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Local behavior of Riemann's function

Stéphane Jaffard

ABSTRACT: We investigate how local Hölder regularity and local scalings of a function can be determined from its wavelet transform; we apply these results to Riemann's function.

1. Introduction

Let $x_0 \in \mathbb{R}$; by definition, a function f is $C^\alpha(x_0)$ if there exists a polynomial P of order at most α such that

$$|f(x_0+h) - P(h)| \leq C|h|^\alpha. \tag{1}$$

Even if this upper bound is optimal, it does not describe the behavior of f in the neighbourhood of x_0. For instance *trigonometric chirps* studied in [10] describe a very oscillatory behavior, of the form

$$f(x_0+h) - P(h) \sim |h|^\alpha \sin \frac{1}{|h|^\beta}.$$

A less oscillatory behavior appears when f is "approximately selfsimilar",i.e. satisfies (for $0 < \alpha < 1$ and $\lambda < 1$)

$$f(x_0+h) - P(h) = \lambda^{-\alpha}\left(f(x_0+\lambda h) - P(\lambda h)\right) + o(|h|^\alpha).$$

This equality is clearly equivalent to

$$f(x_0+h) - P(h) = h^\alpha G_\pm(\log(\pm h)) + o(|h|^\alpha) \tag{2}$$

where $\pm$ is the sign of h, and G_+ and G_- are $\log\lambda$ periodic. Thus, if any of these conditions is satisfied, we will say that f has a *logarithmic chirp* of order (α, λ) at x_0; partial necessary conditions on the wavelet transform of f can be found in [6].

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If the functions G_+ and G_- are C^γ, the chirp is said to have the regularity γ. Our purpose in the first part is to find simple necessary or sufficient conditions of selfsimilarity on the wavelet transform of f. We will then apply these criteria to several examples. Let us mention two of them; the first one is the "selfsimilar functions" introduced in [8], for which we will determine the regularity of the chirps and an explicit formula for the Fourier coefficients of the functions G_+ and G_-; for instance, Weierstrass functions

$$\sum_1^\infty 2^{-\alpha j} \sin 2\pi 2^j x$$

have logarithmic chirps at the rational points where the Fourier coefficients of G_+ and G_- are values of the Gamma function (given by (44)). The second example is "Riemann's function"

$$\varphi(x) = \sum_1^\infty \frac{1}{n^2} \sin \pi n^2 x.$$

In this case, J.J.Duistermaat proved in [1] the existence of logarithmic chirps at the *quadratic irrationals* (the irrational numbers roots of a polynomial of degree two with integer coefficients). We will complete this study by showing that the chirps have a regularity 1/2 at each of these points. We will also show how to determine the Hölder regularity of Riemann's function at every point. It may be interesting to recall briefly the history of this function.

Riemann proposed it as an example of continuous nowhere differentiable function. Actually, Hardy and Littlewood proved that φ is nowhere $C^{3/4}$ except perhaps at the rational points of the form $(2p+1)/(2q+1)$, $p, q \in \mathbb{Z}$ ([4]). Their proof anticipates wavelet methods: they remark that the function

$$C(a,b) = \frac{a}{2}(\theta(b+ia) - 1)$$

(where θ is the Jacobi function $\theta(z) = \sum_{n \in \mathbb{Z}} e^{i\pi n^2 z}$) is the convolution of φ with contractions (by a factor a) of

$$\psi(x) = \frac{1}{\pi(x-i)^2}.$$

Since ψ has a vanishing integral, an Abel type theorem (which we will state precisely in Proposition 1) shows that, if φ is smooth at x_0, $C(a,b)$ must have a certain decay when $a \to 0$ and $b \to x_0$. This method will thus yield upper bounds for the pointwise Hölder exponent

$$\alpha(x_0) = \sup\{\beta : \varphi \in C^\beta(x_0)\}.$$

This method actually yields the following more precise result which relates the pointwise behavior of φ at x_0 to the Diophantine approximation properties of x_0. Let $x_0 \notin \mathbb{Q}$, let p_n/q_n be its continued fraction expansion and define

$$\tau(x_0) = \sup\{\tau : | x_0 - \frac{p_m}{q_m} | \leq \frac{1}{q_m^\tau}\}$$

for infinitely many m's such that p_m and q_m are not both odd. Then

$$\alpha(x_0) \leq \frac{1}{2} + \frac{1}{2\tau(x_0)}.$$

a result which is actually stated by J.J.Duistermaat [1] where a more direct proof is given.

Converse results, which would yield an information about the pointwise behavior of φ from estimates on its convolutions are more difficult to obtain since they are of tauberian type. We will state such results in Proposition 1.

Finally Gerver proved the differentiability at the rational points of the form $(2p+1)/(2q+1)$ [2] (where we now know that φ is exactly $C^{3/2}$, see [10]). The analysis of the behavior of φ near such a rational point x_0 has been considerably sharpened since; in [10], a complete "chirp" asymptotic expansion which describes the oscillations of φ is exhibited:

$$\varphi(x) = u(x) + \sum_{n\geq 0}(x-x_0)^{\frac{3}{2}+n} v_+^n\left(\frac{1}{x-x_0}\right) \text{ if } x \geq x_0$$

$$\varphi(x) = u(x) + \sum_{n\geq 0}|x-x_0|^{\frac{3}{2}+n} v_-^n\left(\frac{1}{|x-x_0|}\right) \text{ if } x \leq x_0$$

where u is C^∞, the $v_\pm^n$ are 2π periodic, with a vanishing integral, and are $C^{\frac{1}{2}+n}$ (we will actually show that these points are the only ones where a chirp expansion exists).

The results of Hardy and Gerver left open the problem of the determination of the exact regularity of φ at irrational points; one of our purposes is to do this determination using the wavelet method we sketched.

Our main results concerning Riemann's function are stated in the following theorem.

Theorem 1 *Let $x \notin \mathbb{Q}$ and let p_n/q_n be the sequence of its approximations by continued fractions. Let*

$$\tau(x) = \sup\{\tau : \mid x - \frac{p_m}{q_m} \mid \leq \frac{1}{q_m^\tau}\}$$

for infinitely many m's such that p_m and q_m are not both odd.
Then

$$\alpha(x) = \frac{1}{2} + \frac{1}{2\tau(x)}.$$

If x is a quadratic irrational, φ has a logarithmic chirp at x of regularity $1/2$.

Thus Riemann's function is interesting not only for historical reasons but also because it displays many different local behaviors: a whole range of different Hölder exponents, chirps at some rationals and logarithmic chirps at quadratic irrationals. Thus it is a perfect example on which one can test the efficiency of local analysis methods; and, in our opinion, it clearly shows the efficiency of wavelet methods.

This paper is partly a rewiew paper and partly contains new results: the analysis of the local regularity by wavelet methods can be found in [7] and the pointwise Hölder exponent of Riemann's function is determined in [9]. The results concerning the analysis of local scalings is new, and its application for Riemann's function sharpens some results of J.J.Duistermaat in [1].

2. The tools: Wavelet transform and two-microlocalization.

Let $\alpha > 0$; it will be the largest order of Hölder regularity we will be interested in. Let $k = [\alpha]$. Suppose that a function ψ is nonvanishing and satisfies the following asumptions

$$|\psi(x)| + |\psi'(x)| + \cdots + |\psi^{(k+1)}(x)| \leq C(1+|x|)^{-k-2}, \tag{3}$$

$$\int \psi(x)dx = \int x\psi(x)dx = \cdots = \int x^k\psi(x)dx = 0 \tag{4}$$

and either

$$\int_0^\infty |\hat{\psi}(\xi)|^2 \frac{d\xi}{\xi} = \int_0^\infty |\hat{\psi}(-\xi)|^2 \frac{d\xi}{\xi} = 1 \tag{5}$$

or

$$\hat{\psi}(\xi) = 0 \text{ if } \xi < 0 \text{ and } \int_0^\infty |\hat{\psi}(\xi)|^2 \frac{d\xi}{\xi} = 1 \tag{6}$$

In the last case the wavelet is said to be analytic. The wavelet transform of an L^∞ function f is defined by

$$C(a,b)(f) = \frac{1}{a} \int f(t)\bar{\psi}(\frac{t-b}{a})dt.$$

We will consider the three following settings: In the first one the analyzed function f is real valued, and the wavelet satisfies (6); in the second one, f is complex valued and the wavelet satisfies (5); in the third one, $\hat{f}(\xi) = 0$ if $\xi < 0$ and the wavelet satisfies (6).

In each case, the following results concerning the relationships between the size of the wavelet transform and the regularity of the function hold.

Proposition 1 *Under the previous hypotheses if a function f is $C^\alpha(x_0)$,*

$$|C(a,b)(f)| \leq Ca^\alpha(1 + \frac{|b-x_0|}{a})^\alpha \tag{7}$$

Conversely, if

$$|C(a,b)(f)| \leq Ca^\alpha(1 + \frac{|b-x_0|}{a})^{\alpha'} \text{ for an } \alpha' < \alpha \tag{8}$$

then

$$|f(x) - f(x_0)| \leq C|x-x_0|^\alpha \tag{9}$$

for all x such that $|x - x_0| \leq 1/2$. Furthermore, large O's can be replaced by little o's in these results, i.e. if

$$f(x_0 + h) - P(h) = o(|h|^\alpha)$$

then

$$C(a,b)(f) = o\left(a^\alpha (1 + \frac{|b - x_0|}{a})^\alpha\right); \tag{10}$$

and conversely, if

$$C(a,b)(f) = o\left(a^\alpha (1 + \frac{|b - x_0|}{a})^{\alpha'}\right) \text{ for an } \alpha' < \alpha. \tag{11}$$

then

$$f(x_0 + h) - P(h) = o(|h|^\alpha).$$

Let us now recall the definition of the two-microlocal spaces $C^{\alpha,\alpha'}(x_0)$ (see [7]). The function f belongs to $C^{\alpha,\alpha'}(x_0)$ if its wavelet transform satisfies

$$|C(a,b)(f)| \leq C a^\alpha (1 + \frac{|b - x_0|}{a})^{-\alpha'}.$$

The conditions above are clearly expressed in terms of two-microlocal spaces; the connexion will even be more obvious for the conditions that imply the existence of a logarithmic chirp of a given regularity.

Proof of Proposition 1: We only consider the case $0 < \alpha < 1$. The reader will easily extend it to any $\alpha > 0$. If $f \in C^\alpha(x_0)$,

$$\begin{aligned}
|C(a,b)(f)| &= \frac{1}{a}\left|\int f(x)\psi(\frac{x-b}{a})dx\right| \\
&= \frac{1}{a}\left|\int (f(x) - f(x_0))\psi(\frac{x-b}{a})dx\right| \\
&\leq \frac{C}{a}\int |x - x_0|^\alpha \left(\frac{1}{1 + |\frac{x-b}{a}|}\right)^2 dx \\
&\leq \frac{C}{a}\int \frac{|x-b|^\alpha}{(1 + |\frac{x-b}{a}|)^2}dx + |b - x_0|^\alpha \frac{C}{a}\int \frac{dx}{(1 + |\frac{x-b}{a}|)^2} \\
&\leq C a^\alpha \left(1 + |\frac{b - x_0}{a}|\right)^\alpha
\end{aligned}$$

Suppose now that we are in the first or the third case. In that case, f is reconstructed from its wavelet transform by

$$f(x) = \int\int C(a,b)(f)\psi(\frac{x-b}{a})\frac{da db}{a^2}.$$

Let

$$\omega(a,x) = \int C(a,b)(f)\psi(\frac{x-b}{a})\frac{db}{a};$$

if (8) holds,

$$|\omega(a,x)| \leq Ca^{\alpha}(1+\frac{|x-x_0|}{a})^{\alpha'}$$

and

$$|\frac{\partial\omega(a,x)}{\partial x}| \leq Ca^{\alpha-1}(1+\frac{|x-x_0|}{a})^{\alpha'}.$$

Using the second estimate (and the mean value theorem) for $a \geq |x-x_0|$ and the first estimate for $a \leq |x-x_0|$, we obtain

$$|f(x)-f(x_0)| \leq \int_{a\geq|x-x_0|} Ca^{\alpha-1}\left(1+|\frac{x-x_0}{a}|\right)^{\alpha'} |x-x_0|\frac{da}{a}\ldots$$

$$+\int_{a\leq|x-x_0|} Ca^{\alpha}\left(1+|\frac{x-x_0}{a}|\right)^{\alpha'} \frac{da}{a} \leq C|x-x_0|^{\alpha}.$$

This also implies the result in the second case by superposing reconstruction formulas for $\xi \geq 0$ and $\xi \leq 0$. The proof for little o's instead of big o's is exactly the same, and we leave it to the reader.

There exists some similar results for negative values of α. The reader is reffered to [8] for a discussion on negative Hölder exponents, and the corresponding conditions on the wavelet transform.

3. Logarithmic chirps and wavelets

We will prove the following result.

Theorem 2 *If f has a logarithmic chirp of order (α,λ) and of regularity $\gamma \geq 0$ at x_0, then its wavelet transform satisfies*

$$C(a,b) = a^{\alpha}H(\log a, \frac{b-x_0}{a}) + o(a^{\alpha}+(\frac{|b-x_0|}{a})^{\alpha}). \tag{12}$$

Where H is $Log\lambda$ periodic in the first variable and satisfies

$$\| H(.,y) \|_{L^\infty} \leq C(1+|y|)^{\alpha-\gamma} \tag{13}$$

$$\| H(.,y) \|_{C^\gamma} \leq C(1+|y|)^{\alpha} \tag{14}$$

Conversely, if

$$C(\lambda a, x_0+\lambda b) = \lambda^{-\alpha}C(a,x_0+b) + o(a^{\alpha}(1+\frac{|b-x_0|}{a})^{\alpha'}) \tag{15}$$

where $\lambda < 1$, and $\alpha' < \alpha$; and if furthermore $f \in C^{\alpha,-\alpha+\gamma}(x_0)$, then f has a logarithmic chirp of order (α,λ) and of regularity $\gamma' \geq 0$ at x_0, for any $\gamma' < \gamma$.

Let us make a few comments. First, we can formulate the "algebraic" selfsimilarity condition satisfied by the wavelet transform either by a selfsimilarity condition (as in (15)) or through the existence of a periodic function (as in (12)). This equivalence is the purpose of the following lemma. The interesting point about using the periodic function formulation is that we can write explicit regularity asumptions about this function.

Lemma 1 *Suppose that there exists* $\lambda < 0$ *and* $\alpha, \alpha' > 0$ *such that*

$$C(a, x_0 + b) = \lambda^{-\alpha} C(\lambda a, x_0 + \lambda b) + O(a^{\alpha'} + b^{\alpha'}) \tag{16}$$

then

$$C(a, b) = a^{\alpha} H(\log a, \frac{b - x_0}{a}) + O(a^{\alpha'} + b^{\alpha'}) \tag{17}$$

where H *is periodic of period* $\log \lambda$ *in the first variable.*

Proof: In order to simplify the notations, suppose that $x_0 = 0$. The sequence $\lambda^{-n\alpha} C(\lambda^n a, \lambda^n b)$ converges uniformy on any compact that does not contain a point of the real axis $a = 0$ because (16) implies that

$$\lambda^{-n\alpha} C(\lambda^n a, \lambda^n b) - \lambda^{-(n+1)\alpha} C(\lambda^{n+1} a, \lambda^{n+1} b) = O((\lambda^n a)^{\alpha'} + (\lambda^n b)^{\alpha'})$$

and the limit function $d(a, b)$ satisfies

$$d(a, b) = \lambda^{-\alpha} d(\lambda a, \lambda b) \tag{18}$$

so that the lemma reduces to the change of notations

$$H(x, y) = e^{-\alpha x} d(e^x, y e^x).$$

We remark that this proof shows that we will be able to obtain size and regularity estimates on H from similar uniform estimates on the sequence $\lambda^{-n\alpha} C(\lambda^n a, \lambda^n b)$.

Let us now prove the direct part in Theorem 2.

Suppose that (2) holds. Because ψ has vanishing moments,

$$C(a, b) = \frac{1}{a} \int (f(x) - P(x - x_0)) \psi(\frac{x - b}{a}) dx.$$

Consider the integral for $x \geq x_0$ (the other part is similar); it is written

$$\frac{1}{a} \int |x - x_0|^{\alpha} G_+(\log |x - x_0|) \psi(\frac{x - b}{a}) dx + o\left(\frac{1}{a} \int |x - x_0|^{\alpha} |\psi(\frac{x - b}{a})| dx\right).$$

The second term is clearly bounded by $o(a^{\alpha} + |b - x_0|^{\alpha})$ and the first one is

$$a^{\alpha} \int |u + \frac{b - x_0}{a}|^{\alpha} G_+(\log a + \log |u + \frac{b - x_0}{a}|) \psi(u) du \tag{19}$$

which is of the form $a^{\alpha} H(\log a, \frac{b - x_0}{a})$ where H is $\log \lambda$ periodic in the first variable. Furthermore, this term is bounded by

$$C a^{\alpha} \int (|u|^{\alpha} + |\frac{b - x_0}{a}|^{\alpha}) \frac{du}{1 + u^{k+2}} \leq C a^{\alpha} (1 + |\frac{b - x_0}{a}|)^{\alpha}$$

In order to prove (13), we can thus suppose that $|y| \geq 1$.

Once (12) is proved, it is clear that (13) is just a two-microlocal estimate for the term $a^\alpha H(\log a, \frac{b-x_0}{a})$. Since such estimates do not depend on the wavelet (see [7]), we can suppose that $supp\psi \subset [-1,1]$. From (19), we get

$$H(x,y) = \int |u+y|^\alpha G_+(x + \log|u+y|)\psi(u)du.$$

Integrating $[\gamma]$ times by parts, we obtain

$$H(x,y) = \int |u+y|^{\alpha'} \tilde{G}(x + \log|u+y|)\tilde{\psi}(u)du$$

where $\alpha' = \alpha - [\gamma]$, $\tilde{\psi}$ is the ψ integrated $[\gamma]$ times, and

$$\tilde{G}(x + \log|u+y|) = \sum_0^{[\gamma]} c_l G_+^{(l)}(x + \log|u+y|),$$

so that $\tilde{G}$ is $C^{\gamma-[\gamma]}$. We have

$$|u+y|^{\alpha'} = |y|^{\alpha'} + ug(u,y)$$

where

$$|g(u,y)| \leq Cy^{\alpha'-1} \text{ if } u \leq y$$

(which is always the case because $lu| \leq 1 \leq |y|$. Thus H is the sum of two terms, the second one being

$$\int ug(u,y)\tilde{G}(x + \log|u+y|)\tilde{\psi}(u)du$$

which is bounded if $u \leq y$ by

$$\int_{u \leq y} |y|^{\alpha'-1} u|\tilde{\psi}(u)|du \leq C|y|^{\alpha'-1}(1 + |\log y|).$$

The first term of H is

$$|y|^{\alpha'} \int \tilde{G}(x + \log|u+y|)\tilde{\psi}(u)du. \tag{20}$$

If $|y| \leq 1$, it is bounded by a constant, and if $|y| \geq 1$, we write

$$\tilde{G}(x + \log|u+y|) = \tilde{G}(x + \log|y| + \log|1 + \frac{u}{y}|);$$

after substracting $\tilde{G}(x + \log|y|)$; we bound (20) by

$$C|y|^{\alpha'} \int |\log(1 + \frac{u}{y})|^{\gamma-[\gamma]} |\tilde{\psi}(u)|du \leq C|y|^{\alpha-\gamma}.$$

Concerning the second estimate of Theorem 2, we have

$$H(z,y) - \sum_{|k|\le[\alpha]} \frac{h^k}{k!}\partial_x^k H(x,y) =$$

$$\int |u+y|^\alpha \left(G_+(z+\log|u+y|) - \sum_{|k|\le[\alpha]} \frac{h^k}{k!}\partial_x^k G_+(x+\log|u+y|) \right) \psi(u)du$$

which is bounded by

$$C\int |u+y|^\alpha |x-z|^\gamma \frac{du}{1+u^{k+2}} \le C|x-z|^\gamma |1+y|^\alpha.$$

In order to prove the converse part of Theorem 2, we will need the following proposition, which is interesting in its own right.

Proposition 2 *Suppose there exists $\alpha' < \alpha$ such that*

$$C(a,b) = a^\alpha H(\log a, \frac{b-x_0}{a}) + o\left(a^\alpha (1+\frac{b-x_0}{a})^{\alpha'} \right) \tag{21}$$

where H is $\log\lambda$ periodic in the first variable and satisfies (13) and (14), then f has a logarithmic chirp of order (α,λ) and of regularity γ' at x_0 for any $\gamma' < \gamma$.

Proof: f is reconstructed using the formula

$$f(x) = \int C(a,b)\psi(\frac{x-b}{a})\frac{dadb}{a^2}$$

By a classical argument (see [7]) the second term in (21) brings a contribution to $f(x) - P(x-x_0)$ which is $o(|x-x_0|^\alpha)$; it can thus be neglected.

We choose for P:

$$P(x-x_0) = \sum_{l=0}^{[\alpha]} \frac{(x-x_0)^l}{l!} \int C(a,b)\psi^{(l)}(\frac{x_0-b}{a})\frac{dadb}{a^{l+2}}$$

(one easily checks that these coefficients are well defined). Thus

$$\begin{aligned} &f(x) - P(x-x_0) \\ &= \int\int a^\alpha H(\log a, \frac{b-x_0}{a}) \left(\psi(\frac{x-b}{a}) - \sum \frac{1}{l!}(\frac{x-x_0}{a})^l \psi^{(l)}(\frac{x_0-b}{a}) \right) \frac{dadb}{a^2} \\ &= \int\int (\frac{x-x_0}{u})^\alpha H(\log|x-x_0| - \log u, b) \left(\psi(u-b) - \sum \frac{u^l}{l!}\psi^{(l)}(-b) \right) \frac{dudb}{u} \end{aligned}$$

which is of the form $|x-x_0|^\alpha G(\log|x-x_0|)$ with g periodic of period $\log\lambda$. Let us prove the estimates on G. We have

$$G(x) = \int\int H(\log x - \log u, b) \left(\psi(u-b) - \sum \frac{u^l}{l!}\psi^{(l)}(-b) \right) \frac{dudb}{u^{\alpha+1}}$$

First, let us prove that G is bounded. In the integral, if $u \leq 1$, we bound $\psi(u-b) - \sum \frac{u^l}{l!}\psi^{(l)}(-b)$ by $\frac{Cu^{k+1}}{(1+|b|)^{k+2}}$ (here $k = [\alpha]$); and the bound follows from (13). If $u \geq 1$, it is bounded by $\frac{C}{(1+|u-b|)^{l+2}} + C\sum \frac{u^l}{(1+|b|)^{k+2}}$ and the bound still follows from (13). In order to estimate $\| G(.,y) \|_{C^{\gamma'}}$, one cannot use directly (14) because the factor $(1+|y|)^\alpha$ yields divergent integrals; one remarks that (13) and (14) imply that $\forall \gamma' < \gamma$, there exists $\alpha' < \alpha$ such that

$$\| H(.,y) \|_{C^{\gamma'}} \leq C(1+|y|)^{\alpha'}$$

and the bound for $\| G(.,y) \|_{C^{\gamma'}}$ is obtained as above.

Let us now check that the converse part of Theorem 2 is an easy consequence of Proposition 2. Suppose that (15) holds. Lemma 1 shows that the limit

$$\lim_{n\to\infty} \lambda^{-n\alpha} C(\lambda^n a, \lambda^n b) = d(a,b)$$

exists and that

$$H(x,y) = e^{-\alpha x} d(e^x, ye^x)$$

is $\log\lambda$ periodic in the first variable. We still have to check (13) and (14). The two-microlocal hypothesis implies that

$$|\lambda^{-n\alpha} C(\lambda^n a, \lambda^n b)| \leq C\lambda^{-n\alpha}(\lambda^n a)^\alpha (1+|\frac{b}{a}|)^{\alpha'} = Ca^\alpha(1+|\frac{b}{a}|)^{\alpha'},$$

hence the same estimate for $d(a,b)$, so that

$$|H(x,y)| \leq Ce^{-\alpha x}e^{\alpha x}(1+|y|)^{\alpha'} \leq C(1+|y|)^{\alpha'}. \quad (22)$$

Let us now estimate the regularity of H in the first variable. Here too we write the proof only for $\alpha < 1$, the general case follows easily. Remark that

$$\frac{\partial}{\partial b} C(a,b) = \frac{1}{a} C_1(a,b),$$

where C_1 is the wavelet transform of f using the admissible wavelet $-\psi'$. Similarly

$$\frac{\partial}{\partial a} C(a,b) = \frac{1}{a} C_2(a,b),$$

where C_2 is the wavelet transform of f using the admissible wavelet $-\psi - x\psi'$. Since two-microlocal estimates are independant of the wavelet that is chosen (see [7]),

$$\nabla d(a,b) = \lim_{n\to\infty} \lambda^{-n\alpha} \nabla C(\lambda^n a, \lambda^n b)$$

so that

$$|\nabla d(a,b)| \leq \lambda^{-n\alpha}\lambda^n \frac{1}{\lambda^n a}(\lambda^n a)^\alpha (1+|\frac{b}{a}|)^{\alpha'} = a^{\alpha-1}(1+|\frac{b}{a}|)^{\alpha'}$$

But

$$\partial_x H(x,y) = -\alpha H(x,y) + e^{(1-\alpha)x}\left(\partial_1 d(e^x, ye^x) + y\partial_2 d(e^x, ye^x)\right).$$

So that

$$|\partial_x H(x,y)| \le C(1+|y|)^{\alpha'} + e^{(1-\alpha)x}\left(e^{(\alpha-1)x}(1+|y|)^{\alpha'} + ye^{(\alpha-1)x}(1+|y|)^{\alpha'}\right).$$

Thus

$$|\partial_x H(x,y)| \le C(1+|y|)^{\alpha'+1}$$

Thus (14) follows from this estimate and (22). (The proof for higher order derivatives is exactly similar). We are now in the position to apply Proposition 2 , hence Theorem 2.

4. Theta Jacobi function and continued fractions

In this section we establish or recall some prerequisites about the *Jacobi theta function* which will be necessary for the determination of the Hölder regularity of φ and of its local scalings.

Using Cauchy's formula, we obtain that (using the wavelet $\psi(x) = (x-i)^{-2}$) the wavelet transform of $\varphi(x)$ is $2ia(\theta(b+ia)-1)/2$. Since we want to determine Hölder exponents between 1/2 and 3/4, because of (8) we can add a term ia and the study of the pointwise regularity of φ reduces to obtaining estimates similar to (7) for the function

(23) $$C(a,b) = a\theta(b+ia).$$

The *theta modular group* is obtained by composing the two transforms

$$x \to x+2 \text{ and } x \to -1/x$$

It is composed of the fractional linear transformations

$$\gamma(x) = \frac{rx+s}{qx-p}$$

where $rp+sq=-1$, r,s,p,q are integers and the matrix

(24) $$\begin{pmatrix} r & s \\ q & p \end{pmatrix} \text{ is of the form } \begin{pmatrix} even & odd \\ odd & even \end{pmatrix} \text{ or } \begin{pmatrix} odd & even \\ even & odd \end{pmatrix}$$

When γ belongs to the theta modular group, θ is transformed following the formula (cf [1])

(25) $$\theta(z) = \theta(\gamma(z))e^{im\pi/4}q^{-1/2}(z-\frac{p}{q})^{-1/2}$$

where m is an integer which depends on r,s,p,q.

Let $\rho \notin \mathbb{Q}$ and p_n/q_n the sequence of its approximations by continued fractions. The idea of the proof of Theorem 1 is to use (25), which will allow us to deduce the behavior of $\theta(z)$ near p_n/q_n (hence near ρ) from its behavior near 0 or 1. Because of (24), we will have to separate two cases depending on whether p_n and q_n are both odd or not; but let us first derive some straightforward estimates for θ near 0 and 1.

First remark that

(26) $$|\theta(z)-1| \le \frac{1}{2} \text{ if } Imz \ge 1$$

because in this case,

$$|\theta(z)-1| \le 2\sum_{n\ge 1} e^{-\pi n^2 Imz} \le \frac{2e^{-\pi Imz}}{1-e^{-\pi Imz}} \le \frac{1}{2}$$

We also have

$$|\theta(z)| \le C|Imz|^{-1/2} \text{ if } Im(z) \le 1 \tag{27}$$

because $|\theta(z)| \le \sum e^{-n^2 Im(z)}$; the sum for $n \le Im(z)^{-1/2}$ is bounded trivially by $Im(z)^{-1/2}+1$ and the same bound holds for $n > Im(z)^{-1/2}$ (by comparison with an integral).

Let us now obtain the behavior of θ near the point 1. Recall that θ satisfies

$$\theta(1+z) = \sqrt{\frac{i}{z}}\left(\theta(-\frac{1}{4z}) - \theta(-\frac{1}{z})\right) \tag{28}$$

so that

$$\theta(1+z) = 2\sqrt{\frac{i}{z}}\,(A(4z)-A(z))$$

where $A(z) = \sum_1^\infty e^{-i\pi n^2/z}$. If $Im(\frac{-1}{z}) \ge 1$, then

$$|A(z)| \le 2\exp(-\pi Im(\frac{-1}{z})),$$

so that in that case

$$|\theta(1+z)| \le C|z|^{-1/2}\exp(-\pi Im(\frac{-1}{z})). \tag{29}$$

Proposition 3 *Let $\frac{p_n}{q_n}$ be the sequence of approximations of ρ by continued fractions; let τ_n be defined by*

$$|\rho - \frac{p_n}{q_n}| = (\frac{1}{q_n})^{\tau_n}. \tag{30}$$

If a, b, and n are such that

$$3|\rho - \frac{p_n}{q_n}| \le |b-\rho+ia| \le 3|\rho - \frac{p_{n-1}}{q_{n-1}}|, \tag{31}$$

the following estimates hold:

If p_n and q_n are not both odd but p_{n-1} and q_{n-1} are both odd

$$|C(a,b)| \le Ca^{\frac{1}{2}+\frac{1}{2\tau_n}}(1+\frac{|b-\rho|}{a})^{\frac{1}{2\tau_n}}. \tag{32}$$

If p_n and q_n are not both odd and p_{n-1} and q_{n-1} are not both odd

$$|C(a,b)| \le Ca^{\frac{1}{2}+\frac{1}{2\tau_n}}(1+\frac{|b-\rho|}{a})^{\frac{1}{2\tau_n}} \tag{33}$$

or

$$|C(a,b)| \le Ca^{\frac{1}{2}+\frac{1}{2\tau_{n-1}}}(1+\frac{|b-\rho|}{a})^{\frac{1}{2\tau_{n-1}}}. \tag{34}$$

If p_n and q_n are both odd

$$|C(a,b)| \leq Ca^{\frac{1}{2}+\frac{1}{2\tau_{n-1}}}(1+\frac{|b-\rho|}{a})^{\frac{1}{2\tau_{n-1}}}. \tag{35}$$

Furthermore, if p_n and q_n are not both odd, these estimates are optimal, which means that there exists a point in the domain (31) where (32) or (33) are equalities.

We remark that, since $\tau_n \geq 2$, this result together with Proposition 1 implies Hardy's result that $\varphi(x)-\varphi(x_0)$ is nowhere $o(|x-x_0|)^{3/4}$ except perhaps at the rational points that are quotients of two odd numbers. More precisely, we have

Corollary 1 *Let $\rho \notin \mathbb{Q}$; if there exists an infinity of integers n such that p_n and q_n are not both odd and such that $\tau_n \geq \tau$, then*

$$\varphi(x)-\varphi(x_0) \quad \textit{is not} \quad o(|x-x_0|)^{\frac{1}{2}+\frac{1}{2\tau}},$$

but if there exists N such that $\tau_n \leq \tau$ for any $n \geq N$ (for which p_n and q_n are not both odd), then

$$\varphi(x)-\varphi(x_0) = O(|x-x_0|)^{\frac{1}{2}+\frac{1}{2\tau}}.$$

Define $\Gamma^\alpha(x_0)$ as the set of functions f such that

$$\begin{cases} \forall\beta>\alpha & f \notin C^\beta(x_0) \\ \forall\beta<\alpha & f \in C^\beta(x_0). \end{cases}$$

If $\eta(\rho) = \limsup \tau_n(\rho)$, where the lim sup bears only on the n's such that p_n and q_n are not both odd, this result implies that $\varphi \in \Gamma^{\frac{1}{2}+\frac{1}{2\eta(\rho)}}(\rho)$. We will prove this proposition in the two next sections, and in the following one, we will show how to derive the local scalings at quadratic irrationals.

5. The case when p_n and q_n are not both odd

Let us now recall a few properties of approximations by continued fractions. Since

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}, \tag{36}$$

we have

$$\frac{1}{q_n q_{n+1}} = |\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}| \geq |\rho - \frac{p_n}{q_n}|$$

because (see [10]) $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$ are not on the same side of ρ; on the other hand,

$$\frac{1}{q_n q_{n+1}} = |\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}| \leq 2|\rho - \frac{p_n}{q_n}|$$

so that

$$(\frac{1}{q_n})^{\tau_n-1} \leq \frac{1}{q_{n+1}} \leq 2(\frac{1}{q_n})^{\tau_n-1}. \tag{37}$$

We first determine $\gamma_n = \dfrac{r_n x + s_n}{q_n x - p_n}$ in the theta modular group such that the pole of γ_n is p_n/q_n. Because of (36) if p_{n-1} and q_{n-1} are not both odd, we can choose

$$r_n = (-1)^n q_{n-1},\ s_n = (-1)^{n+1} p_{n-1};$$

the corresponding transform satisfies (24) and thus belongs to the theta modular group, and if p_{n-1} and q_{n-1} are both odd, we can choose

$$r_n = (-1)^n q_{n-1} + q_n,\ s_n = (-1)^{n+1} p_{n-1} - p_n.$$

Since

$$\gamma_n(\frac{p_n}{q_n} + z) = \frac{r_n}{q_n} - \frac{1}{q_n^2 z},$$

applying (25) to $\dfrac{p_n}{q_n} + z$ and γ_n leads to

$$|\theta(\frac{p_n}{q_n} + z)| = |\theta(\frac{r_n}{q_n} - \frac{1}{q_n^2 z})| \frac{1}{\sqrt{q_n|z|}} \tag{38}$$

Since $Im(\dfrac{-1}{q_n^2 z}) = \dfrac{Im(z)}{q_n^2|z|^2}$, we consider the two following cases.

FIRST CASE: $\dfrac{Im(z)}{q_n^2|z|^2} \geq 1$; then (38) and (26) imply that

$$|\theta(\frac{p_n}{q_n} + z)| \sim \frac{1}{\sqrt{q_n|z|}}$$

so that

$$|C(a,b)| \sim \frac{Ca}{\sqrt{q_n(a + |b-\rho|)}}$$

(note that here and hereafter, $\sim$ means that the two quantities are equivalent, the constants in the equivalence being independent of n). Because of (31),

$$(a + |b - \rho|) \geq \frac{1}{q_n^{\tau_n}},$$

so that

$$|C(a,b)| \leq C a^{\frac{1}{2\tau_n} + \frac{1}{2}} (1 + \frac{|b-\rho|}{a})^{\frac{1}{2\tau_n} - \frac{1}{2}};$$

and because of (26) this upper bound becomes an equality if we choose $a = \frac{1}{q_n^{\tau_n}}$, $b = 0$. This proves (32) and (33) in this case, as well as their optimality.

SECOND CASE: $\dfrac{Im(z)}{q_n^2|z|^2} \leq 1$; we separate this case into two subcases:

First Subcase: p_{n-1} and q_{n-1} are not both odd; then

$$|\theta(\frac{p_n}{q_n} + z)| \leq \frac{1}{\sqrt{q_n|z|}} \left(\frac{Im(z)}{q_n^2|z|^2}\right)^{-1/2} = \sqrt{\frac{q_n|z|}{Im(z)}} \tag{39}$$

so that, since $|z| \geq 2|\rho - \frac{p_n}{q_n}|$,

$$|C(a,b)| \leq 2\sqrt{aq_n(a + |b-\rho|)}. \tag{40}$$

Because of (31),

$$a + |b - \rho| \leq 6|\rho - \frac{p_{n-1}}{q_{n-1}}| \leq 6\left(\frac{1}{q_{n-1}}\right)^{\tau_{n-1}} \leq 6\left(\frac{1}{q_n}\right)^{\frac{\tau_{n-1}}{\tau_{n-1}-1}};$$

thus

$$\begin{aligned} |C(a,b)| &\leq Caq_n^{1/2}(1 + \frac{|b-\rho|}{a})^{1/2} \\ &\leq Ca(\frac{1}{a+|b-\rho|})^{\frac{\tau_{n-1}-1}{2\tau_{n-1}}}(1 + \frac{|b-\rho|}{a})^{1/2} \\ &\leq Ca^{\frac{1}{2}+\frac{1}{2\tau_{n-1}}}(1 + \frac{|b-\rho|}{a})^{\frac{1}{2\tau_{n-1}}}; \end{aligned} \tag{41}$$

which proves (34).

Second Subcase: p_{n-1} and q_{n-1} are both odd; then

$$r_n = (-1)^n q_{n-1} + q_n, \; s_n = (-1)^{n+1} p_{n-1} - p_n.$$

We now want to estimate θ near the points p_n/q_n where p_n and q_n are both odd; we will deduce this estimate from (29).

We see that (38) becomes

$$|\theta(\frac{p_n}{q_n} + z)| = |\theta(\frac{(-1)^n q_{n-1}}{q_n} - \frac{1}{q_n^2 z} + 1)| \frac{1}{\sqrt{q_n|z|}}$$

let

$$g_n(z) = \frac{(-1)^n q_n q_{n-1} z - 1}{q_n^2 z}$$

From (28), we get

$$|\theta(\frac{p_n}{q_n} + z)| = \frac{1}{\sqrt{q_n|z||g_n(z)|}} |\theta(\frac{-1}{4g_n(z)}) - \theta(\frac{-1}{g_n(z)})|.$$

Remark that

$$Im(\frac{-1}{g_n(z)}) = \frac{Im(g_n(z))}{|g_n(z)|^2} = \frac{Im(z)}{q_n^2|z|^2|g_n(z)|^2}.$$

If $Im(\frac{-1}{g_n(z)}) \geq 1$, using (29),

$$\begin{aligned} |\theta(\frac{p_n}{q_n}+z)| &\leq \frac{1}{\sqrt{q_n|z||g_n(z)|}} \exp(-\pi \frac{Im(z)}{q_n^2|z|^2|g_n(z)|^2}) \\ &\leq \frac{1}{\sqrt{q_n|z||g_n(z)|}} \left(\frac{q_n^2|z|^2|g_n(z)|^2}{Im(z)}\right)^{1/4} \\ &\leq \left(\frac{1}{Im(z)}\right)^{1/4} \end{aligned}$$

so that $|C(a,b)| \leq a^{3/4}$; hence (32) in that case.

Suppose now that $Im(\frac{-1}{g_n(z)}) \leq 1$. Then

$$\begin{aligned} |\theta(\frac{p_n}{q_n}+z)| &\leq \frac{1}{\sqrt{q_n|z||g_n(z)|}} \left(Im(\frac{-1}{g_n(z)})\right)^{-1/2} \\ &\leq \frac{1}{\sqrt{q_n|z||g_n(z)|}} \frac{q_n|z||g_n(z)|}{\sqrt{Im(z)}} \\ &\leq \sqrt{\frac{q_n|z||g_n(z)|}{Im(z)}}. \end{aligned}$$

Because of (31), $|z| \leq \frac{6}{q_n q_{n-1}}$, so that $|g_n(z)| \leq \frac{7}{q_n^2|z|}$ and

$$|\theta(\frac{p_n}{q_n}+z)| \leq \sqrt{\frac{q_n|z|}{q_n^2|z|Im(z)}} \leq \frac{1}{\sqrt{q_n Im(z)}}.$$

Thus

$$|C(a,b)| \leq \frac{\sqrt{a}}{\sqrt{q_n}}$$

From (31), we have $(1/q_n)^{\tau_n} \leq (a+|b-\rho|)$ so that

$$|C(a,b)| \leq a^{\frac{1}{2}+\frac{1}{2\tau_n}} (1+\frac{|b-\rho|}{a})^{1\frac{1}{2\tau_n}},$$

which proves (32) in this case.

6. The case when p_n and q_n are not both odd

Following the same procedure as in the previous section, we first determine $\gamma_n = \frac{ax+b}{cx+d}$ such that $\gamma_n(p_n/q_n) = 1$. We choose either $r_n = q_{n+1}$, $s_n = p_{n+1}$ or $r_n = -q_{n+1}$, $s_n = -p_{n+1}$ such that

$$p_n r_n - s_n q_n = 1$$

(which is possible because of (36)). Now γ_n is defined by the coefficients

$$\begin{cases} a = & q_n + r_n \quad b = & -p_n - s_n \\ c = & r_n \quad d = & -s_n. \end{cases}$$

One easily checks that (24) holds and thus γ_n belongs to the theta modular group;

$$\gamma_n(\frac{p_n}{q_n} + z) = \frac{1 + q_n(r_n + q_n)z}{1 + r_n q_n z} = 1 + f_n(z)$$

with

$$f_n(z) = \frac{q_n^2 z}{1 + r_n q_n z}.$$

Because of (25), it follows that

$$|\theta(\frac{p_n}{q_n} + z)| = |\theta(1 + f_n(z))| \frac{1}{r_n^{1/2}(z + \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}})^{1/2}};$$

on the other hand (31) implies

$$|z| \geq 3|\rho - \frac{p_n}{q_n}| \geq \frac{3}{2}|\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}|,$$

so that

(42) $$|\theta(\frac{p_n}{q_n} + z)| \leq \frac{|\theta(1 + f_n(z))|}{r_n^{1/2}|z|^{1/2}}$$

Remark that the condition $Im(\frac{-1}{f_n(z)}) \geq 1$ is equivalent to $Im(z) \geq q_n^2|z|^2$. Thus we now consider the two following cases.

First Case: $Im(z) \geq q_n^2|z|^2$. In this case, because of (29),

$$\begin{aligned} |\theta(\frac{p_n}{q_n} + z)| &\leq C \exp(\frac{-\pi Im(z)}{q_n^2|z|^2}) \left(\frac{1}{|f_n(z) r_n z|}\right)^{1/2} \\ &\leq C \exp(\frac{-\pi Im(z)}{q_n^2|z|^2}) \frac{|1 + r_n q_n z|^{1/2}}{(|r_n z|)^{1/2}|q_n^2 z|^{1/2}}. \end{aligned}$$

Because of (30), $\frac{1}{|r_n q_n|} \leq (\frac{2}{q_n^{r_n}})$ and thus $|r_n q_n z| \geq 3/2$. Thus

$$\begin{aligned} |\theta(\frac{p_n}{q_n} + z)| &\leq C \exp(\frac{-\pi Im(z)}{q_n^2|z|^2}) \frac{1}{|z|^{1/2} q_n^{1/2}} \\ &\leq C \left(\frac{q_n^2|z|^2}{Im(z)}\right)^{\frac{1}{4}} \frac{1}{|z|^{1/2} q_n^{1/2}} = \frac{C}{Im(z)^{1/4}} \end{aligned}$$

so that $|C(a, b)| \leq a^{3/4}$ which proves (35) in this case

Second Case: $Im(z) \leq q_n^2|z|^2$. In this case, from (27), (28) and (42), we obtain

$$\begin{aligned} |\theta(\frac{p_n}{q_n}+z)| &\leq \frac{1}{\sqrt{|f_n(z) r_n z|}}|\theta(\frac{-1}{4f_n(z)})-\theta(\frac{-1}{f_n(z)})| \\ &\leq C\frac{q_n|z|^{1/2}}{\sqrt{r_n|f_n(z)|Im(z)}} \\ &\leq C\frac{|1+r_nq_nz|^{1/2}}{\sqrt{r_nIm(z)}} \leq C\frac{|q_nz|^{1/2}}{\sqrt{Im(z)}}. \end{aligned}$$

As above $|r_nq_nz| \geq 3/2$, so that

$$|C(a,b)| \leq Ca^{1/2}\sqrt{q_n(a+|b-\rho|)}.$$

Because of (31), $a+|b-\rho| \leq 3(\frac{1}{q_n})^{\frac{\tau_{n-1}}{\tau_{n-1}-1}}$ so that

$$|C(a,b)| \leq Ca^{\frac{1}{2}+\frac{1}{2\tau_{n-1}}}(1+\frac{|b-\rho|}{a})^{\frac{1}{2\tau_{n-1}}}.$$

Thus (35) holds in this case and Proposition 3 is proved.

The fact that (32) and (35) cannot be improved in a cone

$$Im(z-\rho) \geq CRe(z-\rho)$$

yields a slightly more precise information than the fact that φ is not smoother than $\frac{1}{2}+\frac{1}{2\eta(\rho)}$ at ρ because it shows that φ has no chirp expansion at an irrational point ρ (see [9]); thus the only points where φ has a chirp expansion are the rationals of the form odd / odd.

The existence of these sets of smooth points is not only a consequence of the kind of lacunarity introduced by the frequencies n^2, but also of the very special coefficents that are chosen, which creates an exceptional behavior, as shown by the following remark: if the coefficients were multiplied by Independant Identically Distributed Gaussians or Rademacher series (± 1), Theorem 4 (chap 8) of [11] shows that the corresponding random function would be almost surely nowhere $C^{1/2}$. It is hard to have an idea about how specific coefficients modify the spectrum or even the regularity at a given point. Under this respect, the following example is interesting. Consider the function

$$\varphi_\tau(x) = \sum e^{i\pi n\tau}\frac{1}{n^2}e^{i\pi n^2 x}$$

where τ is a parameter. For $\tau = 0$ we recover φ (or rather its analytic part, but this does not change its regularity at any point). For a generic τ one might think that we introduce phases which behave quite randomly. This is not the case and actually φ_τ has the same regularity as φ at any point. Let us sketch the proof of this result.

Using the same wavelet as above, we obtain that the wavelet transform of φ_τ is (with the same notations)

$$C(a,b) = a \sum e^{i\pi n^2 z} e^{i\pi n \tau}.$$

Let $\theta(z,\tau) = \sum e^{i\pi n^2 z} e^{i\pi n \tau}$; the changes $n \to n + l$ and $\tau \to \tau + 2m$ show that

$$|\theta(z,\tau)| = |\theta(z, \tau + 2lz + 2m)|$$

If z is irrational, the values of $lz + 2m$ are dense; hence by continuity,

$$|\theta(z,\tau)| = |\theta(z,0)|$$

which, again by continuity also holds if z is rational. Thus the wavelet transforms of φ_τ and φ have the same modulus everywhere, so that these functions have everywhere the same regularity.

7. Logarithmic chirps of Riemann's function and selfsimilar functions

We will prove first the following theorem.

Theorem 3 *Riemann's function has at every quadratic irrational a logarithmic chirp of regularity 1/2.*

Proof: If ω is a quadratic irrational, there exists $\gamma(z)$ in the theta modular group,

$$\gamma(z) = \frac{rz+s}{pz+q},$$

such that ω is invariant by γ (see [1]). Using (25), we see that

$$\theta(\omega + z) = \theta(\omega + \frac{r - p\omega}{q + p\omega} z + O(z^2)) \frac{e^{im\pi/4}}{\sqrt{q\omega + p}} (1 + O(z))$$

since

$$|\theta'(z)| \leq \frac{C}{(Im\ z)^{3/2}},$$

we have

$$\theta(\omega + z) = A\theta(\omega + Bz) + O(\frac{z^2}{(Im\ z)^{3/2}})$$

where B is a real number different from 1 or -1 (because $B = \pm 1$ would imply that ω is a rational). Iterating this relation, we can suppose that A is real. Thus, if $z = b - \omega + ia$,

$$\begin{aligned} C(a, \omega + b) &= a\theta(\omega + z) \\ &= Aa\theta(\omega + Bz) + O(a\frac{z^2}{a^{3/2}}) \\ &= \frac{A}{B} C(BA, \omega + Bb) + a^{-1/2} O(a^2 + b^2) \end{aligned}$$

These last two terms are $O(a^{1/2})$ because φ is $C^{1/2}$ on $\mathbb{R}$, so that the rest is simultaneously $O(a^{1/2})$ and $O(a^{-1/2}(a^2+b^2))$. Hence by a weighted geometric average, it is $O(a^\alpha(1+\frac{|b|}{a})^{\alpha'}$, where we can choose $\alpha' < \alpha$ and α arbitrarily close to 1. Since Proposition 3 implies that φ is $C^{3/4,-1/4}$ at quadratic irrationals, Theorem 2 applies, and the conclusion of Theorem 3 follows.

Let us now consider the following functions

$$f(x) = \sum_1^\infty \lambda^{-\alpha j} g(\lambda^j x) \tag{43}$$

where g has some Hölder continuity, vanishes at the origin and has at most polynomial growth. This situation includes of course Weierstrass functions (where $g(x) = \sin 2\pi x$), but also many examples which appear in the literature about "fractal functions". They are particular examples of the "selfsimilar functions introduced and studied in [8], and the reader will easily check that the following study immediately extends to this setting.

Proposition 4 *The function (43) has a logarithmic chirp at the origin of order (α,λ) with the same regularity as g. The Fourier coefficients of the associated function G are given by the Mellin transform Mg of g as follows:*

$$c_n = Mg(-\alpha - \frac{2i\pi n}{\log \lambda})$$

Proof: Note that f is almost written as a logarithmic chirp, so that by inspection, one gets

$$G(x) = \sum_{j=-\infty}^{+\infty} \lambda^{-\alpha j} e^{-\alpha x} g(\lambda^{\alpha j} e^x)$$

Since G is obtained by a periodization of $e^{-\alpha x} g(e^x)$, its Fourier coefficients are

$$c_n = \int_{\mathbb{R}} e^{-\alpha x} g(e^x) e^{-2i\pi n x \log \lambda} dx = \int_0^\infty u^{-\alpha} g(u) u^{-2i\pi n \log \lambda} \frac{du}{u}$$

Thus for Weierstrass functions

$$w_\alpha(x) = \sum_1^\infty 2^{-\alpha j} \sin 2\pi 2^j x$$

and we obtain in this case

$$c_n = \Gamma(-\alpha - \frac{2i\pi n}{\log 2}) \sin \frac{\pi}{2}(-\alpha - \frac{2i\pi n}{\log 2}) \tag{44}$$

One easily checks that, if g is periodic of period 1 (which is the case for Weierstrass functions), f has logarithmic chirps at the rational points whose Fourier coefficients are given by the Mellin transform of a finite linear combination of g and some of its dilations and translations.

References

1. J.J.Duistermaat, *Selfsimilarity of Riemann's non-differentiable function.* Nieuw Archief voor Wiskunde 9 (1991) 303-337.

2. J. Gerver, *The differentiability of the Riemann function at certain rational multiples of π.* Am. J. Math. 92 (1970), 33-55.

3. G.H.Hardy, *Weierstrass's non-differentiable function .* Trans. AMS 17 (1916), 301-325 .

4. G.H.Hardy and J.E.Littlewood, *Some problems of Diophantine approximation.* Acta Mathematica 37 (1914), 193-239.

5. M. Holschneider & P. Tchamitchian, *Pointwise analysis of Riemann's "non differentiable" function.* Inventiones Mathematicae 105 (1991), 157-176.

6. M. Holschneider, *On the wavelet transformation of fractal objects* Journal of Statistical Physics 5/6 (1988), 963-993.

7. S.Jaffard, *Pointwise smoothness, two-microlocalization and wavelet coefficients* Publicacions Matematiques 35 (1991), p.155-168.

8. S.Jaffard, *Multifractal Formalism for functions Part I: Results valid for all functions and Part II: Selfsimilar functions* Preprint (1993).

9. S.Jaffard S, *The spectrum of singularities of Riemann's function* Preprint (1994).

10. S.Jaffard and Y.Meyer Y, *Pointwise behavior of functions* Preprint (1993).

11. J.P.Kahane, *Some Random Series of Functions* Cambridge Studies in Advanced Mathematics 5 (1993).

12. G.Valiron, *Théorie des fonctions* Masson (1942).

C.E.R.M.A., Ecole Nationale des Ponts et Chaussées, La Courtine, 93167 Noisy-le-Grand, France
and C.M.L.A., E.N.S. Cachan, 61 av. du Président Wilson, 94235 Cachan Cedex, France
E-mail address: sj@cerma.enpc.fr