PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 10, October 1997, Pages 2943–2951
S 0002-9939(97)03974-9

A DIMENSION RESULT ARISING FROM THE L^q-SPECTRUM OF A MEASURE

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We give a rigorous proof of the following heuristic result: Let μ be a Borel probability measure and let $\tau(q)$ be the L^q-spectrum of μ. If $\tau(q)$ is differentiable at $q = 1$, then the Hausdorff dimension and the entropy dimension of μ equal $\tau'(1)$. Our result improves significantly some recent results of a similar nature; it is also of particular interest for computing the Hausdorff and entropy dimensions of the class of self-similar measures defined by maps which do not satisfy the open set condition.

1. Introduction

Let μ be a Borel probability measure on $\mathbb{R}^d$ with bounded support and let $\operatorname{supp}(\mu)$ denote the support of μ. For a finite Borel partition $\mathcal{P}$ of $\operatorname{supp}(\mu)$, we let $|\mathcal{P}|$ be the maximum of the diameters of elements of $\mathcal{P}$. Define

$$h(\mu, \mathcal{P}) = -\sum_{A\in\mathcal{P}} \mu(A) \ln \mu(A).$$

For $\delta > 0$, let

$$h(\mu, \delta) = \inf\{h(\mu, \mathcal{P}) : \ \mathcal{P} \text{ is a finite Borel partition of } \operatorname{supp}(\mu),\ |\mathcal{P}| \le \delta\}.$$

The *entropy dimension* (or *Rényi dimension* [Re]) of μ is defined as

$$\dim_e(\mu) = \lim_{\delta\to 0^+} \frac{h(\mu,\delta)}{-\ln\delta}.$$

Also, we let $\dim_H(E)$ denote the Hausdorff dimension of a set E and define the *Hausdorff dimension* of μ as

$$\dim_H(\mu) = \inf\{\dim_H(E) : \ \mu(\mathbb{R}^d \setminus E) = 0\}.$$

Young [Y] proved that if

$$\lim_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \alpha \quad \text{for } \mu \text{ a.e. } x \in \operatorname{supp}(\mu), \tag{1.1}$$

then

$$\dim_H(\mu) = \dim_e(\mu) = \alpha. \tag{1.2}$$

Received by the editors February 28, 1996 and, in revised form, May 7, 1996.
1991 *Mathematics Subject Classification.* Primary 28A80; Secondary 28A78.
Key words and phrases. Entropy dimension, Hausdorff dimension, L^q-spectrum.
Research supported by a postdoctoral fellowship of the Chinese University of Hong Kong.

An important sufficient condition for (1.1) to hold is when μ is a self-similar measure defined by

$$\mu = \sum_{i=1}^{m} p_i \mu \circ S_i^{-1},$$

where $\{S_i\}_{i=1}^m$ is a family of contractive similitudes satisfying the *open set condition* ([Hut], [F]), and the p_i's are the probability weights satisfying $p_i > 0$ and $\sum_{i=1}^m p_i = 1$. In this case (1.1) holds for

$$\alpha = \sum_{i=1}^{m} p_i \ln p_i / \sum_{i=1}^{m} p_i \ln \rho_i, \tag{1.3}$$

where ρ_i is the contraction ratio of S_i. If we let

$$G = \Big\{ x \in \operatorname{supp}(\mu) : \lim_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \alpha \Big\},$$

then $\dim_H(G) = \alpha$ also. This theorem was proved by Geronimo and Hardin [GH] for $\{S_i\}_{i=1}^m$ satisfying the *strong open set condition* (and also implicitly by Cawley and Mauldin [CM]). It was also proved by Strichartz [S] by using the law of iterated algorithm for $\{S_i\}_{i=1}^m$ satisfying the open set condition.

Another sufficient condition to obtain (1.1) comes from the L^q-spectrum. For $\delta > 0$ and $q \in \mathbb{R}$, the L^q-*(moment) spectrum* of μ is defined as

$$\tau(q) = \varliminf_{\delta\to 0^+} \frac{\ln \sup \sum_i \mu(B_\delta(x_i))^q}{\ln \delta}, \tag{1.4}$$

where $\{B_\delta(x_i)\}_i$ is a family of disjoint closed δ-balls with center $x_i \in \operatorname{supp}(\mu)$ and the supremum is taken over all such families. The function $\tau(q)$ is an important function in multifractal theory; under suitable conditions, its Legendre transform equals the *dimension spectrum* of the measure μ ([H], [F]). Moreover, it is suggested in the physics literature that $\tau'(1)$ is equal to the entropy dimension of the measure ([HP], [H], [F]). Falconer [F] gives a heuristic argument for such equality. The purpose of this note is to give a rigorous proof of such a folklore theorem. Specifically, we prove

Theorem 1.1. *Let μ be a Borel probability measure on $\mathbb{R}^d$ with bounded support. Then*

(a) *for μ a.e. $x \in \operatorname{supp}(\mu)$, we have*

$$\tau'_+(1) \le \varliminf_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \le \varlimsup_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \le \tau'_-(1).$$

(b) *If $\tau(q)$ is differentiable at $q = 1$, then*

$$\lim_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \tau'(1) \quad \textit{for } \mu \textit{ a.e. } x \in \operatorname{supp}(\mu).$$

Consequently, μ is concentrated on $G = \Big\{ x \in \operatorname{supp}(\mu) : \lim_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \tau'(1) \Big\}$, and

$$\dim_H(G) = \dim_H(\mu) = \dim_e(\mu) = \tau'(1).$$

We will prove Theorem 1.1 in Section 3. The main idea is to show that the set of points $x \in \mathrm{supp}(\mu)$ such that

$$\varliminf_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} < \tau'_+(1) \quad \text{or} \quad \tau'_-(1) < \varlimsup_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta}$$

has μ measure zero. The proof of this relies on estimations of some counting functions (Lemma 2.2) together with a standard covering lemma. For the special case of self-similar measures defined by contractive similitudes satisfying the open set condition, $\tau(q)$ is given by

$$\sum_{i=1}^{m} p_i^q \rho_i^{-\tau(q)} = 1.$$

Moreover, $\tau(q)$ is differentiable and $\tau'(1) = \alpha$, where α is given by (1.3) (see [CM]). Such results have also been proved for some extensions of the self-similar measures [AP], [R] (with the open set condition), and for equilibrium measures of Hölder continuous conformal expanding maps [PW]. The equality of $\dim_H(\mu)$ and $\tau'(1)$, under the assumption that $\tau(q)$ is differentiable at $q = 1$, was recently studied by Fan for a certain class of infinite product measures [Fa]. An additional example is the infinitely convolved Bernoulli measure associated with the golden ratio. This is a good illustration and the main motivation for our result because the open set condition fails. This will be discussed in Section 4.

2. Preliminaries

Let $\tau : \mathbb{R} \to [-\infty, \infty)$ be a concave function. We define the *effective domain* of τ as

$$\mathrm{Dom}\ \tau = \{x : \ -\infty < \tau(x) < \infty\}.$$

The *concave conjugate* (or the *Legendre transform*) of τ is the function $\tau^* : \mathbb{R} \to [-\infty, \infty)$ defined by

$$\tau^*(\alpha) = \inf\{\alpha x - \tau(x) : \ x \in \mathbb{R}\}.$$

For $x \in \mathrm{Dom}\ \tau$, we let $\partial\tau(x) \subseteq \mathbb{R}$ be the *subdifferential* of τ at x, i.e.,

$$\partial\tau(x) = \{\alpha : \ \tau(y) \le \tau(x) + \alpha(y - x) \quad \text{for all} \ \ y \in \mathbb{R}\}.$$

Then $\tau^*(\alpha) + \tau(x) = \alpha x$ for $\alpha \in \partial\tau(x)$ [Ro]. If $\tau(x)$ is differentiable at x, then $\partial\tau(x)$ is the singleton $\tau'(x)$. Otherwise, $\partial\tau(x)$ is a closed interval. We will denote the special subdifferentials $\partial\tau(0)$ and $\partial\tau(1)$ respectively by $[\alpha_0^-, \alpha_0^+]$ and $[\alpha_1^-, \alpha_1^+]$.

It is known (e.g. [LN1, Proposition 2.3]) that $\mathrm{Dom}\ \tau^*$ is an interval and $(\mathrm{Dom}\ \tau^*)^o = (\alpha_{\min}, \alpha_{\max})$, where

$$\alpha_{\min} := \inf\{\alpha : \ \alpha \in \partial\tau(x), \ x \in \mathrm{Dom}\ \ \tau\},$$
$$\alpha_{\max} := \sup\{\alpha : \ \alpha \in \partial\tau(x), \ x \in \mathrm{Dom}\ \ \tau\}.$$

For the rest of this note, we assume that $\tau(q)$ is the L^q-spectrum of a Borel probability measure μ defined by (1.4). It is known that $\tau(q)$ is increasing, concave and $\tau(1) = 0$ (see Figure 1). Moreover, it is proved in [LN1] that

$$\text{(2.1)} \qquad \alpha_{\min} = \varliminf_{\delta\to 0^+} \frac{\ln(\sup_x \mu(B_\delta(x)))}{\ln \delta} \quad \text{and} \quad \alpha_{\max} = \varlimsup_{\delta\to 0^+} \frac{\ln(\inf_x \mu(B_\delta(x)))}{\ln \delta},$$

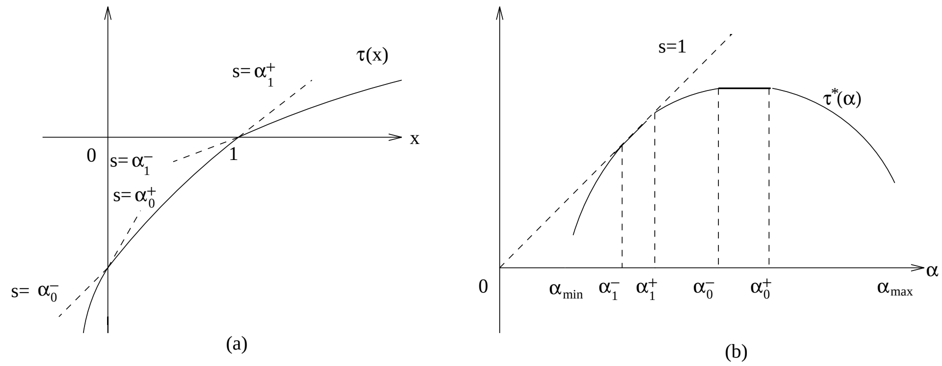

FIGURE 1. A concave function τ and its concave conjugate τ^* (s means slope)

where the supremum and infimum are taken over all $x \in \operatorname{supp}(\mu)$. Define

$$\tau^*(\alpha_{\min}) := \lim_{q\to\infty} \tau^*(\alpha),$$

where $\alpha \in \partial\tau(q)$. The following proposition will be used in the proof of Lemma 3.1.

Proposition 2.1. *Assume that* $\alpha_{\min} < \alpha_1^-$. *Then* $\alpha_{\min} > \tau^*(\alpha_{\min})$.

Proof. Let $\alpha_{\min} < \tilde{\alpha} < \alpha_1^-$ and $q \in \partial\tau^*(\tilde{\alpha})$ (i.e., $\tilde{\alpha} \in \partial\tau(q)$). Consider the line with slope $\tilde{\alpha}$ passing through the point $(q, \tau(q))$. This line intersects the vertical line $q = 1$ at $(1, \tau(q) - (q-1)\tilde{\alpha})$. By using the identity $\tau(q) + \tau^*(\tilde{\alpha}) = q\tilde{\alpha}$ together with the facts that τ is concave with $\tau(1) = 0$ and $\tilde{\alpha} < \alpha_1^-$, we have

$$\tilde{\alpha} - \tau^*(\tilde{\alpha}) = \tau(q) - (q-1)\tilde{\alpha} > 0.$$

The same argument shows that $\alpha - \tau^*(\alpha)$ is an increasing function of q and hence

$$\alpha - \tau^*(\alpha) \geq \tilde{\alpha} - \tau^*(\tilde{\alpha}) \quad \text{for all } \alpha \leq \tilde{\alpha}.$$

The result follows by letting $q \to \infty$. □

Let $\mathcal{B}_\delta$ denote a disjoint family of closed balls of radii δ centered at points in $\operatorname{supp}(\mu)$. For $\alpha \in (\operatorname{Dom} \tau^*)^\circ$, we define the counting functions

$$N_\delta(\alpha) = \sup_{\mathcal{B}_\delta} \#\{B : B \in \mathcal{B}_\delta,\ \mu(B) \geq \delta^\alpha\},$$

$$\widetilde{N}_\delta(\alpha) = \sup_{\mathcal{B}_\delta} \#\{B : B \in \mathcal{B}_\delta,\ \mu(B) < \delta^\alpha\}.$$

The following lemma is proved in [LN1, Lemma 4.2].

Lemma 2.2. *Let* $\alpha_{\min} < \alpha < \alpha_0^+$, $q \in \partial\tau^*(\alpha)$ *and* $\xi > 0$. *Then for any* $\epsilon > 0$, *there exists* $\delta_\epsilon > 0$ *such that for all* $0 < \delta < \delta_\epsilon$,

$$N_\delta(\alpha \pm \epsilon) \leq \delta^{-\tau^*(\alpha) - (\xi \pm q)\epsilon}.$$

For $\alpha_0^+ \leq \alpha < \alpha_{\max}$, *the above holds with* $\widetilde{N}_\delta$ *replacing* N_δ.

Lemma 2.2 and the counting functions play a key role in the proof of the main theorem.

3. Proof of the main theorem

We need two lemmas.

Lemma 3.1. *Let* μ *be a Borel probability measure on* $\mathbb{R}^d$ *with bounded support. Then*

$$\mu\Big\{x \in \operatorname{supp}(\mu):\ \alpha_{\min} \le \varliminf_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} < \alpha_1^-\Big\} = 0.$$

Proof. *Part 1.* We claim that $\mu\Big\{x \in \operatorname{supp}(\mu):\ \alpha_{\min} < \varliminf_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} < \alpha_1^-\Big\} = 0$. Let $\alpha_{\min} < \alpha < \alpha_1^-$ and $q \in \partial\tau^*(\alpha)$. Then $q > 1$. Since τ is increasing, concave, $\alpha < \alpha_1^-$, and since $\tau(1) = 0$, we have $(\tau(q) - \tau(1))/(q-1) \ge \tau'_-(q) \ge \alpha > 0$. We choose $\epsilon > 0$ small enough so that

$$\sigma := (\tau(q) - (q-1)\alpha)/2 \le \tau(q) - (q-1)\alpha - (2+q)\epsilon. \tag{3.1}$$

(This implies that $\alpha + \epsilon < \alpha_1^-$.) Define

$$L_\epsilon(\alpha) = \Big\{x \in \operatorname{supp}(\mu):\ \alpha - \frac{\epsilon}{3} \le \varliminf_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \le \alpha + \frac{\epsilon}{3}\Big\}.$$

We will show that $\mu(L_\epsilon(\alpha)) = 0$. Putting $\xi = 1$ in Lemma 2.2, then there exists $\delta_\epsilon > 0$ such that for all $0 < \delta \le \delta_\epsilon$,

$$N_\delta(\alpha + \epsilon) \le \delta^{-\tau^*(\alpha) - (1+q)\epsilon}. \tag{3.2}$$

Fix $m \in \mathbb{N}$ satisfying

$$2^{-m} < \delta_\epsilon \qquad \text{and} \qquad m \ge 3\alpha/\epsilon + 2. \tag{3.3}$$

For each $x \in L_\epsilon(\alpha)$, we let n_x be the smallest integer satisfying the following conditions:

(i) $n_x \ge m$;
(ii) $\mu(B_\delta(x)) < \delta^{\alpha-\epsilon}$ for all $0 < \delta \le 2^{-(n_x-2)}$;
(iii) there exists $\delta_x > 0$ such that

$$2^{-(n_x+1)} < \delta_x \le 2^{-n_x} \qquad \text{and} \qquad \mu(B_{\delta_x}(x)) > \delta_x^{\alpha+2\epsilon/3}.$$

Note that n_x is uniquely determined by x. Partition $L_\epsilon(\alpha)$ into a countable disjoint union of subsets $L_\epsilon^n(\alpha)$ where $L_\epsilon^n(\alpha) = \{x \in L_\epsilon(\alpha):\ n_x = n\}$. Then

$$L_\epsilon(\alpha) = \bigcup_{n=m}^{\infty} L_\epsilon^n(\alpha). \tag{3.4}$$

Clearly for each $n \ge m$,

$$L_\epsilon^n(\alpha) \subseteq \bigcup_{x \in L_\epsilon^n(\alpha)} B_{2^{-n}}(x).$$

By a standard covering lemma (see [F, Lemma 4.8]), there exists a finite sequence $\{x_i\}_{i=1}^{\ell}$ in $L_\epsilon^n(\alpha)$ such that $\{B_{2^{-n}}(x_i)\}_{i=1}^{\ell}$ is a disjoint family and

$$L_\epsilon^n(\alpha) \subseteq \bigcup_{i=1}^{\ell} B_{2^{-(n-2)}}(x_i). \tag{3.5}$$

For $1 \le i \le \ell$, condition (iii) and (3.3) imply that

$$\mu(B_{2^{-n}}(x_i)) > 2^{-(n+1)(\alpha+2\epsilon/3)} \ge 2^{-n(\alpha+\epsilon)}.$$

Hence by (3.2),

$$\ell \leq 2^{-n(-\tau^*(\alpha)-(1+q)\epsilon)}. \tag{3.6}$$

Combining condition (ii), (3.5), (3.6) and (3.1), we have

$$\mu(L_\epsilon^n(\alpha)) \leq \sum_{i=1}^{\ell} \mu(B_{2^{-(n-2)}}(x_i)) \leq 2^{-(n-2)(\alpha-\epsilon)} \cdot 2^{-n(-\tau^*(\alpha)-(1+q)\epsilon)}$$
$$\leq C \cdot 2^{-n(\tau(q)-(q-1)\alpha-(2+q)\epsilon)} \leq C \cdot 2^{-n\sigma}.$$

(C is a constant independent of n.) Using this and (3.4), we have

$$\mu(L_\epsilon(\alpha)) \leq \sum_{n=m}^{\infty} \mu(L_\epsilon^n(\alpha)) \leq C \sum_{n=m}^{\infty} 2^{-n\sigma} = C\frac{2^{-\sigma m}}{1-2^{-\sigma}}.$$

Letting $m \to \infty$, we get $\mu(L_\epsilon(\alpha)) = 0$. The claim follows easily by taking a countable cover for $(\alpha_{\min}, \alpha_1^-)$ by sets of the form $L_\epsilon(\alpha)$.

Part 2. We will show that if $\alpha_{\min} < \alpha_1^-$, then

$$\mu\Big\{x \in \operatorname{supp}(\mu): \ \varliminf_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \alpha_{\min}\Big\} = 0.$$

By Proposition 2.1, we may choose $\epsilon > 0$ sufficiently small and $\alpha \in (\mathrm{Dom}\ \tau^*)^\circ$ sufficiently close to $\alpha_{\min}$ such that

$$0 < \sigma := (\alpha_{\min} - \tau^*(\alpha))/2 \leq \alpha_{\min} - \tau^*(\alpha) - (2+q)\epsilon,$$

where $q \in \partial\tau^*(\alpha)$. By Lemma 2.2, there exists $\delta_\epsilon > 0$ such that for all $0 < \delta \leq \delta_\epsilon$,

$$N_\delta(\alpha+\epsilon) \leq \delta^{-\tau^*(\alpha)-(1+q)\epsilon}.$$

Now choose m and n_x as in the proof of Part 1 but replace conditions (ii) and (iii) respectively by

(ii)' $\mu(B_\delta(x)) < \delta^{\alpha_{\min}-\epsilon}$ for all $0 < \delta \leq 2^{-(n_x-2)}$;

(iii)' there exists $\delta_x > 0$ such that

$$2^{-(n_x+1)} < \delta_x \leq 2^{-n_x} \qquad \text{and} \qquad \mu(B_{\delta_x}(x)) > \delta_x^{\alpha_{\min}+\epsilon/2}.$$

The same proof yields the result and the lemma follows by combining the above two parts. □

Lemma 3.2. *Under the same hypotheses of Lemma 3.1, then*

$$\mu\Big\{x \in \operatorname{supp}(\mu): \ \alpha_1^+ < \varlimsup_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta}\Big\} = 0.$$

Proof. Again we divide the proof into two parts.

Part 1. $\mu\Big\{x \in \operatorname{supp}(\mu): \ \alpha_1^+ < \varlimsup_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} < \alpha_0^+\Big\} = 0$. Let $\alpha_1^+ < \alpha < \alpha_0^+$ and $q \in \partial\tau^*(\alpha)$. The condition $\tau(q) - (q-1)\alpha > 0$ still holds by the assumption $\alpha > \alpha_1^+$ and by the fact that τ is increasing and concave. Instead of $L_\epsilon(\alpha)$, we define

$$U_\epsilon(\alpha) = \Big\{x \in \operatorname{supp}(\mu): \ \alpha - \frac{\epsilon}{3} \leq \varlimsup_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \alpha + \frac{\epsilon}{3}\Big\}.$$

Let $\delta_\epsilon > 0$ and $m \in \mathbb{N}$ be as in the proof of Lemma 3.1. For each $x \in U_\epsilon(\alpha)$, we let n_x be chosen as in Lemma 3.1 but replace conditions (ii) and (iii) by (ii)' and (iii)' respectively as follows:

(ii)' For all $0 < \delta \le 2^{-(n_x-1)}$, $\mu(B_\delta(x)) \ge \delta^{\alpha+\epsilon}$;
(iii)' there exists $\delta_x > 0$ such that

$$2^{-n_x} < \delta_x \le 2^{-(n_x-1)} \qquad \text{and} \qquad \mu(B_{\delta_x}(x)) \le \delta_x^{\alpha-\epsilon}.$$

Then apply the same technique.

Part 2. We need to show that if $\alpha > \alpha_1^+$ and $\alpha_0^+ \le \alpha \le \alpha_{\max}$, then

$$\mu\Big\{x \in \operatorname{supp}(\mu) : \varlimsup_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} > \alpha\Big\} = 0.$$

Choose $\epsilon > 0$ as in the proof of Part 1 of Lemma 3.1 and define

$$U_\epsilon'(\alpha) = \Big\{x \in \operatorname{supp}(\mu) : \alpha - \frac{\epsilon}{3} \le \varlimsup_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta}\Big\}.$$

Using Lemma 2.2, we can replace inequality (3.2) by $\tilde{N}_\epsilon(\alpha-\epsilon) \le \delta^{-\tau^*(\alpha)-(1-q)\epsilon}$. A similar argument yields $\mu(U_\epsilon'(\alpha)) = 0$ and the result follows. □

We now proof the main theorem by combining Lemmas 3.1 and 3.2.

Proof of Theorem 1.1. (a) It follows easily from (2.1) that for each $x \in \operatorname{supp}(\mu)$,

$$\alpha_{\min} \le \varliminf_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \le \varlimsup_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \le \alpha_{\max}.$$

Consequently, Lemma 3.1 implies that

$$\mu\Big\{x \in \operatorname{supp}(\mu) : \varliminf_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} < \alpha_1^-\Big\} = 0.$$

By Lemma 3.2,

$$\mu\Big\{x \in \operatorname{supp}(\mu) : \varlimsup_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} > \alpha_1^+\Big\} = 0.$$

Part (a) now follows.

(b) The assumption that $\tau(q)$ is differentiable at $q = 1$ implies that $\partial\tau(1)$ is a singleton, i.e., $\alpha_1^- = \alpha_1^+ = \tau'(1)$. Part (a) now implies that for μ a.e. $x \in \operatorname{supp}(\mu)$,

$$\lim_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \tau'(1).$$

The result follows from Theorem 4.4 in [Y]. □

4. Infinite Bernoulli convolutions

Let $0 < \rho < 1$, $S_1(x) = \rho x$, $S_2(x) = \rho x + (1-\rho)$, and let μ_ρ be the self-similar measure defined by S_1, S_2, i.e.,

$$\mu_\rho = \frac{1}{2}\mu_\rho \circ S_1^{-1} + \frac{1}{2}\mu_\rho \circ S_2^{-1}.$$

μ_ρ is known as an *infinitely convolved Bernoulli measure* (ICBM) because it can be identified with the distribution of the random variable $(1-\rho)\sum_{n=0}^\infty \rho^n \epsilon_n$ where $\{\epsilon_n\}$ are i.i.d. random variables each taking values 0 or 1 with probability 1/2. Such measures have been studied extensively since the 30's. For $1/2 < \rho < 1$, $\{S_1, S_2\}$ does not satisfy the open set condition and hence the dimension result stated in (1.2) (with $\alpha = \tau'(1)$) does not cover such measures. An important result of Erdös says that if ρ^{-1} is a P.V. number, then μ is singular [E]. (Recall that an algebraic integer $\beta > 1$ is a *P.V. number* if all of its conjugates have moduli strictly less than 1.)

We will consider the special P.V. number $\rho_o^{-1} = (\sqrt{5}+1)/2$ (the golden ratio), which is so far the best understood case. The Hausdorff and entropy dimensions of this particular measure have been studied by a number of authors ([AY], [AZ], [LP], [La]). It is known that these two dimensions are equal and it is conjectured that they are equal to 0.99571312... [AZ]. In [LN2], a closed formula which defines the corresponding $\tau(q)$ for all $q > 0$ is derived. Moreover, it is proved that $\tau(q)$ is differentiable on $(0, \infty)$ and

$$\tau'(1) = \frac{1}{9 \ln \rho_o} \sum_{k=0}^{\infty} \sum_{|J|=k} c_J \ln c_J, \tag{4.1}$$

where

$$c_J = \frac{1}{8 \cdot 4^k} \begin{bmatrix} 1, & 1 \end{bmatrix} P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad P_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

and $P_J = P_{j_1} \cdots P_{j_k}$, with $j_i = 0$ or 1. Theorem 1.1 implies that $\tau'(1)$ is equal to the Hausdorff and entropy dimensions of the measure. Numerical calculations using (4.1) suggest that $\tau'(1) \approx 0.9957$, agreeing with the result obtained in [AY], [AZ] and [La]. It is an open question how to obtain the L^q-spectrum $\tau(q)$ for other P.V. numbers.

Acknowledgements

The author would like to thank Mr. Y. F. Seid for providing the numerical computation of $\tau'(1)$ for the Bernoulli convolution associated with the golden ratio. He is also grateful to Professor K.-S. Lau for many valuable comments.

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