A DIMENSION RESULT ARISING FROM THE $L^q$-SPECTRUM OF A MEASURE

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Abstract. We give a rigorous proof of the following heuristic result: Let $\mu$ be a Borel probability measure and let $\tau(q)$ be the $L^q$-spectrum of $\mu$. If $\tau(q)$ is differentiable at $q = 1$, then the Hausdorff dimension and the entropy dimension of $\mu$ equal $\tau'(1)$. Our result improves significantly some recent results of a similar nature; it is also of particular interest for computing the Hausdorff and entropy dimensions of the class of self-similar measures defined by maps which do not satisfy the open set condition.

1. Introduction

Let $\mu$ be a Borel probability measure on $\mathbb{R}^d$ with bounded support and let $\text{supp}(\mu)$ denote the support of $\mu$. For a finite Borel partition $\mathcal{P}$ of $\text{supp}(\mu)$, we let $|\mathcal{P}|$ be the maximum of the diameters of elements of $\mathcal{P}$. Define

$$h(\mu, \mathcal{P}) = - \sum_{A \in \mathcal{P}} \mu(A) \ln \mu(A).$$

For $\delta > 0$, let

$$h(\mu, \delta) = \inf \{ h(\mu, \mathcal{P}) : \mathcal{P} \text{ is a finite Borel partition of } \text{supp}(\mu), |\mathcal{P}| \leq \delta \}.$$ 

The entropy dimension (or Rényi dimension [Re]) of $\mu$ is defined as

$$\dim_e(\mu) = \lim_{\delta \to 0^+} \frac{h(\mu, \delta)}{-\ln \delta}.$$ 

Also, we let $\dim_H(E)$ denote the Hausdorff dimension of a set $E$ and define the Hausdorff dimension of $\mu$ as

$$\dim_H(\mu) = \inf \{ \dim_H(E) : \mu(\mathbb{R}^d \setminus E) = 0 \}.$$ 

Young [Y] proved that if

$$\lim_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \alpha \quad \text{for } \mu \text{ a.e. } x \in \text{supp}(\mu),$$

then

$$\dim_H(\mu) = \dim_e(\mu) = \alpha.$$
An important sufficient condition for (1.1) to hold is when $\mu$ is a self-similar measure defined by
$$
\mu = \sum_{i=1}^{m} p_i \mu \circ S_i^{-1},
$$
where $\{S_i\}_{i=1}^{m}$ is a family of contractive similitudes satisfying the open set condition ([Hut], [F]), and the $p_i$’s are the probability weights satisfying $p_i > 0$ and $\sum_{i=1}^{m} p_i = 1$. In this case (1.1) holds for
$$
\alpha = \sum_{i=1}^{m} p_i \ln p_i / \sum_{i=1}^{m} p_i \ln \rho_i,
$$
where $\rho_i$ is the contraction ratio of $S_i$. If we let
$$
G = \left\{ x \in \text{supp}(\mu) : \lim_{\delta \to 0^+} \frac{\ln \mu(B_{\delta}(x))}{\ln \delta} = \alpha \right\},
$$
then $\dim_H(G) = \alpha$ also. This theorem was proved by Geronimo and Hardin [GH] for $\{S_i\}_{i=1}^{m}$ satisfying the strong open set condition (and also implicitly by Cawley and Mauldin [CM]). It was also proved by Strichartz [S] by using the law of iterated algorithm for $\{S_i\}_{i=1}^{m}$ satisfying the open set condition.

Another sufficient condition to obtain (1.1) comes from the $L^q$-spectrum. For $\delta > 0$ and $q \in \mathbb{R}$, the $L^q$-(moment) spectrum of $\mu$ is defined as
$$
\tau(q) = \lim_{\delta \to 0^+} \ln \sup_{i} \frac{\mu(B_\delta(x_i))}{\ln \delta},
$$
where $\{B_\delta(x_i)\}_i$ is a family of disjoint closed $\delta$-balls with center $x_i \in \text{supp}(\mu)$ and the supremum is taken over all such families. The function $\tau(q)$ is an important function in multifractal theory; under suitable conditions, its Legendre transform equals the dimension spectrum of the measure $\mu$ ([H], [F]). Moreover, it is suggested in the physics literature that $\tau'(1)$ is equal to the entropy dimension of the measure ([HP], [H], [F]). Falconer [F] gives a heuristic argument for such equality. The purpose of this note is to give a rigorous proof of such a folklore theorem. Specifically, we prove

**Theorem 1.1.** Let $\mu$ be a Borel probability measure on $\mathbb{R}^d$ with bounded support. Then

(a) for $\mu$ a.e. $x \in \text{supp}(\mu)$, we have
$$
\tau'_+(1) \leq \lim_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \lim_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \tau'_-(1).
$$

(b) If $\tau(q)$ is differentiable at $q = 1$, then
$$
\lim_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \tau'(1)
$$
for $\mu$ a.e. $x \in \text{supp}(\mu)$.

Consequently, $\mu$ is concentrated on $G = \left\{ x \in \text{supp}(\mu) : \lim_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \tau'(1) \right\}$, and
$$
\dim_H(G) = \dim_H(\mu) = \dim_e(\mu) = \tau'(1).
$$
We will prove Theorem 1.1 in Section 3. The main idea is to show that the set of points \( x \in \text{supp}(\mu) \) such that
\[
\lim_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} < \tau'_+(1) \quad \text{or} \quad \tau'_-(1) < \lim_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta}
\]
has \( \mu \) measure zero. The proof of this relies on estimations of some counting functions (Lemma 2.2) together with a standard covering lemma. For the special case of self-similar measures defined by contractive similitudes satisfying the open set condition, \( \tau(q) \) is given by
\[
\sum_{i=1}^m p_i^q \alpha_i^{-\tau(q)} = 1.
\]
Moreover, \( \tau(q) \) is differentiable and \( \tau'(1) = \alpha \), where \( \alpha \) is given by (1.3) (see [CM]). Such results have also been proved for some extensions of the self-similar measures [AP], [R] (with the open set condition), and for equilibrium measures of Hölder continuous conformal expanding maps [PW]. The equality of \( \dim_H(\mu) \) and \( \tau'(1) \), under the assumption that \( \tau(q) \) is differentiable at \( q = 1 \), was recently studied by Fan for a certain class of infinite product measures [Fa]. An additional example is the infinitely convolved Bernoulli measure associated with the golden ratio. This is a good illustration and the main motivation for our result because the open set condition fails. This will be discussed in Section 4.

2. Preliminaries

Let \( \tau : \mathbb{R} \to [-\infty, \infty) \) be a concave function. We define the effective domain of \( \tau \) as
\[
\text{Dom } \tau = \{ x : -\infty < \tau(x) < \infty \}.
\]
The concave conjugate (or the Legendre transform) of \( \tau \) is the function \( \tau^* : \mathbb{R} \to [-\infty, \infty) \) defined by
\[
\tau^*(\alpha) = \inf \{ \alpha x - \tau(x) : x \in \mathbb{R} \}.
\]

For \( x \in \text{Dom } \tau \), we let \( \partial \tau(x) \subseteq \mathbb{R} \) be the subdifferential of \( \tau \) at \( x \), i.e.,
\[
\partial \tau(x) = \{ \alpha : \tau(y) \leq \tau(x) + \alpha(y-x) \text{ for all } y \in \mathbb{R} \}.
\]
Then \( \tau^*(\alpha) + \tau(x) = \alpha x \) for \( \alpha \in \partial \tau(x) \) [Ro]. If \( \tau(x) \) is differentiable at \( x \), then \( \partial \tau(x) \) is the singleton \( \tau'(x) \). Otherwise, \( \partial \tau(x) \) is a closed interval. We will denote the special subdifferentials \( \partial \tau(0) \) and \( \partial \tau(1) \) respectively by \( [\alpha_0^-, \alpha_0^+] \) and \( [\alpha_1^-, \alpha_1^+] \).

It is known (e.g. [LN1, Proposition 2.3]) that \( \text{Dom } \tau^* \) is an interval and \( (\text{Dom } \tau^*)^o = (\alpha_{\min}, \alpha_{\max}) \), where
\[
\alpha_{\min} := \inf \{ \alpha : \alpha \in \partial \tau(x), x \in \text{Dom } \tau \},
\]
\[
\alpha_{\max} := \sup \{ \alpha : \alpha \in \partial \tau(x), x \in \text{Dom } \tau \}.
\]

For the rest of this note, we assume that \( \tau(q) \) is the \( L^q \)-spectrum of a Borel probability measure \( \mu \) defined by (1.4). It is known that \( \tau(q) \) is increasing, concave and \( \tau(1) = 0 \) (see Figure 1). Moreover, it is proved in [LN1] that
\[
\alpha_{\min} = \lim_{\delta \to 0^+} \frac{\ln(\sup_x \mu(B_\delta(x)))}{\ln \delta} \quad \text{and} \quad \alpha_{\max} = \lim_{\delta \to 0^+} \frac{\ln(\inf_x \mu(B_\delta(x)))}{\ln \delta},
\]
Theorem 3.1. Assume that \( \alpha \in \partial \tau(q) \). The following proposition will be used in the proof of Lemma 3.1.

**Proposition 2.1.** Assume that \( \alpha_{\min} < \alpha < \alpha_{\max} \). Then \( \alpha_{\min} > \tau^*(\alpha_{\min}) \).

**Proof.** Let \( \alpha_{\min} < \tilde{\alpha} < \alpha^+ \) and \( q \in \partial \tau^*(\tilde{\alpha}) \) (i.e., \( \tilde{\alpha} \in \partial \tau(q) \)). Consider the line with slope \( \tilde{\alpha} \) passing through the point \((q, \tau(q))\). This line intersects the vertical line \( q = 1 \) at \((1, \tau(q) - (q - 1)\tilde{\alpha})\). By using the identity \( \tau(q) + \tau^*(\tilde{\alpha}) = q\tilde{\alpha} \) together with the facts that \( \tau \) is concave with \( \tau(1) = 0 \) and \( \tilde{\alpha} < \alpha_1^- \), we have

\[
\tilde{\alpha} - \tau^*(\tilde{\alpha}) = \tau(q) - (q - 1)\tilde{\alpha} > 0.
\]

The same argument shows that \( \alpha - \tau^*(\alpha) \) is an increasing function of \( q \) and hence

\[
\alpha - \tau^*(\alpha) \geq \tilde{\alpha} - \tau^*(\tilde{\alpha}) \quad \text{for all} \quad \alpha \leq \tilde{\alpha}.
\]

The result follows by letting \( q \to \infty \).

Let \( \mathcal{B}_\delta \) denote a disjoint family of closed balls of radii \( \delta \) centered at points in \( \supp(\mu) \). For \( \alpha \in (\text{Dom } \tau^*)^\circ \), we define the counting functions

\[
N_\delta(\alpha) = \sup_{\mathcal{B}_\delta} \# \{ B : B \in \mathcal{B}_\delta, \mu(B) \geq \delta^\alpha \},
\]

\[
\widetilde{N}_\delta(\alpha) = \sup_{\mathcal{B}_\delta} \# \{ B : B \in \mathcal{B}_\delta, \mu(B) < \delta^\alpha \}.
\]

The following lemma is proved in [LN1, Lemma 4.2].

**Lemma 2.2.** Let \( \alpha_{\min} < \alpha < \alpha_0^+ \), \( q \in \partial \tau^*(\alpha) \) and \( \xi > 0 \). Then for any \( \epsilon > 0 \), there exists \( \delta_\epsilon > 0 \) such that for all \( 0 < \delta < \delta_\epsilon \),

\[
N_\delta(\alpha \pm \epsilon) \leq \delta^{-\tau^*(\alpha) - (\xi \pm q)\epsilon}.
\]

For \( \alpha_0^+ \leq \alpha < \alpha_{\max} \), the above holds with \( \widetilde{N}_\delta \) replacing \( N_\delta \).

Lemma 2.2 and the counting functions play a key role in the proof of the main theorem.
3. Proof of the main theorem

We need two lemmas.

**Lemma 3.1.** Let $\mu$ be a Borel probability measure on $\mathbb{R}^d$ with bounded support. Then

$$\mu\left\{ x \in \text{supp}(\mu) : \alpha_{\text{min}} \leq \lim_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} < \alpha_1^- \right\} = 0.$$

**Proof.** Part 1. We claim that $\mu\left\{ x \in \text{supp}(\mu) : \alpha_{\text{min}} < \lim_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} < \alpha_1^- \right\} = 0$. Let $\alpha_{\text{min}} < \alpha < \alpha_1^-$ and $q \in \partial \tau^+(\alpha)$. Then $q > 1$. Since $\tau$ is increasing, concave, $\alpha < \alpha_1^-$, and since $\tau(1) = 0$, we have $(\tau(q) - \tau(1))/(q - 1) \geq \alpha > 0$. We choose $\epsilon > 0$ small enough so that

$$\sigma := (\tau(q) - (q - 1)\alpha)/2 \leq \tau(q) - (q - 1)\alpha - (2 + q)\epsilon.$$  
(This implies that $\alpha + \epsilon < \alpha_1^-$.)

Define

$$L_\epsilon(\alpha) = \left\{ x \in \text{supp}(\mu) : \alpha - \frac{\epsilon}{3} \leq \lim_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \alpha + \frac{\epsilon}{3} \right\}.$$

We will show that $\mu(L_\epsilon(\alpha)) = 0$. Putting $\xi = 1$ in Lemma 2.2, then there exists $\delta_x > 0$ such that for all $0 < \delta \leq \delta_x$,

$$N_{\delta}(\alpha + \epsilon) \leq \delta^{-\gamma(\alpha)-(1+q)\epsilon}.$$

Fix $m \in \mathbb{N}$ satisfying

$$2^{-m} < \delta_x \quad \text{and} \quad m \geq 3\alpha/\epsilon + 2.$$

For each $x \in L_\epsilon(\alpha)$, we let $n_x$ be the smallest integer satisfying the following conditions:

(i) $n_x \geq m$;
(ii) $\mu(B_\delta(x)) < \delta^{\alpha - \epsilon}$ for all $0 < \delta \leq 2^{-(n_x - 2)}$;
(iii) there exists $\delta_x > 0$ such that $2^{-(n_x + 1)} < \delta_x \leq 2^{-n_x}$ and $\mu(B_{\delta_x}(x)) > \delta_x^{\alpha + 2\epsilon/3}$.

Note that $n_x$ is uniquely determined by $x$. Partition $L_\epsilon(\alpha)$ into a countable disjoint union of subsets $L_n^\epsilon(\alpha)$ where $L_n^\epsilon(\alpha) = \{ x \in L_\epsilon(\alpha) : n_x = n \}$. Then

$$L_\epsilon(\alpha) = \bigcup_{n=m}^{\infty} L_n^\epsilon(\alpha).$$

Clearly for each $n \geq m$,

$$L_n^\epsilon(\alpha) \subseteq \bigcup_{x \in L_n^\epsilon(\alpha)} B_{2^{-n}}(x).$$

By a standard covering lemma (see [F, Lemma 4.8]), there exists a finite sequence \{x_i\}_{i=1}^\ell in $L_n^\epsilon(\alpha)$ such that \{B_{2^{-n}}(x_i)\}_{i=1}^\ell is a disjoint family and

$$L_n^\epsilon(\alpha) \subseteq \bigcup_{i=1}^\ell B_{2^{-(n+\epsilon)}(x_i)}.$$  

For $1 \leq i \leq \ell$, condition (iii) and (3.3) imply that

$$\mu(B_{2^{-(n+\epsilon)}}(x_i)) > 2^{-2(n+1)(\alpha+2\epsilon/3)} \geq 2^{-n(n+\epsilon)}.$$
Hence by (3.2),
\[ \ell \leq 2^{-n(-\tau^*(\alpha)-(1+q)\epsilon)}. \]
Combining condition (ii), (3.5), (3.6) and (3.1), we have
\[ \mu(L_\epsilon^n(\alpha)) \leq \sum_{i=1}^{\ell} \mu(B_{2^{-(n-2)}(x_i)}) \leq 2^{-(n-2)(\alpha-\epsilon)} \cdot 2^{-n(-\tau^*(\alpha)-(1+q)\epsilon)} \]
\[ \leq C \cdot 2^{-n(q-\alpha-1)-2(1+q)\epsilon} \leq C \cdot 2^{-n\sigma}. \]
\((C \text{ is a constant independent of } n.)\) Using this and (3.4), we have
\[ \mu(L_\epsilon(\alpha)) \leq \sum_{n=m}^{\infty} \mu(L_\epsilon^n(\alpha)) \leq C \sum_{n=m}^{\infty} 2^{-n\sigma} = C \frac{2^{-\sigma m}}{1-2^{-\sigma}}. \]
Letting \(m \to \infty\), we get \(\mu(L_\epsilon(\alpha)) = 0\). The claim follows easily by taking a countable cover for \((\alpha_{\min}, \alpha^-)\) by sets of the form \(L_\epsilon(\alpha)\).

Part 2. We will show that if \(\alpha_{\min} < \alpha^-\), then
\[ \mu\left\{ x \in \text{supp}(\mu) : \lim_{\delta \to 0^+} \frac{\ln \mu(B_{\delta}(x))}{\ln \delta} = \alpha_{\min} \right\} = 0. \]
By Proposition 2.1, we may choose \(\epsilon > 0\) sufficiently small and \(\alpha \in \text{Dom } \tau^*\) sufficiently close to \(\alpha_{\min}\) such that
\[ 0 < \sigma := (\alpha_{\min} - \tau^*(\alpha))/2 \leq \alpha_{\min} - \tau^*(\alpha) - (2+q)\epsilon, \]
where \(q \in \partial \tau^*(\alpha)\). By Lemma 2.2, there exists \(\delta_{\epsilon} > 0\) such that for all \(0 < \delta \leq \delta_{\epsilon}\),
\[ N_{\delta}(\alpha + \epsilon) \leq \delta^{-\tau^*(\alpha)-(1+q)\epsilon}. \]
Now choose \(m\) and \(n_x\) as in the proof of Part 1 but replace conditions (ii) and (iii) respectively by
(iii)' \(\mu(B_{\delta}(x)) < \delta_{\alpha_{\min}^{-}\epsilon}\) for all \(0 < \delta \leq 2^{-(n_x-2)}\);
(ii)' there exists \(\delta_{x} > 0\) such that
\[ 2^{-(n_x-1)} < \delta_{x} \leq 2^{-n_x} \quad \text{and} \quad \mu(B_{\delta_x}(x)) > \delta_{x}^{\alpha_{\min}+\epsilon}/2. \]
The same proof yields the result and the lemma follows by combining the above two parts.

\begin{lemma}
Under the same hypotheses of Lemma 3.1, then
\[ \mu\left\{ x \in \text{supp}(\mu) : \alpha_{\epsilon}^+ < \lim_{\delta \to 0^+} \frac{\ln \mu(B_{\delta}(x))}{\ln \delta} \right\} = 0. \]
\end{lemma}

\textbf{Proof}. Again we divide the proof into two parts.

\textbf{Part 1}. \(\mu\left\{ x \in \text{supp}(\mu) : \alpha_{\epsilon}^+ < \lim_{\delta \to 0^+} \frac{\ln \mu(B_{\delta}(x))}{\ln \delta} < \alpha_{\epsilon}^+ \right\} = 0.\) Let \(\alpha_{\epsilon}^+ < \alpha < \alpha_{0}^+\) and \(q \in \partial \tau^*(\alpha)\). The condition \(\tau(q) - (q-1)\alpha > 0\) still holds by the assumption \(\alpha > \alpha_{1}^+\) and by the fact that \(\tau\) is increasing and concave. Instead of \(L_\epsilon(\alpha)\), we define
\[ U_\epsilon(\alpha) = \left\{ x \in \text{supp}(\mu) : \alpha - \frac{\epsilon}{3} \leq \lim_{\delta \to 0^+} \frac{\ln \mu(B_{\delta}(x))}{\ln \delta} \leq \alpha + \frac{\epsilon}{3} \right\}. \]
Let \(\delta_{\epsilon} > 0\) and \(m \in \mathbb{N}\) be as in the proof of Lemma 3.1. For each \(x \in U_\epsilon(\alpha)\), we let \(n_x\) be chosen as in Lemma 3.1 but replace conditions (ii) and (iii) by (ii)' and (iii)' respectively as follows:
(ii') For all \( 0 < \delta \leq 2^{-(n-1)} \), \( \mu(B_\delta(x)) \geq \delta^{\alpha + \epsilon} \);

(iii') there exists \( \delta_x > 0 \) such that
\[
2^{-n_x} < \delta_x \leq 2^{-(n-1)} \quad \text{and} \quad \mu(B_{\delta_x}(x)) \leq \delta_x^{\alpha - \epsilon}.
\]
Then apply the same technique.

\textbf{Part 2.} We need to show that if \( \alpha > \alpha_1^+ \) and \( \alpha_0^+ \leq \alpha \leq \alpha_{\max} \), then
\[
\mu \{ x \in \text{supp}(\mu) : \lim_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} > \alpha \} = 0.
\]
Choose \( \epsilon > 0 \) as in the proof of Part 1 of Lemma 3.1 and define
\[
U'_\epsilon(\alpha) = \{ x \in \text{supp}(\mu) : \alpha - \frac{\epsilon}{3} \leq \lim_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \}.
\]
Using Lemma 2.2, we can replace inequality (3.2) by \( \tilde{N}_\epsilon(\alpha - \epsilon) \leq \delta^{-\tau'(\alpha)-(1-\eta)\epsilon} \).
A similar argument yields \( \mu(U'_\epsilon(\alpha)) = 0 \) and the result follows. \( \square \)

We now proof the main theorem by combining Lemmas 3.1 and 3.2.

\textbf{Proof of Theorem 1.1.} (a) It follows easily from (2.1) that for each \( x \in \text{supp}(\mu) \),
\[
\alpha_{\min} \leq \lim_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \lim_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \alpha_{\max}.
\]
Consequently, Lemma 3.1 implies that
\[
\mu \{ x \in \text{supp}(\mu) : \lim_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} < \alpha_1^- \} = 0.
\]
By Lemma 3.2,
\[
\mu \{ x \in \text{supp}(\mu) : \lim_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} > \alpha_1^+ \} = 0.
\]
Part (a) now follows.

(b) The assumption that \( \tau(q) \) is differentiable at \( q = 1 \) implies that \( \partial \tau(1) \) is a singleton, i.e., \( \alpha_1 = \alpha_1^+ = \tau'(1) \). Part (a) now implies that for \( \mu \) a.e. \( x \in \text{supp}(\mu) \),
\[
\lim_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \tau'(1).
\]
The result follows from Theorem 4.4 in [Y]. \( \square \)

4. INFINITE BERNOULLI CONVOLUTIONS

Let \( 0 < \rho < 1 \), \( S_1(x) = \rho x \), \( S_2(x) = \rho x + (1 - \rho) \), and let \( \mu_\rho \) be the self-similar measure defined by \( S_1, S_2 \), i.e.,
\[
\mu_\rho = \frac{1}{2} \mu_\rho \circ S_1^{-1} + \frac{1}{2} \mu_\rho \circ S_2^{-1}.
\]
\( \mu_\rho \) is known as an \textit{infinitely convolved Bernoulli measure} (ICBM) because it can be identified with the distribution of the random variable \( (1 - \rho) \sum_{n=0}^{\infty} \rho^n \epsilon_n \) where \( \{\epsilon_n\} \) are i.i.d. random variables each taking values 0 or 1 with probability 1/2. Such measures have been studied extensively since the 30's. For \( 1/2 < \rho < 1 \), \( \{S_1, S_2\} \) does not satisfy the open set condition and hence the dimension result stated in (1.2) (with \( \alpha = \tau'(1) \)) does not cover such measures. An important result of Erdős says that if \( \rho^{-1} \) is a P.V. number, then \( \mu \) is singular [E]. (Recall that an algebraic integer \( \beta > 1 \) is a \textit{P.V. number} if all of its conjugates have moduli strictly less than 1.)
We will consider the special P.V. number
\[ \rho_0^{-1} = \left( \sqrt{5} + 1 \right)/2 \]
(the golden ratio), which is so far the best understood case. The Hausdorff and entropy dimensions of this particular measure have been studied by a number of authors ([AY], [AZ], [LP], [La]). It is known that these two dimensions are equal and it is conjectured that they are equal to 0.99571312... [AZ]. In [LN2], a closed formula which defines the corresponding \( \tau(q) \) for all \( q > 0 \) is derived. Moreover, it is proved that \( \tau(q) \) is differentiable on \((0, \infty)\) and
\[
\tau'(1) = \frac{1}{9 \ln \rho_0} \sum_{k=0}^{\infty} \sum_{|J|=k} c_J \ln c_J,
\]
where
\[ c_J = \frac{1}{8 \cdot 4^k} \left[ 1, 1 \right] P_J \left[ 1 \right], \quad P_0 = \left[ 1 \, 0 \, 1 \right], \quad P_1 = \left[ 1 \, 0 \, 1 \right], \]
and \( P_J = P_{j_1} \cdots P_{j_k} \), with \( j_i = 0 \) or 1. Theorem 1.1 implies that \( \tau'(1) \) is equal to the Hausdorff and entropy dimensions of the measure. Numerical calculations using (4.1) suggest that \( \tau'(1) \approx 0.9957 \), agreeing with the result obtained in [AY], [AZ] and [La]. It is an open question how to obtain the \( L^q \)-spectrum \( \tau(q) \) for other P.V. numbers.

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