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LETTER TO THE EDITOR

Functions with Prescribed Hölder Exponent

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Abstract — We characterize the class of functions that are Hölder exponents, and for each function $\alpha(x)$ of this class we give an explicit wavelet construction of a function f whose Hölder exponent is $\alpha(x)$ for every x. © 1995 Academic Press, Inc.

A bounded function f is $C^\alpha(x_0)(\alpha \geqslant 0)$ if there exists a polynomial P of degree at most $[\alpha]$ such that $|f(x) - P(x-x_0)| \leqslant C|x-x_0|^\alpha$. The *Hölder exponent* of f at x_0 (which we will denote $\alpha(x_0)$) is by definition the supremum of all αs such that f is $C^\alpha(x_0)$; this function takes its values in $[0, +\infty]$ (bounds included). The determination of the Hölder exponent of a function is often a difficult problem since it can be a very erratic function. Wavelet methods give an easy estimation of the Hölder regularity (see Proposition 1 below) and consequently they allowed to determine the exact Hölder exponent of some classical functions, see [7]. Numerically, even when the Hölder exponent is impossible to compute, one often wishes to obtain at least the dimension of the level sets of $\alpha(x)$; the Multifractal Formalism for Functions was introduced for that purpose and wavelet methods also proved their efficiency in this problem ([1, 4, 6]).

Let ψ^i be orthogonal wavelets in $\mathcal{S}(\mathbb{R}^n)$ as in [8]. The wavelet coefficients of f are denoted

$$C^i_{j,k} = \langle f | \psi^i_{j,k}\rangle = \int f(x) 2^{nj}\psi^i(2^jx-k)\,dx \quad (j \in \mathbb{Z}, k \in \mathbb{Z}^n).$$

We note $C^{\log}$ the class of functions such that

$$|C^i_{j,k}| \leqslant C2^{-j/\log j}. \tag{1}$$

It is a slightly stronger assumption than uniform continuity, but it implies no uniform Hölder regularity. More precisely if $\mu(t) = 1/(\log\log(1/t))$

$$\forall x,y \quad |f(x)-f(y)| \leqslant C|x-y|^{\mu(|x-y|)} \Rightarrow f \in C^{\log}$$
$$f \in C^{\log} \Rightarrow \forall x,y \quad |f(x)-f(y)| \leqslant (C/\mu(|x-y|))|x-y|^{\mu(|x-y|)}.$$

THEOREM 1. *A nonnegative function $\alpha(x)$ is a Hölder exponent of a function f in $C^{\log}$ if and only if it can be written as a limit inf of a sequence of continuous functions.*

This theorem answers a question posed by Lévy-Véhel (see [3]). The important point in its proof is the explicit wavelet construction of a function f which has a prescribed spectrum α. Remark that when α has a minimal Hölder regularity, a more natural probabilist construction is supplied by the *Fractional Brownian Motion* studied in [2].

We note $\lambda\ (=\lambda(j,k)) = k2^{-j}$. The following proposition is a slight extension of Theorem 1 of [5] and has a very similar proof, which we leave as an exercice.

PROPOSITION 1. *Suppose that $f \in C^\alpha(x_0)$; if $|k2^{-j} - x_0| \leqslant \frac{1}{2}$ then*

$$|C^i_{j,k}| \leqslant C2^{-\alpha j}(1 + |2^jx_0 - k|)^\alpha. \tag{2}$$

Conversely, if (2) *holds for all j,k such that $|\lambda - x_0| \leqslant 2^{-j/(\log j)^2}$, and if* (1) *holds, there exists a polynomial P of degree at most $[\alpha]$ such that*

$$|f(x) - P(x-x_0)| \leqslant C|x-x_0|^\alpha(\log|x-x_0|)^2. \tag{3}$$

Proof of Theorem 1. Suppose that f satisfies the uniform regularity condition (1); then (2) and (3) imply that

$$\alpha(x) = \liminf_{|\lambda - x| \leqslant 2^{-j/(\log j)^2}} \frac{\log|C^i_{j,k}|}{\log(2^{-j} + |\lambda - x|)}; \tag{4}$$

let $\theta_0(x)$ be a C^∞ positive function such that $\theta_0(x) = 0$ if $|x| \leqslant 1$ and $\theta_0(x) = 1$ if $|x| \geqslant 2$. We define $\theta_1(x) = 1 - \theta_0(x)$ and

$$\alpha_j(x) = \inf_{i,k}\left\{\frac{\log(2^{-j\log j} + |C^i_{j,k}|)}{\log(2^{-j} + |\lambda - x|)}\,\theta_0\left(\frac{|\lambda - x|}{2^{-j/(\log j)^2}}\right) + j\theta_1\left(\frac{|\lambda - x|}{2^{-j/(\log j)^2}}\right)\right\};$$

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$\alpha_j(x)$ is continuous and (4) implies that $\alpha(x) = \liminf \alpha_j(x)$ so that a Hölder exponent of a function f in $C^{\log}$ is a limit inf of a sequence of continuous functions. Remark that if $0 \leqslant \alpha(x) \leqslant 1$,

$$\alpha(x) = \liminf_{j\to\infty} \inf_{2^{-j}\leq|h|\leq 2.2^{-j}} \frac{\log(|f(x+h) - f(x)| + 2^{-j^2})}{\log h}$$

so that the characterization of Theorem 1 holds for all continuous functions if $0 \leqslant \alpha(x) \leqslant 1$.

Let us now prove the converse result; we suppose that $\beta(x)$ is a limit inf of a sequence of continuous functions $\beta_n(x)$; the problem is local so that we can suppose that the β_n are uniformly continuous; thus there exist C^1 functions γ_n such that $|\gamma_n(x) - \beta_n(x)| \leqslant C/n$; let $A(n) = n + \sup|\nabla\gamma_n(x)|$; we define the wavelet coefficients of f as follows. If j is one of the numbers $[A(n)]$,

$$C^i_{j,k} = \inf(2^{-j/\log j}, 2^{-j\gamma_n(\lambda)})$$

else we take $C^i_{j,k} = 0$. Let $\alpha(x)$ be the Hölder exponent of f at x. The direct part of Proposition 1 obviously implies that

$$\begin{aligned}\alpha(x) &\leqslant \liminf_{[\lambda-2^{-j},\lambda+2^{-j}]\ni x} \gamma_n(\lambda)\\ &\leqslant \liminf \gamma_n(x) + 2^{-j}A(n) = \liminf \beta_n(x)\end{aligned}$$

so that $\alpha(x) \leqslant \beta(x)$. In order to prove the converse estimate remark that f satisfies (1) so that we can use the second part of Proposition 1. We have

$$\begin{aligned}C^i_{j,k} &= \inf(2^{-j/\log j}, 2^{-j\gamma_n(\lambda)})\\ &\leqslant \inf(2^{-j/\log j}, 2^{-j\gamma_n(x)}2^{j|x-\lambda|A(n)}).\end{aligned}$$

Since $|\lambda - x| \leqslant 2^{-j/(\log j)^2}, 2^{j|x-\lambda|A(n)} \leqslant 2$ for n (hence j) large enough and Proposition 1 implies that Hölder exponent at x is exactly $\liminf \gamma_n(x)$.

EXAMPLE 1. Let $\alpha(x)$ be a positive measurable function in L^1. After perhaps redefining α on a set of measure 0, we can suppose that

$$\forall x \quad \alpha(x) = \liminf_{r\to 0} \frac{1}{Vol(B(x,r))}\int_{B(x,r)} \alpha(u)\,du.$$

Let $\phi(x)$ be a positive C^∞ function supported in $[-1,1]$ of integral 1; then $\alpha(x) = \liminf \alpha * (1/n)\phi(\cdot/n)$ so that $\alpha(x)$ is a Hölder exponent.

EXAMPLE 2. Let $\beta_1 > \beta_0 > 0$; there exists a function f such that $\alpha(x) = \beta_1$ is x is a rational, and $\alpha(x) = \beta_0$ if x is an irrational.

We define the wavelet coefficients of f as follows. If there exists a rational p/q such that $|\lambda - p/q| \leqslant 2^{-j/(\log j)^2}$ and $j \geqslant q$, then $C_\lambda = 2^{-\beta_1 j}$, else $C_\lambda = 2^{-\beta_0 j}$.

EXAMPLE 3. If $\beta_0 > \beta_1 > 0$, there exists no function f such that $\alpha(x) = \beta_1$ if x is a rational, and $\alpha(x) = \beta_0$ if x is an irrational.

Suppose such a function exists and let β_2 be such that $\beta_0 > \beta_2 > \beta_1 > 0$. We construct a decreasing sequence of closed intervals I_n with nonempty interiors: suppose I_{n-1} is constructed. There exists a dyadic interval $\lambda \subset I_{n-1}$ such that $|C_\lambda| \geqslant 2^{-\beta_2 j}$ (else f would be C^{β_2} on an open subinterval of I_{n-1}). We choose for I_n a subinterval of λ which contains no rational of denominator less than n. The sequence I_n decreases towards an irrational ρ and $\alpha(\rho) \leqslant \beta_2$.

Remark that in this counterexample, we do not need to make any uniform regularity assumption on f.

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