested the study of the Riemann function R, and guessed its multifractal nature. I use this opportunity to mention that I am very indepted to Jean-Pierre Kahane's "Fourier series and wavelets, Part I"; most of the historical material I use is borrowed from there.

2 The integrable Riemann Function

2.1 Upper bounds of the Hölder exponent

We start with an easy lemma which yields an upper bound for the Hölder exponent of functions having a dense set of singularities.

Lemma 1 *Let f be a function discontinuous on a dense set of points, $x_0 \in \mathbb{R}$ and let r_n be a sequence converging to x_0 such that, at each point r_n, f has a right and a left limit; denote by s_n the jump of f at r_n. Then*

$$h_f(x_0) \leq \liminf \frac{\log s_n}{\log |r_n - x_0|}.$$

Proof of Lemma 1: Let P be a polynomial; since

$$|(f(r_n^+) - P(r_n - x_0)) - (f(r_n^-) - P(r_n - x_0))| = s_n,$$

there exists r_n' arbitrarily close to r_n such that

$$|f(r_n') - P(r_n' - x_0)| \geq s_n/2$$

and $|r_n' - x_0| \geq \frac{1}{2}|r_n - x_0|$; we choose $h = |r_n' - x_0|$ and we deduce Lemma 1.

We will now apply this lemma to Riemann's function R. Since R has discontinuities at the rationals $p/2q$, we expect R to be irregular at points "well approximated" by these rationals, and to be smooth at points badly approximated. (Actually we used this property of bad approximation to prove regularity at rationals with an odd denominator). The Hölder regularity of F at a point x_0 will thus depend on properties of diophantine approximation of x_0.

Proposition 1 *Let x_0 be an irrational number; then*

$$h_R(x_0) \leq \frac{2}{\tau(x_0)}$$

(in particular, for any x, $0 \leq h(x) \leq 1$).

Proof of Proposition 1: First we consider the case where A is infinite. Let $n \in A$;

$$R(\frac{p_n}{q_n}^+) - R(\frac{p_n}{q_n}^-) = \frac{\pi^2}{8q_n^2}.$$

Since $|x_0 - \frac{p_n}{q_n}| = \frac{1}{q_n^{\tau_n}}$, Lemma 1 implies that

$$h(x_0) \leq \liminf \frac{2}{\tau_n} = \frac{2}{\tau(x_0)}.$$

Suppose now that q_n is odd for all $n \geq N$. In that case we consider

$$r_n = \frac{p_n + p_{n+1}}{q_n + q_{n+1}}.$$

Since $p_n q_{n+1} - q_n p_{n+1} = (-1)^{n+1}$, this fraction is under irreducible form; thus it has an even denominator. The jump of R at r_n is

$$\frac{\pi^2}{8(q_n + q_{n+1})^2} \geq \frac{1}{4q_{n+1}^2}. \tag{12}$$

On the other hand

$$|x_0 - r_n| = \frac{1}{q_n + q_{n+1}} |(p_n - x_0 q_n) + (p_{n+1} - x_0 q_{n+1})|;$$

but $|p_n - x_0 q_n| \leq 1/q_{n+1}$, so that

$$|x_0 - r_n| \leq \frac{1}{q_{n+1}} \left(\frac{1}{q_{n+1}} + \frac{1}{q_{n+2}}\right) \leq \frac{2}{q_{n+1}^2}. \tag{13}$$

Using Lemma 1, we obtain $h_R(x_0) \leq 1$, hence Proposition 1 in this case.

2.2 An estimate of the Hölder exponent

In this section, we show how the determination of the Hölder exponent at irrationals can be reduced to a problem of Diophantine approximation that we will solve in the following section. Since $h_R \leq 1$, we have to estimate $R(x_0 + h) - R(x_0)$. Suppose that $h > 0$ and let $N = [1/h]$;

$$\sum_{n=N+1}^{\infty} \frac{(nx)}{n^2} \leq \frac{C}{N} \leq Ch$$

so that

$$R(x_0 + h) - R(x_0) = \sum_{n=1}^{N} \frac{(n(x_0 + h)) - (nx_0)}{n^2} + O(h)$$

(and the term $O(h)$ can be neglected since the Hölder exponent of R is at most 1).

Let $E(x_0, h)$ be the set of rationals $r = p/q$ such that

$$\left.\begin{array}{l} q \quad \text{is even} \\ r \in [x_0, x_0 + h] \\ q \leq N \ \ (= [1/h]). \end{array}\right\} \tag{14}$$

Each function (nx) is linear on $[x_0, x_0 + h]$, with perhaps one jump (at most) of amplitude $\pi^2/8q^2$ if $r \in E(x_0, h)$. Thus

$$\begin{aligned} R(x_0 + h) - R(x_0) &= \sum_{n=1}^{N} \frac{nh}{n^2} - \frac{\pi^2}{8} \sum_{r \in E(x_0,h)} \frac{1}{q^2} + O(h) \\ &= -\frac{\pi^2}{8} \sum_{r \in E(x_0,h)} \frac{1}{q^2} + O(h \log h). \end{aligned} \tag{15}$$

The determination of the Hölder exponent of R at x_0 is thus reduced to the estimation of

$$\sum_{r \in E(x_0,h)} \frac{1}{q^2}. \tag{16}$$

We separate two contributions in this sum. The first one comes from the rationals that are convergents of x_0, and the second one from the other rationals.

Let us first compute the contribution of the convergents. Since the q_n grow at least geometrically, the order of magnitude of the sum is given by its first term, so that, if $h = |x_0 - \frac{p_n}{q_n}|$,

$$h^{2/\tau_n} = \frac{1}{q_n^2} \leq \sum_{r \in E(x_0,h)} \frac{1}{q^2} \leq \frac{C}{q_n^2} = Ch^{2/\tau_n} \tag{17}$$

(where the sum in the middle is restricted to convergents). And the estimate

$$\sum_{r\in E(x_0,h)} \frac{1}{q^2} \leq Ch^{2/\tau_n}$$

a fortiori holds if

$$|x_0 - \frac{p_n}{q_n}| \leq h < |x_0 - \frac{p_{n-1}}{q_{n-1}}|.$$

We denote by $E'(x_0, h)$ the subset of $E(x_0, h)$ composed of rationals that are not convergents. If p/q is not a convergent,

$$|\frac{p}{q} - x_0| \geq \frac{1}{2q^2}.$$

Furthermore, if $p/q \in E'(x_0, h)$, then $q \geq 1/\sqrt{2h}$. Thus the denominators of the rationals of $E'(x_0, h)$ satisfy

$$\frac{1}{\sqrt{h}} \leq q \leq \frac{1}{h}.$$

In order to estimate $\sum_{r\in E'(x_0,h)} \frac{1}{q^2}$, we take a large integer m, and we split the interval $[1/2, 1]$ of the exponents of h into m subintervals

$$I_k = [\gamma_k, \gamma_{k+1}) = [\frac{1}{2} + \frac{k}{2m}, \frac{1}{2} + \frac{k+1}{2m}).$$

The following proposition will be proved in the following section.

Proposition 2 *Denote by $E(m, k)$ the set of rationals $r \in E'(x_0, h)$ such that*

$$\frac{1}{h^{\gamma_k}} \leq q \leq \frac{1}{h^{\gamma_{k+1}}}. \tag{18}$$

The number $N(h, k)$ of elements of $E(m, k)$ is bounded by

$$\frac{1}{2 - \frac{1}{\gamma_{k+1}}} h^{1-2\gamma_{k+1}}.$$

Using Proposition 2, $\sum_{r\in E(m,k)} \frac{1}{q^2}$ is bounded by

$$\frac{2}{2 - \frac{1}{\gamma_{k+1}}} h^{2\gamma_k+1-2\gamma_{k+1}} \leq \frac{2}{2 - \frac{1}{\gamma_{k+1}}} h^{1-1/m}$$

and the following bound holds:

$$\sum_{r\in E(m,k)} \frac{1}{q^2} \leq C(m)h^{1-\frac{1}{m}}.$$

Using this bound, we deduce that

$$\sum_{r\in E'(x_0,h)} \frac{1}{q^2} \leq C(m)h^{1-\frac{1}{m}}$$

for all integers m, so that

$$\sum_{r\in E'(x_0,h)} \frac{1}{q^2} = O(h^{1-\epsilon}) \;\; \forall \epsilon > 0.$$

This estimate, together with (18), proves the first part of Theorem 1 (the proof is exactly the same if $h < 0$). In order to determine the number of rationals satisfying (18), we need to make an excursion into diophantine approximation.

2.3 Some diophantine approximation

The results we use in this section can all be found in in Serge Lang's book [9]. Let x_0 be an irrational number, and g an increasing function, larger than 1.

Definition 1 *The number x_0 is said to be of type less than g if for any B large enough, there exists a solution of the system*

$$|x_0 - \frac{p}{q}| < \frac{1}{q^2} \tag{19}$$

$$\frac{B}{g(B)} \leq q < B; \tag{20}$$

(p and q having no common factor).

Firstnote that the convergents $\frac{p_n}{q_n}$ satisfy (19). If $B = q_{n+1}$, then (20) holds if $\frac{B}{g(B)} \leq q_n$; hence if $g(q_{n+1}) \geq \frac{q_{n+1}}{q_n}$. But

$$|\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}| \leq 2|x_0 - \frac{p_n}{q_n}|,$$

which can also be written $\frac{1}{q_n q_{n+1}} \leq \frac{2}{q_n^{\tau_n}}$ or $q_{n+1} \geq \frac{1}{2} q_n^{\tau_n - 1}$; thus

$$g(q_{n+1}) \geq 2(q_{n+1})^{\frac{\tau_n - 2}{\tau_n - 1}}. \tag{21}$$

If we choose g increasing and satisfying (21), a fortiori,

$$g(B) \geq 2B^{\frac{\tau_n - 2}{\tau_n - 1}}$$

if $B \in (q_n, q_{n+1}]$. Thus, the following corollary holds.

Corollary 1 *Let $\tau' > \tau(x_0)$; the number x_0 is of type less than $t^{\frac{\tau' - 2}{\tau' - 1}}$.*

Let $\psi(t)$ be a positive decreasing function such that

$$\sum_{q=1}^{\infty} \psi(q) = +\infty.$$

Let

$$\theta(N) = \int_1^{\infty} \psi(t) dt,$$

and $\lambda(N)$ be the number of solutions of the inequalities

$$0 < x_0 - \frac{p}{q} < \frac{\psi(q)}{q} \quad 1 \leq q < N.$$

Finally, let $\omega(t) = t\psi(t)$. The following result holds (Theorem 8 of [9]).

Theorem 4 *Let x_0 be an irrational of type less than g. If ω satisfies the three following properties:*

- $g = o(\omega)$
- *ω is increasing and tends to $+\infty$*
- *$\sqrt{\omega(t) g(t)}/t$ is decreasing for t large enough.*

Then

$$\lambda(N) = \theta(N) + o(\theta(N)).$$

Recall that we want to estimate the number $N(h,k)$ of rationals $r = p/q$ satisfying

$$\left.\begin{array}{l} r \in [x_0, x_0 + h] \\ \\ \dfrac{1}{h^{\gamma_k}} \le q < \dfrac{1}{h^{\gamma_{k+1}}}. \end{array}\right\}$$

These two conditions imply that

$$0 < x_0 - \frac{p}{q} \le \left(\frac{1}{q}\right)^{\frac{1}{\gamma_{k+1}}} \tag{22}$$

so that $N(h,k)$ is bounded by

$$\lambda_{k+1}\left(\frac{1}{h^{\gamma_{k+1}}}\right) \tag{23}$$

where $\lambda_{k+1}(N)$ is the counting function associated with

$$\psi_{k+1}(q) = q\left(\frac{1}{q}\right)^{\frac{1}{\gamma_{k+1}}}.$$

First, when $\tau(x_0) > 2$, we will estimate $\lambda_{k+1}(N)$ by using a function g of the form $g(t) = t^\beta$.

Let us check the hypotheses of Theorem 4. First $\gamma_{k+1} \in (1/2, 1]$, so that ψ is positive decreasing and satisfies $\sum \psi(q) = +\infty$. Since $\omega(t) = t^{2-\frac{1}{\gamma_{k+1}}}$, the hypotheses of Theorem 4 will be satisfied if

$$\left.\begin{array}{l} \beta \ge 0 \\ \\ 2 - \frac{1}{\gamma_{k+1}} > \beta \\ \\ 2 - \frac{1}{\gamma_{k+1}} > 0 \\ \\ \dfrac{1}{2}\left(2 - \dfrac{1}{\gamma_{k+1}} + \beta\right) - 1 \le 0 \\ \\ \beta < \dfrac{\tau(x_0) - 2}{\tau(x_0) - 1}. \end{array}\right\} \tag{24}$$

Since $1/2 < \gamma_{k+1} \le 1$, we can choose any β satisfying

$$0 < \beta < \inf\left(\frac{\tau(x_0) - 2}{\tau(x_0) - 1}, 2 - \frac{1}{\gamma_{k+1}}\right)$$

(as long as $\tau(x_0) > 2$).

Thus $\lambda_{k+1}(N) \sim \frac{1}{2-\frac{1}{\gamma_{k+1}}} N^{2-\frac{1}{\gamma_{k+1}}}$, so that, using Theorem 4, $\lambda_{k+1}(\frac{1}{h^{\gamma_{k+1}}})$ is bounded by

$$\frac{2}{2 - \frac{1}{\gamma_{k+1}}}\left(\frac{1}{h^{\gamma_{k+1}}}\right)^{2-\frac{1}{\gamma_{k+1}}} = \frac{2}{2 - \frac{1}{\gamma_{k+1}}} h^{1-2\gamma_{k+1}}$$

hence Proposition 2 in this case.

If $\tau(x_0) = 2$, we take for g an increasing function satisfying

$$g(q_{n+1}) \ge \frac{q_{n+1}}{q_n};$$

g increases slower than any positive power of t, so that the hypotheses of Theorem 4 still hold, and Proposition 2 holds in this case.

2.4 The spectrum of singularities of R

Let H_τ be the set of all real numbers ρ such that

$$\exists C>0, \quad |\rho - \frac{p_n}{q_n}| \leq \frac{C}{q_n^\tau}$$

for an infinity of values of n such that q_n is even (and if there is only a finite number of values of n such that q_n is even, we decide that $\rho \in H_2$). Let us prove that the $\mathcal{H}^{2/\tau}$ Hausdorff measure of H_τ is positive.

First, if $\tau = 2$, then $H_\tau = \mathbb{R}$ and the result holds. If $\tau > 2$, one uses the following classical lemma.

Lemma 2 *If p and q have no common factor, and if*

$$|qx_0 - p| < \frac{1}{2q}$$

then p/q is a convergent of x_0.

Let F_τ be the set of all real numbers x_0 such that

$$\exists C>0, \ |x_0 - \frac{p_n}{q_n}| \leq \frac{C}{q_n^\tau}$$

for an infinity of values of n such that p_n and q_n are both odd.

The $\mathcal{H}^{2/\tau}$ Hausdorff measure of F_τ satisfies (see [4])

$$\mathcal{H}^{2/\tau}(F_\tau) > 0.$$

But, if $x_0 \in F_\tau$, $x_0/2 \in H_\tau$, $p_n/2q_n$ is an irreducible fraction and

$$|\frac{x_0}{2} - \frac{p_n}{2q_n}| \leq \frac{C}{2p_n^\tau}$$

and Lemma 2 implies that p_n/q_n is a convergent of $x_0/2$, thus $\mathcal{H}^{2/\tau}(H_\tau) > 0$.

Consider the set

$$H_\tau - \bigcup_{\tau'>\tau} H_{\tau'}.$$

The $\mathcal{H}^{2/\tau}$ measure of $\bigcup_{\tau'>\tau} H_{\tau'}$ vanishes; since H_τ has a $\mathcal{H}^{2/\tau}$ measure positive, $H_\tau - \bigcup_{\tau'>\tau} H_{\tau'}$ has dimension $2/\tau$.

If $x_0 \in H_\tau - \bigcup_{\tau'>\tau} H_{\tau'}$, since $x_0 \in F_\tau$, the first part of Theorem 1 implies that R is not smoother than $\frac{2}{\tau}$ at x_0 and since $x_0 \notin \bigcup_{\tau'>\tau} H_{\tau'}$, φ is $C^{\frac{2}{\tau}-\epsilon}(x_0)$ $\forall \epsilon > 0$; thus $h_R(x_0) = \frac{2}{\tau}$ and the dimension of $\{x_0 : \ h_R(x_0) = \frac{2}{\tau}\}$ is at least $2/\tau$.

Suppose that $h_R(x_0) = \frac{2}{\tau}$; then R is $C^{\frac{2}{\tau}-\epsilon}(x_0)$ $\forall \epsilon > 0$ and thus $x_0 \in H_{\tau'}$ $\forall \tau' < \tau$; thus

$$\{x_0 : \ h_R(x_0) = \frac{2}{\tau}\} \subset \bigcup_{\tau'>\tau} H_{\tau'}$$

and the dimension of $\{x_0 : \ h_R(x_0) = \frac{2}{\tau}\}$ is bounded by $2/\tau$ hence the second part of Theorem 1.

3 Jordan's function

The study of Jordan's function differs from the study of Riemann's integrable function only at rationals with an odd denominator, and we will detail that point.

Recall that

$$J(x) = \sum \frac{(nx)}{n^3}$$

is continuous except at rationals $p/2q$, where J has a right and a left limit, the jump of J at such a point being $7\zeta(3)/8q^3$. Let $r = p/(2q+1)$ be a rational with an odd denominator. In order to bound the regularity of J at r, we use Lemma 1; since J has a discontinuity at a distance $h = C/n$ of r, the jump being of C'/n^3, we obtain that the Hölder exponent of J at r is at most 3. In contrast with Riemann's function R, this upper bound will turn out to be the right exponent at these rationals. Inded, using the same notations as in Section 2.2,

$$J(\frac{p}{2q+1}+h) - J(\frac{p}{2q+1}) = \sum_{n=1}^{N-1} \frac{h}{n^2} + \sum_{n=N}^{\infty} \frac{(n(\frac{p}{2q+1}+h)) - (n\frac{p}{2q+1})}{n^3} \tag{25}$$

We split each term $(n(\frac{p}{2q+1}+h)) - (n\frac{p}{2q+1}))$ as a sum of a linear term nh and a certain number $\xi(n)$ of jumps; thus (25) is the sum of two terms, the first one being

$$\sum_{n=1}^{N-1} \frac{h}{n^2} + \sum_{n=N}^{\infty} \frac{h}{n^2} = \frac{h\pi^2}{6}$$

and the second one being

$$\sum_{n=N}^{\infty} \frac{\xi(n)}{n^3}. \tag{26}$$

Since (x) has discontinuities at $(2k+1)/2$ ($k \in \mathbf{Z}$), t is a point where $(n(\frac{p}{2q+1}+t))$ jumps if there exists k such that

$$(n(\frac{p}{2q+1}+t)) = \frac{2k+1}{2}$$

hence

$$t = \frac{q - np + \frac{1}{2} + k(2q+1)}{n(2q+1)}.$$

Suppose that $h > 0$. The numerator is always larger than $1/2$; thus if

$$n < \frac{1}{2(2q+1)h} \quad (= A)$$

the function $(n(\frac{p}{2q+1}+t))$ has no jump on $[0,h]$.

If $A \le n < 3A$, the only contributions to (26) come from the values of n satisfying

$$q - np \equiv 0 \mod.\ 2q+1. \tag{27}$$

There exists a unique solution of (27) between A and $A+2q$; the other solutions form an arithmetic sequence of reason $2q+1$. The contribution of this whole sequence to (26) is thus between

$$\sum_{m=0}^{\infty} \frac{1}{(A+m(2q+1))^3} \quad \text{and} \quad \sum_{m=0}^{\infty} \frac{1}{(A+2q+m(2q+1))^3},$$

and the value of these two sums is

$$\frac{1}{2(2q+1)A^2} + O(\frac{1}{A^3}).$$

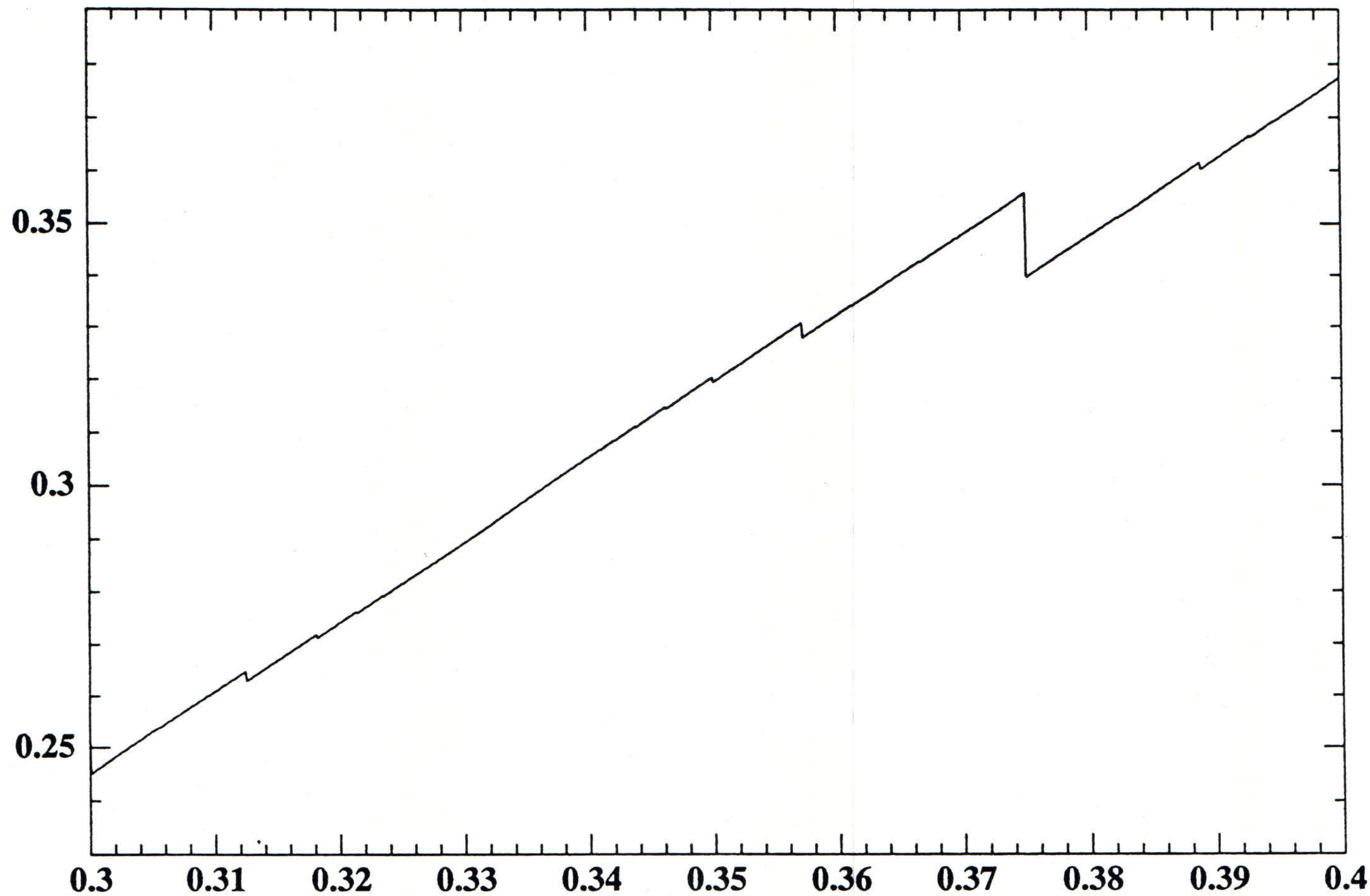

Fig.4: Jordan's function near $1/3$.

If $3a \leq n < 5A$, the values of n satisfying

$$q - np \equiv 1 \mod. \quad 2q+1. \tag{28}$$

also contribute to (26). As above, this contribution amounts to

$$\frac{1}{2(2q+1)(3A)^2} + O(\frac{1}{(3A)^3}).$$

The same argument works for all possible values of $q - np + 1/2$, and finally (26) is equal to

$$\sum_{l=0}^{\infty} \frac{1}{2(2q+1)(2l+1)^2A^2} + O(\frac{1}{(2l+1)^3A^3}) = \frac{\pi^2(2q+1)}{4}h^2 + O(h^3).$$

We just proved the following proposition.

Proposition 3 *The Hölder exponent of J at $\frac{p}{2q+1}$ is 3, and the following expansion holds:*

$$J(\frac{p}{2q+1} + h) = J(\frac{p}{2q+1}) + \frac{\pi^2}{6}h - \frac{\pi^2(2q+1)}{4}h^2 + O(h^3).$$

Suppose now that x_0 is an irrational number. Proposition 1 becomes

$$h(x_0) \leq \frac{3}{\tau(x_0)} \tag{29}$$

The Hölder exponents at irrationals are thus between 0 and 3/2.

Since

$$\sum_{N+1}^{\infty} \frac{(nx)}{n^3} \leq \frac{C}{N^2} \leq Ch^2,$$

$$J(x_0+h) - J(x_0) = \sum_{n=1}^{N} \frac{(n(x_0+h)) - (nx_0)}{n^3} + O(h^2)$$

and (15) becomes

$$\begin{aligned} J(x_0+h) - J(x_0) &= \sum_{n=1}^{N} \frac{nh}{n^3} - \frac{7\zeta(3)}{8} \sum_{r\in E(x_0,h)} \frac{1}{q^3} + O(h^2) \\ &= \frac{\pi^2}{6}h - \frac{7\zeta(3)}{8} \sum_{r\in E(x_0,h)} \frac{1}{q^3} + O(h^2) \end{aligned}$$

The contributions of the convergents to

$$\sum_{r\in E(x_0,h)} \frac{1}{q^3} \tag{30}$$

is bounded by Ch^{3/τ_n}, and, using the same method as in Section 2.2, the contribution to (30) of the rationals satisfying (22) is bounded by

$$Ch^{3\gamma_k}\left(\frac{1}{h^{\gamma_{k+1}}}\right)^{2-\frac{1}{\gamma_{k+1}}} \leq Ch^{\gamma_k+1-\frac{1}{m}};$$

since $\gamma_k \geq 1/2$,

$$\sum_{r\in E(m,h)} \frac{1}{q^3} \leq Ch^{\frac{3}{2}-\frac{1}{m}}.$$

We deduce Theorem 2 as in the case of Riemann's function.

Note that the derivative of $J(x) - \frac{\pi^2}{6}x$ vanishes at the points where it exists, so that it is an example of increasing singular function. However, the term in h^2 in the Taylor expansion of J at rationals with an odd denominator does not vanish, so the the spectrum of the singular measure μ_J differs slightly form the spectrum of J: they are the same except that the value $d(3) = 0$ in the spectrum of J is replaced for μ_J by $d(2) = 0$.

4 From Rajchman function to renormalized measures

We now consider Rajchman function R_a, its derivative, the measure $\zeta(3) \sum_{p\wedge q=1} \frac{1}{q^3}\delta_{p/q}$, and more generally the distributions

$$\mu_\beta = \sum_{p\wedge q=1} \frac{1}{q^\beta}\delta_{p/q}.$$

The study of the pointwise regularity of R_a is very similar to the study of Jordan's function. The exception is that, here, we do not have to make a specific study at rationals. The reader will easily check that Theorem 2 must be reformulated as follows.

If x_0 is an irrational number, let

$$\tau''(x_0) = \limsup_{n\to\infty} \tau_n;$$

the Hölder exponent of Rajchman's function R_a at x_0 is

$$h_{R_a}(x_0) = \frac{3}{\tau''(x_0)}.$$

More generally, if $\beta > 2$, the same analysis as above yields an exponent

$$h_{\mu_\beta}(x_0) = \frac{\beta}{\tau''(x_0)}. \tag{31}$$

Using Lemma 1, (31) is also an upper bound for the Hölder exponent of the primitive f_β of μ_β, so that the Hölder exponents of f_β and μ_β coincide everywhere.

The spectrum of singularities of μ_β is calculated using the following remark: Let E_τ be the set of all real numbers x_0 such that

$$\exists C > 0, \quad |x_0 - \frac{p_n}{q_n}| \le \frac{C}{q_n^\tau}$$

for an infinity of values of n . The $\mathcal{H}^{2/\tau}$ Hausdorff measure of H_τ satisfies

$$\mathcal{H}^{2/\tau}(H_\tau) > 0$$

(see [1] or [4]). The spectrum of singularities of μ_β is thus obtained exactly as in the case of Riemann's function R:

$$\begin{aligned} d_{\mu_\beta}(\alpha) \quad &= \frac{2\alpha}{\beta} \quad \text{for} \quad \alpha \in [0, \beta/2] \\ &= -\infty \quad \text{elsewhere} \end{aligned}$$

We now consider the renormalized measures

$$\sum_{p \wedge q = 1} \frac{1}{q^\beta} \delta_{p/q} \quad \text{when} \quad 1 < \beta \le 2.$$

Recall that this must be understood as the distribution

$$\lim_{N\to\infty} \sum_{q=1}^{N} \left(\frac{1}{q^\beta} \langle \sum_p \delta_{p/q} | \psi \rangle - \frac{1}{q^{\beta-1}} \int \psi \right) \quad (= \lim \langle S_N | \psi \rangle) \tag{32}$$

(if ψ is C^∞ and 1-periodic). In order to be able to define an Hölder exponent of this distribution, we must check if we can take for ψ the characteristic function of an interval. If $\psi = 1_{[a,b]}$,

$$\frac{1}{q^\beta} \langle \sum_q \delta_{p/q} | \psi \rangle - \frac{1}{q^{\beta-1}} \int \psi = \frac{|b-a|q + r}{q^\beta} - \frac{|b-a|}{q^{\beta-1}} = \frac{r}{q^\beta} \quad \text{with} \quad r \in \{-1, 0, 1\} \tag{33}$$

and the limit in (32) exists. We can now try to determine the order of magnitude of (32) when $\psi = 1_{[x_0, x_0+h]}$.

Thus, we denote by $A(x_0, h)$ the limit of (32) when $\psi = 1_{[x_0, x_0+h]}$, and we define the Hölder exponent of the measure $\sum_{p \wedge q = 1} \frac{1}{q^\beta} \delta_{p/q}$ by

$$h_\beta(x_0) = \liminf_{|h|\to 0} \frac{\log |A(x_0, h)|}{\log |h|}. \tag{34}$$

Of course, this is the same a computing the Hölder exponent of the function

$$A(x) = \lim_{N\to\infty} \langle S_N | 1_{[0,x]} \rangle$$

which is the "renormalized primitive" of μ_β. Furthermore, it coincides with the usual definition of the Hölder exponent of a measure when $\beta > 2$. Let us first show that A is continuous at irrationals, and estimate its Hölder exponent.

As usual, $\frac{p_n}{q_n}$ denotes the convergents of x_0, and we consider an increment

$$h = |x_0 - \frac{p_n}{q_n}| = \frac{1}{q_n^{\tau_n}}.$$

Note first that from (33) we deduce that

$$\sum_{q \geq 1/h} \left| \frac{1}{q^\beta} \langle \sum_p \delta_{p/q} | 1_{[x_0, x_0+h]} \rangle - \frac{h}{q^{\beta-1}} \right| \leq \sum_{q \geq 1/h} \frac{1}{q^\beta} \leq \frac{h^{\beta-1}}{\beta - 1} + O(1). \tag{35}$$

Of course

$$\sum_{q<1/h} \frac{h}{q^{\beta-1}} = \frac{h}{2-\beta} \left((\frac{1}{h})^{2-\beta} + O(1) \right) = \frac{h^{\beta-1}}{2-\beta} + O(1) \tag{36}$$

We still have to estimate

$$\sum_{q<1/h} \frac{1}{q^\beta} \langle \sum_p \delta_{p/q} | 1_{[x_0, x_0+h]} \rangle.$$

- If $q < q_n$, because of the best approximation properties of convergents, no Dirac mass $\delta_{p/q}$ is supported in $[x_0, x_0 + h]$.
- The contribution of $q = q_n$ is
$$\frac{1}{q_n^\beta} = h^{\beta/\tau_n} \tag{37}$$
- If $q > q_n$, we have
$$\frac{1}{h^{1/2}} < q < \frac{1}{h}.$$
As usual, we split the intervals $[1/2, 1]$ of the exponents of $1/h$ into arbitrarily small subintervals $[\gamma_k, \gamma_{k+1}]$ and we apply Proposition 2 to each of these subintervals. We obtain a contribution bounded by
$$C_k h^{1-2\gamma_{k+1}} h^{\beta\gamma_{k+1}} \leq C_k h^{\beta-1}.$$

We see that the contribution of the convergents dominates when

$$\frac{\beta}{\tau''(x_0)} < \beta - 1;$$

in this case, the Hölder exponent is β/τ.(The estimation for values of h different from $|x_0 - \frac{p_n}{q_n}| = \frac{1}{q_n^{\tau_n}}$ is straightforward). When $\frac{\beta}{\tau''(x_0)} \geq \beta - 1$, we can only say that the Hölder exponent is larger than $\beta - 1$ (and smaller than $\beta/\tau''(x_0)$); hence Theorem 3. Note that the method we use cannot yield the spectrum between $\beta - 1$ and $\beta/2$.

5 Lévy's function

Paul Lévy introduced

$$\mathcal{L}(x) = \sum_{n=0}^{\infty} \frac{(2^n x)}{2^n}$$

as an illustration of the type of discontinuities that a stochastic process with independant increments can have. $\mathcal{L}$ is clearly continuous at non-dyadic points, and discontinuous with right and left limits at dyadic points; at $\lambda = \frac{2k+1}{2^{n+1}}$, the jump of $\mathcal{L}$ is 2^{-n}.

The regularity of $\mathcal{L}$ at a non-dyadic point x_0 will clearly depend on the quality of approximation of x_0 by dyadics. Let us introduce the notation

$$\Delta_n(x) = dist(x, 2^{-n}\mathbf{Z}).$$

Proposition 4 *The Hölder exponent of $\mathcal{L}$ at x_0 is*

$$h_{\mathcal{L}}(x_0) = \liminf \frac{n}{\log_2 \Delta_n(x)} \tag{38}$$

and the spectrum of singularities of $\mathcal{L}$ is

$$d_{\mathcal{L}}(\alpha) = \alpha \quad \textit{for} \quad \alpha \in [0, 1]. \tag{39}$$

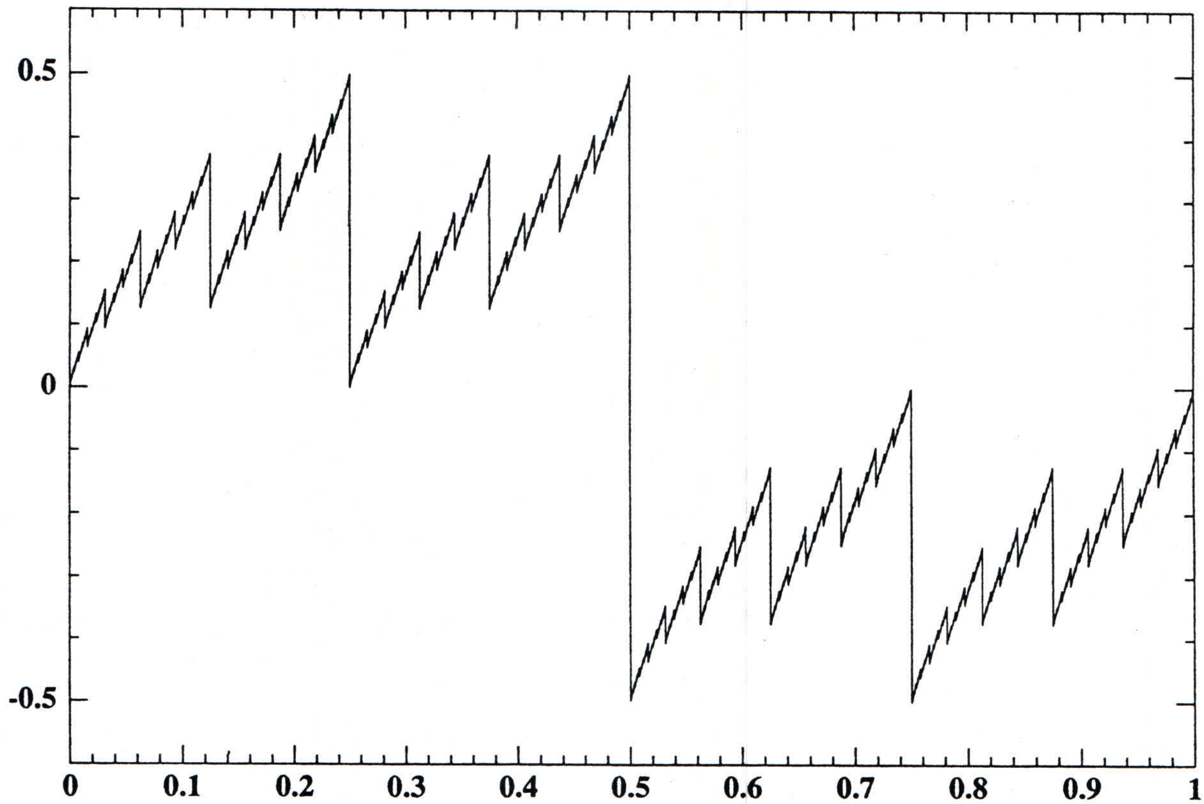

Fig.5: Lévy's function.

Proof of Proposition 4: Let x_0 be given; since $\mathcal{L}$ has, at a distance of $h = \Delta_n(x)$ from x_0, a jump of size at least 2^{-n}, Lemma 1 implies that

$$h_{\mathcal{L}}(x_0) \leq \liminf \frac{n}{\log_2 \Delta_n(x)};$$

in particular $h_{\mathcal{L}}(x_0) \leq 1 \ \forall x_0$.

Note that for $h = \Delta_n(x)$,

$$|\mathcal{L}(x_0 + h) - \mathcal{L}(x_0)| \leq 2^{-n} + O(h) \tag{40}$$

(if n is the first index such that $h = \Delta_n(x)$) and (40) is a fortiori satisfied if h lies between two values Δ_n.

We still have to calculate the dimension of the set of points where $\mathcal{L}$ has a given Hölder exponent.

If $\alpha \geq 1$, let

$$E_\alpha = \limsup_{n\to\infty} \bigcup_k [k2^{-n} - 2^{-n\alpha}, k2^{-n} + 2^{-n\alpha}].$$

Clearly, $dim\ E_\alpha \leq 1/\alpha$; the converse inequality is almost as easy: We pick a very lacunary sequence n_m ($n_m = 2^{n_{m-1}}$ for instance), and we construct a probability measure μ supported by

$$F_\alpha = \bigcap_m \left(\bigcup_k [k2^{-n_m} - 2^{-n_m\alpha}, k2^{-n_m} + 2^{-n_m\alpha}] \right).$$

If $m = 0$, we put on each interval $[k2^{-n_0} - 2^{-n_0\alpha}, k2^{-n_0} - 2^{-n_0\alpha}]$ the same mass 2^{-n_0}. Each of these intervals countains $A(k, n_0) = 2.2^{-n_0\alpha}(2^{n_1} + O(1))$ intervals $[l2^{-n_1} - 2^{-n_1\alpha}, l2^{-n_1} - 2^{-n_1\alpha}]$; on each of these intervals, we put the measure $2^{-n_0}/A(k, n_0)$. We iterate this construction, and thus obtain at the limit a probability measure μ supported by F_α. One easily checks that , $\forall x \in F_\alpha$,

$$\mu([x - h, x + h]) \leq ch^{1/\alpha}.$$

We use Proposition 4.9 of [1] which implies that

$$\mathcal{H}_{1/\alpha}(F_\alpha) > 0 \tag{41}$$

Since $F_\alpha \subset E_\alpha$,

$$dim(E_\alpha) = 1/\alpha. \tag{42}$$

Using (38), $h_\mathcal{L}(x_0) = \beta$ if and only if

$$x_0 \in \bigcap_{\gamma>\beta} E_{1/\gamma} - \bigcup_{\gamma<\beta} E_{1/\gamma}$$

From (41) and (42) we deduce that the dimension of the set of points where $h_\mathcal{L}(x_0) = \beta$ is β; hence Proposition 4.

6 Concluding remarks: direct methods vs. wavelets

Lévy's function can be seen as a modification of the Weierstrass function

$$\sum 2^{-n} \sin 2^n x$$

where the sine function is replaced by (x). Let us compare the determination of the pointwise regularity of these two functions. As regards Lévy's function Lemma 1 immediately yields the right upper bound for the Hölder exponent. As regards Weierstrass function, the oscillations of the sine functions make it difficult to obtain upper bounds for the Hölder exponent. Actually, Weierstrass, using only "by hands" methods didn't get optimal results; Hardy in 1916, had the idea of estimating a convolution product of the analyzed function with a well localized function having one vanishing moment (the derivative of the Poisson kernel). This idea, which announces wavelet methods, yields optimal results (see [3] and [4]). Up to now, wavelet methods were used in order to study multifractal functions (see [4], [5], [6] and references therein). However wavelet methods cannot be applied to the functions we study in this paper, since these functions have a dense set of discontinuities, so that no wavelet criterion could possibly give their pointwise regularity (all these criteria require a minimal uniform Hölder regularity).

Roughly speaking, if f is a series of piecewise linear functions, direct methods for estimating the modulus of continuity usually yield optimal results; and if f has a minimal regularity, one should rather use wavelet methods.

Of course particularly simple cases are the functions which belong to both of these categories. An example is the Takagi function

$$\mathcal{T}(x) = \sum_{n=0}^{\infty} \frac{|(2^n x)|}{2^n}. \tag{43}$$

(Note that $|(x)|$ is the "hat function", which is the first function of the Schauder basis). This function was introduced by Takagi in [15] as a particularly simple example of continuous nowhere differentiable function, and it was rediscoverd by de Rham later ([13]). In order to study this (monofractal) function, we can either compute directly increments of the function, or remark that (43) yields immediately the expansion of $\mathcal{T}(x)$ in Schauder basis, and use a wavelet criterium. Both methods give $h_{\mathcal{T}}(x) = 1$ everywhere.

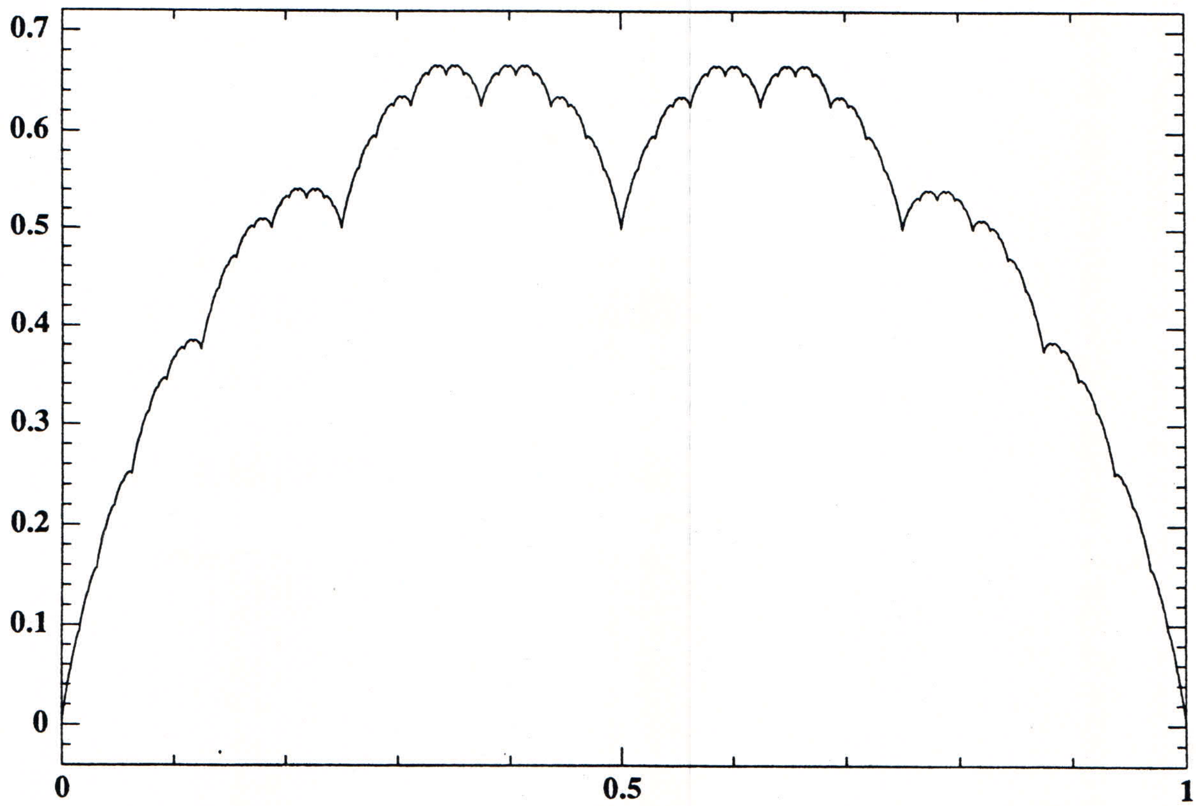

Fig.6: Takagi's function.

More interesting is the case of functions that belong to none of the categories we mentioned. Consider for instance

$$\mathcal{F}(x) = \sum_{n=1}^{\infty} \frac{\phi(2^n x)}{2^n} \tag{44}$$

where ϕ is one-periodic, discontinuous, piecewise smooth, but not piecewise linear; and suppose for instance that it is discontinuous at 1/2. None of the methods we considered applies. The situation is not desperate however: We can write ϕ as a sum of $C.(x)$ and a Lipschitz function. The Hölder exponent is thus the same as for Lévy's function: the only problem might be at the points where both functions have the same exponent; but, in this case it is equal to 1, so that the exponent of F is larger than 1; and it is actually equal to 1 because of Lemma 1. We leave to the reader the amusing cases where the discontinuity of ϕ is not at $1/2$.

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