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MULTIFRACTAL FORMALISM FOR FUNCTIONS PART II: SELF-SIMILAR FUNCTIONS*

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Abstract. In this paper we introduce and study the *self-similar functions.* We prove that these functions have a concave spectrum (increasing and then decreasing) and that the different formulas that were proposed for the multifractal formalism allow us to determine either the whole increasing part of their spectrum or a part of it. One of these methods (the wavelet-maxima method) yields their complete spectrum.

Key words. multifractal formalism, self-similarity, wavelet transform

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1. Introduction.

We proved in Part I that the multifractal formalism always yields an upper bound for the Hölder spectrum. Hence a very natural and important question arises: When does it yield the exact spectrum? Partial results exist for multifractal measures (especially when multinomial or for invariant measures of dynamical systems; see [6] and [8]). Hence we have similar results for the functions that are indefinite integrals of these measures. Apart from these functions, no general results hold. A few examples have been worked out: the scaling functions ϕ that appear in wavelet constructions [12], Riemann's nondifferentiable function [20], and the "peano-type" function of Polya [21]. These three examples exhibit a common feature: their graphs follow locally some self-similarity conditions. This is explicitly stated below in (2.7) for Riemann's function. The recursive definition of Polya's function is an exact self-similarity condition, and this is also the case for the scaling equation of the ϕ functions.

We can thus infer that the multifractal formalism probably holds if the function considered exhibits some kind of self-similarity. Of course, this assertion is very vague, and we are now far from guessing what is the weakest form of local self-similarity that implies the validity of the multifractal formalism. Our purpose is to verify it for a case study, i.e., under some restrictive assumptions for the self-similarity conditions. These assumptions are listed in Definition 2.1 below. In our opinion, this partial result is interesting for two reasons: 1) it is the first proof of the validity of the multifractal formalism for a several-parameter family of functions different from indefinite integrals of measures; 2) we also expect the methods we introduce to extend to more general settings. There is already some evidence for this. Since the first preprint version of this paper, Daubechies showed that some of the results concerning scaling functions can be deduced from our study [11]. Slimane showed that our restricive conditions concerning the contractions S_i in Definition 2.1 can be weakened [4].

2. Basic properties of self-similar functions.

In this section, we recall the definition of self-similar functions, give some examples, and derive their basic properties.

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In the following two sections, we obtain the exact regularity of these functions at any point when the functions considered have uniform minimal regularity.

In section 4, we deduce these functions' Hölder spectra in the aforementioned case.

In section 5, we prove the validity of the multifractal formalism.

In section 6, we study the wavelet-maxima method. We show that after a slight modification, this method yields the complete spectrum, including the part where the infimum in the Legendre transform is obtained for negative values of q.

In section 7, we consider the more general case of unbounded self-similar functions.

We first recall the definition of self-similarity that was established in Part I.

DEFINITION 2.1. *A function $F : \mathbb{R}^m \to \mathbb{R}$ is self-similar* (*of order $k \in \mathbb{R}^+$*) *if the following three conditions hold:*

- *There exists a bounded open set Ω and $S_1, \dots, S_d$ contractive similitudes such that*

$$S_i(\Omega) \subset \Omega, \tag{2.1}$$

$$S_i(\Omega) \cap S_j(\Omega) = \emptyset \quad \textit{if } i \neq j. \tag{2.2}$$

(*The S_i's are the product of an isometry with the mapping $x \to \mu_i x$, where $\mu_i < 1$.*)

- *There exists a C^k function g such that g and its derivatives of order less than k have fast decay and F satisfies*

$$F(x) = \sum_{i=1}^{d} \lambda_i F(S_i^{-1}(x)) + g(x), \tag{2.3}$$

where the λ_i's are real or complex numbers.

- *The function F is not uniformly C^k in a certain closed subset of Ω.*

The first condition was first introduced by Hutchinson (in [17]) in order to study self-similar sets; it is called the "open-set condition." A stronger condition is sometimes required, namely,

$$S_i(\bar{\Omega}) \cap S_j(\bar{\Omega}) = \emptyset \quad \text{if } i \neq j; \tag{2.4}$$

this is called the "separated open-set condition."

Concerning the last point of the definition, if k is an integer, the condition must be understood as follows. Once restricted to a closed subset A of Ω, the derivatives of order $k-1$ of F do not belong to the Zygmund class. Thus for any $k \in \mathbb{R}^+$, this condition is equivalent to the existence of sequences $a_n \to 0$, $b_n \in A$, and $C_n \to \infty$ such that

$$|C(a_n, b_n)| \geq C_n a_n^k. \tag{2.5}$$

(This condition is a straightforward consequence of the wavelet characterization of the spaces $C^s(\mathbb{R}^m)$ that we recall below and of the localization of the wavelets.)

We will see that solutions of (2.3) need not necessarily be functions but can be distributions.

Recall the following notations introduced in Part I. Let

$$\tilde{Z}(a, q) = \int_{\mathbb{R}^m} |C(a, b)|^q db.$$

Then

$$\eta(q) = \liminf \frac{\log \tilde{Z}(a,q)}{\log a}.$$

Let

$$\alpha_{\min} = \inf_{i=1,\ldots,d} \left(\frac{\log \lambda_i}{\log \mu_i} \right), \qquad \alpha_{\max} = \sup_{i=1,\ldots,d} \left(\frac{\log \lambda_i}{\log \mu_i} \right).$$

We use this notation because $\alpha_{\min}$ will turn out to be the smallest pointwise Hölder regularity exponent of F and $\alpha_{\max}$ will be the largest (lower than k). Let τ be the function defined by

$$\sum_{i=1}^{d} \lambda_i^a \mu_i^{-\tau(a)} = 1.$$

The results concerning the multifractal formalism for self-similar functions are summed up in the following theorems.

THEOREM 2.2. *Suppose that F is self-similar. If $\alpha_{\min} > 0$, the function $d(\alpha)$ vanishes outside the interval $[\alpha_{\min}, \alpha_{\max}] \cup [k, +\infty)$ and is analytic and concave on $[\alpha_{\min}, \alpha_{\max}]$. Its maximal value $d_{\max}$ satisfies*

$$\sum \mu_i^{d_{\max}} = 1.$$

Let α_0 be the value for which this maximum is attained. First, suppose that g is C^∞. If $\alpha \le \alpha_0$, $d(\alpha)$ can be obtained by computing the Legendre transform of $\eta(q) - m$.

If g is only C^k, let p_0 be defined by $\eta(p_0) = kp_0$ and let $\alpha_1 < \alpha_0$ be the value of the inverse Legendre transform of $\eta(q) - m$ at p_0. If $\alpha \le \alpha_1$, $d(\alpha)$ can be obtained by computing the Legendre transform of $\eta(q) - m$.

Without any assumption on $\alpha_{\min}$, if $\sum |\lambda_j| \mu_j^m < 1$, the same results hold if we replace $d(\alpha)$ with $D'(\alpha)$, the packing dimension of the wavelet α-singularities (or by $D(\alpha)$, if g and the λ_i's are positive, and furthermore if the separated open-set condition holds).

The corresponding results concerning the *wavelet-maxima method* will be stated and proved in section 6 (see Theorem 2.3).

Before beginning our study of self-similar functions, we consider a few examples.

(1) *Indefinite integrals of multinomial measures in dimension* 1.

Let μ be a probability measure supported by $[0,1]$ and suppose that for any interval I,

$$\forall i = 1, \ldots, d, \quad \mu(S_i(I)) = \lambda_i \mu(I)$$

with $\sum \lambda = D1i = 1$, the S_i's as above, and $\Omega = (0,1)$. Let

$$F(x) = \left(\int_0^x d\mu \right) - x \quad \forall x \in [0,1].$$

F vanishes at 0 and 1, and is smooth outside the intervals $S_i([0,1])$. One immediately checks that F is continuous and

$$\forall x \in S_i([0,1]), \quad F(x) = \lambda_i F(S_i^{-1}(x)) + g_i(x)$$

with g_i linear. Thus F is self-similar.

For any probability measure μ on $\mathbb{R}$, the scaling index of μ at x_0 is the supremum of all values of α such that

$$\exists C > 0, \forall \varepsilon > 0, \quad \mu([x_0 - \varepsilon, x_0 + \varepsilon]) \leq C\varepsilon^{\alpha}.$$

We can easily check that μ has a scaling index α at x_0 if and only if its indefinite integral F defined by $F(x) = \mu([0, x])$ is C^{α} at x_0 (see [2] or [19]). This property allowed Arneodo, Bacry, and Muzy [2] to determine the Hölder spectrum of the indefinite integrals of multinomial measures when the separated open-set condition holds. This remark shows that when F is the indefinite integral of a one-dimensional measure, some results derived in this paper are a consequence of similar results concerning measures (for $\alpha \in [0, 1]$) obtained by Brown, Michon, and Peyrière in [6]. Thus we are particularly interested in the case of functions that are not in bounded variation (BV), in which case Theorem 2.2 cannot be derived from corresponding results for measures.

(2) *Some self-similar fractal sets.* Consider, for instance, the example of the Van Koch set. Since it is a curve, it can be parametrized (in infinite ways) as the image of a mapping $t \to (x(t), y(t))$ from $[0, 1]$ to $\mathbb{R}^2$. This curve has dimension $\log 4/\log 3$ and a corresponding finite nonzero Hausdorff measure. Therefore, a *canonical* parametrization maps intervals of same length on sets of equal Hausdorff measure. The reader will immediately check that with *this* parametrization the Van Koch function is self-similar. Another example is supplied by Polya's function, a continuous mapping defined on $[0, 1]$ whose graph fills the area of a triangle. However, the lack of regularity of the function g in this case requires a specific treatment, and we plan to study the local regularity of this function in a forthcoming paper.

(3) *Lacunary trigonometric series and Riesz products.* Let

$$F_{\alpha}(x) = \sum_{j=0}^{\infty} 2^{-\alpha j} \sin 2\pi 2^j x$$

for $x \in [0, 1]$ and $0 < \alpha \leq 1$. Define

$$\begin{aligned} g(x) &= \sin 2\pi x \quad \text{if} \ \ x \in [0,1] \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Obviously,

$$F_{\alpha}(x) = 2^{-\alpha} F_{\alpha}(2x) + 2^{-\alpha} F_{\alpha}(2x - 1) + g(x)$$

so that the F_{α}'s are self-similar.

Another example is very similar. Consider the Riesz products

$$F_{\alpha,k}(x) = \prod_{j=1}^{\infty} (1 + 2^{-\alpha j} \sin(k^j x)),$$

where $0 < \alpha < 1$ and $k \in \mathbb{N}$, $k \geq 2$. It is a simple exercise to check that $\log(F_{\alpha,k}(x))$ is self-similar (the function g being $C^{\inf(2\alpha,1)}$). Since $F_{\alpha,k}(x)$ is bounded from above and below by strictly positive constants, $F_{\alpha,k}(x)$ and its logarithm share the same function η and the same spectrum so that the results that will be proved for self-similar functions will also hold for the Riesz products $F_{\alpha,k}(x)$.

(4) *Several dimension examples.* In dimension 1, the "geometry" contained in the transforms S_i is poor. In several dimensions, this is sometimes no longer the case. Consider two examples. First, if

$$\Omega = [-1,1]^2, \quad \text{let } i,j \in \left\{\frac{1}{2}, -\frac{1}{2}\right\}, \quad \text{and} \quad S_{i,j}(x) = \frac{1}{2}x + (i,j).$$

The $S_{i,j}$'s map the square Ω on its four subsquares of half size. If the homothety had a ratio smaller than $1/2$, by iterating the $S_{i,j}$'s, we would get a kind of two-dimensional Cantor set. There exist more "exotic" examples. For instance, if

$$S_1^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad S_2^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} + (1,0),$$

the S_i's map a "fractal dragon" on their two self-similar components (see [9] or [16]).

In order to understand the scope and limitations of the model given by self-similar functions, it is interesting to mention a few classical examples of functions that, though not self-similar in the sense that we gave, satisfy functional equations that have similarities with (2.3).

First, the scaling function of the function φ used in the construction of orthonormal wavelet bases satisfies (see [10] or [12])

$$\varphi(x) = \sum a_k \varphi(2x-k),$$

but condition (2.2) does not hold except for some nonsmooth functions φ, such as characteristic functions of sets, in which case there exist examples similar to the fractal dragon that we mentioned above (see [9] or [16]). In [12], Daubechies and Lagarias recently proved that a converse formula to the multifractal formalism holds for these functions (i.e., the function $\eta(p)$ is the Legendre transform of $d(\alpha)$).

Our second example is the Brownian bridge on $[0,1]$. It satisfies

$$B(t) = \frac{1}{\sqrt{2}}B_1(2t) + \frac{1}{\sqrt{2}}B_2(2t-1) + \xi\Lambda(x), \tag{2.6}$$

where B_1 and B_2 are two Brownian bridges that have the same law as B, ξ is a Gaussian, $\Lambda(x) = \sup(x, 1-x)$ on $[0,1]$, and the three terms of the right-hand side of (2.6) are independent. We are in a situation where (2.3) holds "in law." Actually, the self-similar processes studied in, for instance, [3] also verify (2.3) "in law." We will not recall the definition of these processes here but only mention that in dimension 1, they coincide with the fractional Brownian motions. The reader can easily check that the results that we give below easily extend to this probabilistic setting. However, such results would be poor for the following reason. Direct methods yield sharper results for the the pointwise regularity of these processes and from a "multifractal point of view," the spectrum of these processes is not interesting since it vanishes everywhere except at one point.

Our last example is *Riemann's nondifferentiable function*

$$\Phi(x) = \sum_1^\infty \frac{1}{n^2}\sin(\pi n^2 x),$$

which was shown by Duistermaat in [13] to satisfy some functional equations similar to (2.3). For instance

$$\Phi(1+x) = \frac{\pi i}{12} - \frac{x}{2} + e^{i\pi/4}x^{3/2}\Big(4\Phi\Big(\frac{-1}{4x}\Big) - \Phi\Big(\frac{-1}{x}\Big)\Big) + \xi(x), \tag{2.7}$$

where ξ is a smooth function. Also, using the periodicity of Φ, a whole collection of similar equations can be derived. We are in a situation close to (2.3) but where the S_i's are not linear. (See [20] for an extension of the multifractal formalism to this case.)

We now determine the sense in which (2.3) has solutions, and we examine some basic regularity properties of these solutions. They will depend upon the assumptions that we make on the λ_i.

Iterating (2.3), we obtain for any N that

$$\begin{aligned} F(x) &= \sum_{n=0}^{N-1} \sum_{(i_1,\dots,i_n)} \lambda_{i_1}\cdots\lambda_{i_n} g\big(S_{i_n}^{-1}\dots S_{i_1}^{-1}(x)\big) \\ &\quad + \sum_{(i_1,\dots,i_N)} \lambda_{i_1}\cdots\lambda_{i_n} F\big(S_{i_N}^{-1}\cdots S_{i_1}^{-1}(x)\big)\,, \end{aligned} \tag{2.8}$$

so a (formal) solution of (2.3) is given by

$$F(x) = \sum_{n=0}^{\infty} \sum_{(i_1,\dots,i_n)} \lambda_{i_1}\cdots\lambda_{i_n} g\big(S_{i_n}^{-1}\cdots S_{i_1}^{-1}(x)\big)\,. \tag{2.9}$$

Here F is written as a superposition of similar structures at different scales, reminiscent of some possible models of turbulence [5], [15], [26]. This formula also looks like a wavelet decomposition (except that g has no cancellation), and our proof of Proposition 3.2 in the next section will be similar to classical proofs of the regularity of wavelets series; see [18] or [22].

When the (formal) series (2.9) converges in a certain function space, a solution of (2.3) exists in this space. (Actually, it is easy to check that (2.9) converges almost everywhere if the separated open-set condition holds.) We will be particularly interested in three cases: first, when (2.3) has solutions that are locally L^1 functions; second, when the solutions have some global C^α smoothness (this case is important because it is the setting where the multifractal formalism works without any modification); and third, in spaces of distributions where the series converge when we make no assumption on the λ_i's. A good setting to study this last case is supplied by the real Hardy spaces, whose definition we now recall.

Suppose that we use an orthonormal basis of wavelets indexed by dyadic cubes. We denote these wavelets by ψ_λ and the corresponding wavelet coefficients by C_λ. The *real Hardy $\mathcal{H}^p$ space* (cf. [24]) is the set of distributions whose wavelet coefficients satisfy

$$\int \Bigg(\sum_{j,k} |C_\lambda|^2\, 2^{nj}\, \chi_\lambda(x) \Bigg)^{p/2} dx < +\infty, \tag{2.10}$$

assuming that the wavelets that we use are $C^{m(p^{-1}-1)}$ and have vanishing moments up to order $m(p^{-1}-1)$. This is a direct generalization of the space L^p when $p<1$.

Recall that

$$\alpha_{\min} = \inf_{j=1,\dots,d} \frac{\log|\lambda_j|}{\log \mu_j} \quad \text{and} \quad \alpha_{\max} = \sup_{j=1,\dots,d} \frac{\log|\lambda_j|}{\log \mu_j}. \tag{2.11}$$

Proposition 2.3. *Suppose that* $\sum |\lambda_j| \mu_j^m < 1$. *In this case,* (2.3) *has a unique distribution solution, which is an* L^1 *function and is given by the series* (2.9). *Furthermore, if* $0 < \alpha_{\min} < k$, *this function is* $C^{\alpha_{\min}}$.

Suppose that $\sum |\lambda_j| \mu_j^m \geq 1$; *in that case,* (2.3) *may have several distribution solutions. Let* $p < 1$ *such that* $\sum |\lambda_j|^p \mu_j^m < 1$. *If* g *is* C^k *with* $k > m(p^{-1} - 1)$ *and if the moments of* g *of order less than* k *vanish,* (2.9) *converges in the Hardy real space* $\mathcal{H}^p$ *so that* (2.3) *has at least one solution in this space of distributions.*

Furthermore, these results are optimal.

Before proving Proposition 2.3, we begin with some preliminary results concerning the geometry of the mappings S_i. If A is a subset of $\mathbb{R}^n$, we define the mapping S by

$$S(A) = \bigcup_{i=1}^{d} S_i(A)$$

and let K be the set defined by

$$K = \bigcap_{n \in \mathbb{N}} S^n(\bar{\Omega}).$$

K is called the invariant compact set of S. Its Hausdorff dimension is $d_{\max}$ (defined in Theorem 2.2). We introduce some notation. Let i be a finite sequence $i = (i_1, \ldots, i_n)$. We define $x_i = S_{i_1} \cdots S_{i_n}(0)$, and if the sequence i is infinite, $x_i = \lim_{n\to\infty} x_{(i_1,\ldots,i_n)}$. Similarly, let $\mu_i = \mu_{i_1} \cdots \mu_{i_n}$ and $\lambda_i = \lambda_{i_1} \cdots \lambda_{i_n}$. Thus with each sequence $i \in \{1, \ldots, d\}^{\mathbb{N}}$ we associate a unique point x_i in K. This correspondence is, in general, not one to one. (Consider, for instance, the example of lacunary trigonometric series where the dyadic points are the limit of two sequences.) However, the correspondence is clearly one to one if the separated open-set condition holds.

The points of K can also be represented as the limit points of the branches of the following tree T constructed in the "time-scale half-space." The treetop is conventionally the point $(0, 1) \in \mathbb{R}^m \times \mathbb{R}^+$. This treetop is linked to the d first nodes, which are the $(S_j(0), \mu_j)$'s. This point $(S_j(0), \mu_j)$ is linked to $(S_j S_k(0),\ \mu_j \mu_k), \ldots$. If $\mathbb{R}^m$ is identified to $\mathbb{R}^m \times \{0\}$, then clearly the branch indexed by a sequence $i \in \{1, \ldots, d\}^{\mathbb{N}}$ approaches the point x_i (and it is the only one which does so if the mapping $i \to x_i$ is one to one). This tree is related to the wavelet transform of F more precisely in Proposition 4.1. We will show that the order of magnitude of the wavelet transform of F near (x_i, μ_i) is $|\lambda_i|$.

Definition 2.4. *Let* $x \in \mathbb{R}^m$. *A "D-branch over x" is a branch of the tree of length n that starts from the origin* $(0, 1)$, *ends at*

$$(S_{i_1} \cdots S_{i_n}(0), \mu_{i_1} \cdots \mu_{i_n}),$$

and is such that

$$|S_{i_1} \cdots S_{i_n}(0) - x| \leq D \mu_{i_1} \cdots \mu_{i_n}.$$

When $D \leq 10\mathrm{Diam}(\Omega)$, *such a branch is a "main branch over x."*

This requirement means that the endpoint of the branch is—in the time-scale half-space—in a certain cone of width D over x. We often identify a branch with the sequence i that indexes it.

We will need the following lemma, which estimates the number of D-branches over a point x.

Lemma 2.5. *Let $x \in K$ and let $B_{j,D}(x)$ be the set of D-branches $(i_1,\dots,i_n)$ over x such that*

$$2^{-j} \leq \mu_{i_1}\cdots\mu_{i_n} < 2.2^{-j}. \tag{2.12}$$

The cardinality of this set of branches is bounded independentely of x and j by CD^m.

Proof. We can assume that the $S_{i_1}\cdots S_{i_n}(\Omega)$'s are disjoint. If not, the open-set condition implies that the corresponding sequences $(i_1,\dots,i_n)$ and $(i'_1,\dots,i'_m)$ satisfy—if $n \leq m$, for instance—$(i_1,\dots,i_n) = (i'_1,\dots,i'_n)$. (One of the branches is included in the other.) In that case, we keep the longest sequence, dividing the cardinality of $B_{j,D}(x)$ by at most an absolute constant (which depends only on the values of $\mu_1,\dots,\mu_d$). Thus we can assume that the $S_{i_1}\cdots S_{i_n}(\Omega)$'s are disjoint and are all included in $B(x, CD2^{-j})$, so if $B_{j,D}(x)^{\#}$ denotes the cardinality of $B_{j,D}(x)$,

$$B_{j,D}(x)^{\#}2^{-mj}\mathrm{vol}(\Omega) \leq C(D.2^{-j})^m,$$

and thus $B_{j,D}(x)^{\#}$ is bounded by CD^m. Hence Lemma 2.5 follows. ∎

We now prove Proposition 2.3. Existence and uniqueness in the L^1 case are straightforward. The last term in (2.8) tends to zero in L^1, so that (2.9) is the only (possible) solution in L^1, and it is actually in L^1 because the L^1 norm of series (2.9) is bounded by

$$C\sum_{|i|\leq n}|\lambda_i|\mu_i^m = C\sum_{l\leq n}\left(\sum_{j=1}^{d}|\lambda_j|\mu_j^m\right)^l \leq C.$$

We estimate the C^s norm of F using the Littlewood–Paley characterization of this norm. For the reader's convenience, we recall this characterization.

Let ψ be a function in the Schwartz class whose Fourier transform vanishes outside $1 \leq |\xi| \leq 8$ and is equal to 1 on $2 \leq |\xi| \leq 4$. Let $\psi_l(x) = 2^{ml}\psi(2^l x)$. A function F belongs to C^s if and only if $|F * \psi_l(x)| \leq C2^{-sl}$.

We return to Proposition 2.3. We first split F as a sum $F = \sum F_j$, where F_j is series (2.9) restricted to the indices $i \in I_j$ such that

$$2^{-j} \leq \mu_i < 2.2^{-j}.$$

Let $\omega_{l,j} = F_j * \psi_l$. If $l \geq j$, because of the localization and cancellation of ψ, for any N,

$$|\omega_{l,j}(x)| \leq C_N \sum_{i\in I_j}\frac{\lambda_i 2^{-k(l-j)}}{(1+2^j|x-x_i|)^N}.$$

Because of Lemma 2.5, as soon as $N > m$,

$$\sum_{i\in I_j}\frac{1}{(1+2^j|x-x_i|)^N} \leq C$$

so that

$$|\omega_{l,j}(x)| \leq C\sup_{i\in I_j}|\lambda_i|2^{-k(l-j)}.$$

If $j > l$, $|\omega_{l,j}(x)| \le C \sup |F_j(x)|$ so that $|\omega_{l,j}(x)| \le C \sup_{i\in I_j} |\lambda_i|$. Since $\sup_{i\in I_j} |\lambda_i| \le C 2^{-\alpha_{\min} j}$, summing up, we obtain that $|(F * \psi_l)(x)| \le C 2^{-\alpha_{\min} l}$. Hence we have the Hölder regularity of F.

In order to show that F belongs to $\mathcal{H}^p$, first notice that the regularity and cancellation that we requested for g is consistent with the atomic definition of $\mathcal{H}^p$ so that series (2.9) can be interpreted as a "vaguelette" decomposition of F (see [24]). Thus—following [24]—the "$\mathcal{H}^p$ norm" of F_j is bounded by

$$C \left(\int \left(\sum_{i\in I_j} |\lambda_i|^2 1_{|x-x_i|\le \mu_i}(x) \right)^{p/2} dx \right)^{1/p} = C \left(\sum_{i\in I_j} \lambda_i^p \mu_i^m \right)^{1/p}.$$

By the same argument as in the L^1 case, this quantity is exponentially decreasing with j so that F belongs to $\mathcal{H}^p$.

The optimality of Proposition 2.3 can easily be checked via some explicit examples. The optimality of the global Hölder regularity is shown by example (2) above concerning lacunary trigonometric series. We sketch how to obtain the optimality of the L^1 and $\mathcal{H}^p$ criteria.

Let g be supported on $[1, 2]$ and suppose that F satisfies

$$F(x) = \lambda F(2x) + g(x).$$

If $\int g(x)dx \ne 0$ and $\lambda \ge 2$, series (2.9) does not converge in L^1 (or in any distribution space). If g has vanishing moments and $\lambda \ge 2$, the "$\mathcal{H}^p$ norm" of F can be calculated. For instance, when g is the function ψ that generates an orthonormal basis of compactly supported wavelets, ψ is properly contracted in order to be supported on the interval $[1, 2]$.

PROPOSITION 2.6. *If x does not belong to K, F is C^k in a neighborhood of x.*

Proof. Let α be such that $|\alpha| \le k$, and let us show that the series

$$\sum \lambda_i \partial^\alpha (g \circ S_i^{-1}(.))$$

converges uniformly in a neighborhood of x.

This series is bounded in modulus by

$$\sum \frac{|\lambda_i| \mu_i^{-|\alpha|}}{(1 + \mu_i^{-1}|x - x_i|)^N} \le \sum_j \sum_{i\in I_j} \frac{C 2^{C' j}}{(1 + 2^j |x - x_i|)^N},$$

but since $x \notin K$, $|x - x_i| \ge C > 0$ so that

$$\sum_{i\in I_j} \frac{C}{(1 + 2^j |x - x_i|)^N} \le C' 2^{-(N-m)j}.$$

Choosing N large enough, we obtain Proposition 2.6 when $k \in \mathbb{N}$. The verification when k is not an integer is just as easy and is thus left to the reader. ∎

We conclude this section with a study on the uniqueness of solutions of (2.3). First, note that (2.8) holds for any N and that outside K the second term in (2.8) tends to 0 in C^k so that any distribution solution of (2.3) outside K is a function that satisfies (2.8). Thus if (2.3) has two solutions, their difference is a distribution supported by K, which is a solution of the homogeneous equation

$$F = \sum_{j=1}^{d} \lambda_j F \circ S_j^{-1}. \tag{2.13}$$

Since such a distribution is compactly supported, it belongs to a space $L^{p,s}$ (perhaps for a negative s). Note that $\|F \circ S_j^{-1}\|_{L^{p,s}} = \mu_j^{m/p-s} \|F\|_{L^{p,s}}$. Thus (2.13) implies that

$$\|F\|_{L^{p,s}} \leq \left(\sum_{j=1}^{d} |\lambda_j| \mu_j^{m/p-s} \right) \|F\|_{L^{p,s}},$$

and it has a nonvanishing solution in $L^{p,s}$ only if $\sum_{j=1}^{d} |\lambda_j| \mu_j^{(m/p)-s} > 1$.

Suppose that $\sum_{j=1}^{d} |\lambda_j| \mu_j^m < 1$. For all $s < 0$, let $p_0 m/(m-s)$. Then $\sum_{j=1}^{d} |\lambda_j| \mu_j^{(m/p)-s} < 1$ if $p < p_0$ so that (2.13) has no solution in $L^{p,s}$ for $p < p_0$ (hence for any p since F is compactly supported). Hence we have the uniqueness result in Proposition 2.3.

If $\sum_{j=1}^{d} |\lambda_j| \mu_j^{(m/p)-s} > 1$, it is easy to find distributions supported by K and solutions of (2.13). A trivial example is the Dirac mass at the origin, a solution of $\delta(.) = 2^m \delta(2.)$, but multinomial measures, such as the canonical measure on the triadic Cantor set, satisfy such equations. (The self-similar measures supported on K that we construct in section 4 also satisfy such equations.) In the case where $\sum_{j=1}^{d} |\lambda_j| \mu_j^{(m/p)-s} > 1$, we thus have no unique solution of (2.3), and we call (2.9) the *fundamental solution*.

For a given branch indexed by $i = (i_1, \ldots, i_n)$, let

$$\alpha(i) = \frac{\mathrm{Log}|\lambda_i|}{\mathrm{Log}\mu_i} \tag{2.14}$$

and denote the set $B_{j,10diam(\Omega)}(x)$ by $B_j(x)$.

In the next two sections, we prove the following result, which yields the exact regularity of f at each point of K when $\alpha_{\min} > 0$. (Recall that by definition f is Γ^α at x if α is the supremum of all β such that $f \in C^\beta(x)$.)

PROPOSITION 2.7. *Suppose that $\alpha_{\min} > 0$. Let $x \in K$. Then F is $\Gamma^{\alpha(x)}$ at x, where*

$$\alpha(x) = \liminf_{j \to \infty} \inf_{i \in B_j(x)} \frac{\mathrm{Log}|\lambda_i|}{\mathrm{Log}\mu_i}. \tag{2.15}$$

The lower bound for $\alpha(x)$ will be obtained in section 3, and the upper bound will be obtained in section 4. In section 5, we determine the dimension of the set where F is Γ^α for a given α.

A case of special interest is when the separated open-set condition holds. In that case, there is only one branch over x and $i \to x(i)$ is onto so that if $i = (i_1(x), \ldots, i_n(x), \ldots)$ is the only sequence such that $x_0 = x(i)$, (2.14) and (2.15) become

$$\alpha(x_0) = \liminf_{n \to \infty} \frac{\mathrm{Log}|\lambda_{i_1(x_0)}| \cdots |\lambda_{i_n(x_0)}|}{\mathrm{Log}\mu_{i_1(x_0)} \cdots \mu_{i_n(x_0)}}.$$

3. A lower bound for regularity. We will need the following lemma, which yields an estimate for the products $\lambda_{i_1} \cdots \lambda_{i_n}$ on D-branches.

LEMMA 3.1. *Let* $\Lambda_j(x) = \sup_{i \in B_j(x)} |\lambda_i|$ *and* $L_j(x) = \sum_{l=1}^{j} \Lambda_l(x) 2^{-A(j-l)}$, *where* $A > \alpha_{\max}$. *Then*

$$\liminf_{j\to\infty} \frac{\mathrm{Log}(L_j(x))}{-j\log 2} = \liminf_{j\to\infty} \frac{\mathrm{Log}(\Lambda_j(x))}{-j\log 2} \left(= \liminf_{j\to\infty} \inf_{i\in B_j(x)} \frac{\mathrm{Log}(\lambda_i)}{\mathrm{Log}(\mu_i)} \right)$$

and $\forall x \in \mathbb{R}^m$, *if* $\mu_i \sim 2^{-j}$,

$$|\lambda_i| \leq C L_j (1 + 2^j |x - x_i|)^A.$$

In this lemma, we do not make any assumptions on the λ_i's. Let us prove the first assertion. $L_j \geq \Lambda_j$, and if $n(= n(j))$ is such that $n \leq j$ and $\Lambda_n(x)2^{-A(j-n)} = \sup_{l\leq j} \Lambda_l(x) 2^{-A(j-l)}$, then $L_n \leq n\Lambda_n(x)$. $A > \alpha_{\max}$ so that $n(j) \to \infty$ when $j \to \infty$. Hence we have the first assertion.

We now prove the second assertion. First, if i is a main branch, $|\lambda_i| \leq L_j$. Now suppose that i is not a main branch. Let $i = (i_1, \ldots, i_n)$ and let l be the largest integer such that the subbranch $(i_1, \ldots, i_l)$ is a main branch over x. Clearly, $2^l |x - x_i| \sim 10\mathrm{diam}(\Omega)$ and $\lambda_i \leq 2^{A(j-l)}\Lambda_l$ (because all of the λ_j's are $< 2^{\alpha_{\max}}$), so

$$|\lambda_i| \leq L_j \frac{\Lambda_l}{L_j} \leq L_j 2^{A(j-l)} \leq L_j (C 2^j |x - x_i|)^A.$$

Hence Lemma 3.1 follows.

PROPOSITION 3.2. *Let* $x_0 \in K$. *The function* F *is* $C^\beta(x_0)$ *for any* $\beta < \alpha(x_0)$.

Proof. Let $x \in K$ and $P(x - y)$ be the Taylor expansion of order $[\beta]$ of (2.9) at x. We first check that this Taylor expansion yields a convergent series.

Let α be a multiindex such that $|\alpha| < \beta$. We have to check that the series

$$\sum \frac{|\lambda_i| \mu_i^{-|\alpha|}}{(1 + \mu_i^{-1}|x - x_i|)^N} \tag{3.1}$$

is convergent. We split this sum into the sets

$$I_{j,l} = \{ i \in I_j \text{ and } 2^l < \mu_i^{-1}|x - x_i| \leq 2^{l+1} \}.$$

Because of Lemma 2.5, each term has about 2^{lm} elements, and because of Lemma 3.1, on this set $I_{j,l}$,

$$\lambda_i \leq C L_j (1 + 2^l)^A,$$

so series (3.1) is bounded by

$$C \sum_{j,l} L_j \mu_i^{-|\alpha|} (1 + 2^l)^{A-N} 2^{lm} \leq C \sum_{j,l} 2^{(|\alpha|-\beta)j} 2^{l(m-N+A)}$$

(since for j large enough, $L_j \leq C 2^{-\beta j}$), which is bounded because N can be chosen arbitrarily large.

Let $Tg_x(x - y)$ be the Taylor expansion of g of order $[\beta]$ at point x, i.e.,

$$Tg_x(x - y) = \sum_{|\gamma| \leq [\beta]} \frac{\partial^\gamma g(x)}{\gamma!} (x - y)^\gamma.$$

Let J such that $2^{-J} \leq |x-y| \leq 2.2^{-J}$. Using formula (2.8) but stopping the iteration on each branch at the first level such that $\mu_i \leq 2^{-J}$, we obtain

$$\begin{aligned}(3.2)\qquad & F(y) - P(x-y) \\ &= \sum_{j\leq J}\sum_{i\in I_j} \lambda_i \left(g(S_i^{-1}(y)) - Tg_x(S_i^{-1}(x-y))\right) \\ &\quad + \sum_{j=J}\sum_{i\in I_J} \lambda_i F(S_i^{-1}(y)) - \sum_{j>J}\sum_{i\in I_j} \lambda_i Tg_x(S_i^{-1}(x-y)).\end{aligned}$$

The third sum is bounded in modulus by

$$\begin{aligned} & C\sum_{|\gamma|\leq\beta}\sum_{j\geq J} |\lambda_i|\mu_i^{-|\gamma|}|x-y|^{|\gamma|}(1+\mu_i^{-1}|x-x_i|)^{-N} \\ &\leq C\sum_{|\gamma|\leq\beta}|x-y|^{|\gamma|}\sum_{l}\sum_{j\geq J} L_j 2^{Al}2^{|\gamma|j}2^{l(m-N)} \\ &\leq C\sum_{|\gamma|\leq\beta}|x-y|^{|\gamma|}2^{(|\gamma|-\beta)J} \leq C|x-y|^{\beta}\end{aligned}$$

(where we have again split the sum into the sets $I_{j,l}$).

Because of the localization of F, the second sum is bounded by $C\sup_{i\in I_J}|\lambda_i| \leq C2^{-\beta J}$.

We now consider the first sum in (3.2). We consider two cases. Let $D = |x-y|^{-\epsilon}$ for an arbitrarily small ϵ. First, suppose that

$$|x - x_i| \leq D2^{-j}.$$

For each j, the sum has about D^m terms, and using the mean-value theorem, the sum of the corresponding terms is bounded by

$$\begin{aligned} & D^m\sum_{j\leq J}\sum_{i} L_j(1+D)^A|x-y|^{[\beta]+1}\mu_i^{-[\beta]-1} \\ &\leq C|x-y|^{[\beta]+1}\sum_{j\leq J}2^{-\beta j}2^{([\beta]+1)j}D^{m+A} \leq C|x-y|^{\beta}D^{m+A}.\end{aligned}$$

Hence we have the bound that we claimed if we take ϵ small enough.

Now suppose that $|x-x_i| > D2^{-j}$; then $|\lambda_i| \leq CL_j(1+2^j|x-x_i|)^A$. Applying Lemma 2.5 with $D = 2^j|x-x_i|$, the remaining sum is bounded by

$$C\sum_{|\gamma|\leq\beta}\sum_{j\leq J}\frac{L_j(1+2^j|x-x_i|)^A(1+2^j|x-x_i|)^m}{(1+2^j|x-x_i|)^N}|x-y|^{\gamma}2^{\gamma j}$$

$$\leq \sum_{|\gamma|\leq\beta}\sum_{j\leq J}2^{-\beta j}2^{\gamma j}|x-y|^{\gamma}(1+2^j|x-x_i|)^{-N+m+A},$$

and we obtain the bound in this case since $2^j|x-x_i| > |x-y|^{-\epsilon}$. Hence Proposition 3.2 follows.

4. An upper bound of the pointwise Hölder exponent. We will bound the regularity of F at each point in K by estimating the size of the wavelet transform in a neighborhood of such a point. The wavelet transform of F satisfies a functional equation similar to (2.3), which will enable us to obtain this estimate. Let $C(a,b)$ be the wavelet transform of F and $\omega(a,b)$ be the wavelet transform of g.

PROPOSITION 4.1. *There exists $A > 0$ such that $\forall x \in K$, $J \in \mathbb{N}$. There exists $j \in [J - A, J]$, a branch $b = (j_1, \dots, j_n)$ in $B_j(x)$, $a \sim 2^{-j}$, and $t \in \Omega$ such that*

$$|t - x| \le Ca \quad \textit{and} \quad |C(a,t)| \ge C\Lambda_j(x).$$

Note that in this proposition, we do not have to make any assumptions on the uniform regularity of F, and we will actually use the proposition in cases where F is unbounded. Nonetheless, let us first show that if F has some minimal uniform regularity, Proposition 2.7 follows. To this end, we first recall a relation between the regularity of F and the size of the wavelet transform given by the following results (see Part I). Suppose that $s > 0$. If $F \in C^s(x_0)$,

$$|C_{a,b}(F)| \le Ca^s \left(1 + \frac{|b - x_0|}{a}\right)^s . \tag{4.1}$$

Thus Proposition 4.1 together with (4.1) shows that F is not smoother than $C^{\alpha(x)}$ at x. Thus using Proposition 3.2, we will have proved Proposition 2.7.

Proof of Proposition 4.1. We first prove Proposition 4.1 with $j \in [(1-\epsilon)J, J]$ (for an arbitrarily small ϵ). Let $C(a,b)$ be the wavelet transform of F and let $\omega(a,b)$ be the wavelet transform of g. Using (2.8) but stopping the iteration on each branch when $\mu_i \sim 2^{-J}$, we obtain

$$C(a,t) = \sum_{j=1}^{J} \sum_{i \in I_j} \lambda_i \omega\left(\frac{a}{\mu_i}, S_i^{-1}(t)\right) + \sum_{i \in I_J} \lambda_i \, C\left(\frac{a}{\mu_i}, S_i^{-1}(t)\right). \tag{4.2}$$

Let $y \in \Omega$ be a fixed point that will be determined later. Let $x \in K$ and let $b = (j_1, \dots, j_n)$ be a branch over x. Let $t = S_{j_1} = CA \cdots S_{j_n}(y)$. Then

$$|x - t| \le C\mu_{j_1} \cdots \mu_{j_n}.$$

Hence we have the first condition of Proposition 4.1.

We want to show that on the set S_j, the main term in (4.2) corresponds to the branch b. This is intuitively clear because all terms in the first sum decay like a^k because of the smoothness of g, and since F is smooth outside Ω, all terms in the second sum decay also like a^k except precisely the one corresponding to the branch b. We make this argument more precise. We first prove the following bound for the first sum in (4.2):

$$\sum_{i \in I_j} \left|\lambda_i \omega\left(\frac{a}{\mu_i}, S_i^{-1}(t)\right)\right| \le Ca^k 2^{kj} L_j(t). \tag{4.3}$$

First, note that because of the smoothness and decay of g,

$$\forall N \ge 0, \quad |\omega(a,b)| \le \frac{C_N a^k}{(1 + |b|)^N}.$$

Thus

$$\begin{aligned}\sum_{i\in I_j}\left|\lambda_i\omega\left(\frac{a}{\mu_i},S_i^{-1}(t)\right)\right| &\le Ca^k\sum_{i\in I_j}\frac{|\lambda_i|}{\mu_i^k(1+|2^j(t-x_i)|)^N}\\ &\le Ca^k2^{kj}L_j(t)\sum_{i\in I_j}\frac{1}{(1+|2^j(t-x_i)|)^{N-A}}\\ &\le Ca^k2^{kj}L_j(t).\end{aligned}$$

Hence we have (4.3). Thus

$$\sum_{j\le J}\sum_{i\in I_j}\left|\lambda_i\omega\left(\frac{a}{\mu_i},S_i^{-1}(t)\right)\right|\le Ca^k\sum_{j\le J}2^{kj}L_j(t).$$

Since $\sup(\log\lambda_i/\log\mu_i)<k$, this series grows exponentially so that the first term in (4.2) is bounded by $Ca^k2^{kJ}L_J(t)$.

We now estimate the second term in (4.2) when $i\ne b$. Recall that A is the closed subset of Ω where by assumption F is not uniformly C^k. Let $A_\epsilon=A+B(0,\epsilon)$, where ϵ is a constant small enough that $A_\epsilon\subset\Omega$. Thus outside A_ϵ,

$$|C(a,b)|\le C\frac{a^k}{(1+|b|)^N}$$

so that

$$C\left(\frac{a}{\mu_i},S_i^{-1}(t)\right)\le C\left(\frac{a}{\mu_i}\right)^k\frac{1}{(1+2^j|t-x_i|)^N}.$$

Thus we obtain, as above,

$$\sum_{i\in I_J,\, i\ne b}\lambda_iC\left(\frac{a}{\mu_i},S_i^{-1}(t)\right)\le Ca^k2^{kJ}L_J(t).$$

Finally, from (4.2), we get

$$\left|C(a,t)-\lambda_jC\left(\frac{a}{\mu_j},S_j^{-1}(t)\right)\right|\le Ca^k2^{kJ}L_J(t). \tag{4.4}$$

We now estimate the term corresponding to the sequence b. Recall that the last condition in Definition 2.1 is equivalent to the existence of sequences $a_n\to0$, $b_n\in A$, and $C_n\to+\infty$ such that $|C(a_n,b_n)|\ge C_na_n^k$ so that

$$\left|\lambda_bC\left(\frac{a}{\mu_b},S_b^{-1}(t)\right)\right|\ge|\lambda_b|C_n\left(\frac{a}{\mu_b}\right)^k.$$

Recall that $\Lambda_j=\sup_{i\in B_j(x)}|\lambda_i|$. Choosing a branch for which this supremum (taken on a finite number of terms) is attained, we get for this branch that

$$\left|\lambda_bC\left(\frac{a}{\mu_b},S_b^{-1}(t)\right)\right|\ge\Lambda_J(x)C_n2^{kJ}a^k$$

so that

$$|C(a,t)| = 2^{kJ} a^k [C_n \Lambda_J(t) + R],$$

where $|R| \le C L_J(t)$. We choose n such that $C_n \ge 2C$, which determines a value of $a_n = a/\mu_b$. If

$$\Lambda_j(t) \ge \frac{1}{2} L_J(t),$$

the proposition is proved with $j = J$. Otherwise, since

$$L_J(t) = \Lambda_J + 2^{-A}\Lambda_{J-1} + 2^{-2A}\Lambda_{J-2} + \cdots,$$

one of the terms $2^{-lA}\Lambda_{J-l}$ must be large. More precisely, there exists l such that

$$2^{-lA}\Lambda_{J-l} \ge \frac{1}{10l^2}\Lambda_J. \tag{4.5}$$

(If several values of l satisfy (4.5), we choose the smallest.) We can choose the corresponding branch in Proposition 4.1, and since $l = o(J)$, this implies the irregularity of F at x. We found points in the "cone above x" where the wavelet transform is large. The statement of Proposition 4.1 is more precise because we will need precise estimates on the wavelet transform everywhere in order to estimate the integrals of the wavelet transform needed in the multifractal formalism. We have to check that we can choose $l \le C$. We first prove that $l \le \epsilon J$. We have $\Lambda_{J-l} \le 2^{-\alpha_{\min}(J-l)}$ and $\Lambda_J \ge 2^{-\alpha_{\max} J}$ so that if (4.5) holds,

$$2^{-lA} 2^{-\alpha_{\min}(J-l)} \ge \frac{1}{10l^2} 2^{-\alpha_{\max} J},$$

which implies that

$$l \le \left(\frac{\alpha_{\max} - \alpha_{\min}}{A - \alpha_{\min}}\right) J.$$

Choosing A large enough, we have $l \le \epsilon J$ for ϵ arbitrarily small. For this branch b,

$$|C(a,t)| \ge \frac{1}{2}\lambda_b(x) C\left(\frac{a}{\mu_b}, S_b^{-1}(t)\right) \tag{4.6}$$

and $C(a/\mu_b, S_b^{-1}(t)) \sim 1$. Hence we have Proposition 4.1 when $j \in [(1-\epsilon)J, J]$. We now want to prove that the proposition holds for $j \in [J - A, J]$.

Suppose that t is a point inside Ω such that the $S_i(t)$'s do not approach the boundary of Ω. We know that

$$C(a,t) = \omega(a,t) + \sum_{j=1}^{d} \lambda_j C\left(\frac{a}{\mu_j}, S_j^{-1}(t)\right),$$

but $|\omega(a,t)| \le C_1 a^k$ and outside of a certain neighborhood of Ω, $|C(a,t')| \le C_2 a^k$. Let

$$i = (i_1, \dots i_n) \quad \text{and} \quad i' = (i_1, \dots i_n, i_{n+1}).$$

Thus if t and r are such that $|C(\mu_i, t) - r\lambda_i| \le e$, then

$$\begin{aligned}|C(\mu_{i'}, S_{i'}(t)) - r\lambda_{i'}| &\left|\omega(\mu_{i'}, S_{i'}(t)) + \sum_j^d \lambda_j C\left(\frac{\mu_{i'}}{\mu_j}, S_j^{-1}(t)\right) - r\lambda_{i'}\right| \\ &\le C_1(\mu_{i'})^k + \sum_{j\ne i_{n+1}} \lambda_j C\left(\frac{\mu_{i'}}{\mu_j}\right)^k + \lambda_{i_{n+1}}|r\lambda_i - C(\mu_i, t)| \\ &\le \left(C_1 + C\sum_{j\ne i_{n+1}} \frac{\lambda_j}{(\mu_j)^k}\right)\mu_{i'}^k + e\lambda_{i+1} \le C\mu_{i'}^k + e\lambda_{i+1}.\end{aligned}$$

We start with a branch i such that $r \sim 1$ and $e = 0$, which is possible because of the first part of the proof. After one iteration, we obtain an error of $C\mu_i^k$; after two iterations, we get an error of $C\mu_i^k\lambda_{i+1} + C\mu_{i+1}^k, \ldots$; and after j iterations, the error is

$$C\mu_i^k\left(\frac{\lambda_j}{\lambda_i} + \mu_i\frac{\lambda_j}{\lambda_{i+1}} + \cdots\right) \sim C\mu_i^k\frac{\lambda_j}{\lambda_i}$$

so that $|C(\mu_j, t') - r\lambda_j| \le C'\mu_i^k(\lambda_j/\lambda_i) \le \epsilon\lambda_j$, where t' is on the subtree deduced from t. Thus $C(\mu_j, t') \sim r\lambda_j$. Hence Proposition 4.1 follows.

Note that Propositions 3.2 and 4.1 show that the wavelet transform of F is "large" near the tree T, and thus the ramifications of this tree of wavelet maxima reflect the "dynamics" of self-similarity as stated by Arneodo, Bacry, and Muzy in [2].

It is remarkable that these results do not depend on the function g. If g were replaced by another function, the new F would have the same regularity at every point. Only the global smoothness of g is important. It defines a value beyond which one can no longer calculate the regularity of F.

5. Determination of the Hölder spectrum. In this section, we prove that for $\alpha < k$, the Hölder spectrum of a self-similar function is the Legendre transform of the function τ defined by

$$\sum_{i=1}^d \lambda_i^a \mu_i^{-\tau(a)} = 1.$$

PROPOSITION 5.1. *Let $\alpha < k$ and define $d(\alpha)$ as the Hausdorff dimension of the set of points x where F is $\Gamma^\alpha(x)$. Then $d(\alpha)$ is given on $[0, k)$ by*

$$d(\alpha) = \left(\inf_a a\alpha - \tau(a)\right). \tag{5.1}$$

We will need the following proposition (Proposition 4.9 in [25]) in the proof of Proposition 5.1.

PROPOSITION 5.2. *Let $\mathcal{H}^s$ be the Hausdorff measure of dimension s. Let μ be a probability measure on $\mathbb{R}^m$, $F \subset \mathbb{R}^m$, and C be such that $0 < C < +\infty$. Then*

- *if* $\limsup_{r\to 0} \mu(B(x,r))/r^s < C\ \forall x \in F$, $\mathcal{H}^s(F) \ge \mu(F)/C$;
- *If* $\limsup_{r\to 0} \mu(B(x,r))/r^s > C\ \forall x \in F$, $\mathcal{H}^s(F) \le 2^s/C$.

Proof of Proposition 5.1. Let $a \in \mathbb{R}$, $b = -\tau(a)$, and $P_i = \lambda_i^a\mu_i^b$. Thus $\sum P_i = 1$. We first consider on K a probability measure ν such that

$$\forall(i_1, \ldots, i_n), \quad \nu(S_{i_1}\cdots S_{i_n}(K)) = P_{i_1}\cdots P_{i_n}. \tag{5.2}$$

The construction of such a measure by induction is straightforward (see, for instance, [17]). Let $x \in K$, $s > 0$ and $r > 0$ and consider the set $B_j(x)$, where $2^{-j} \leq r < 2.2^{-j}$. Then

$$\frac{\nu(B_r(x))}{r^s} = \sum_{i \in B_j(x)} \frac{\lambda_i^a \mu_i^b}{\mu_i^s} \sim \sup_{i \in B_j(x)} \left(\lambda_i \; \mu_i^{\frac{b-s}{a}} \right)^a$$

(because the number of branches over x in $B_j(x)$ is bounded by an absolute constant).

Suppose that

$$\frac{b-s}{a} < -\alpha(x).$$

Then $\limsup_{r \to 0} \nu(B_r(x))/r^s \to +\infty$ so that, using Proposition 5.2, $\mathcal{H}^s(\Gamma^\alpha) = 0$. Thus $d(\alpha) \leq b + a\alpha$ so that $d(\alpha) \leq -\tau(a) + a\alpha \; \forall a \in \mathbb{R}$.

In order to prove Proposition 5.1, we have to show that the infimum is reached. Using Proposition 5.2, it is sufficient to find a and b such that $\nu(\Gamma^\alpha) > 0$.

Suppose that a and b are solutions of the following system

$$\left.\begin{aligned} \sum_{i=1}^{d} \lambda_i^a \mu_i^b &= 1, \\ \frac{\sum P_i \mathrm{Log} \lambda_i}{\sum P_i \mathrm{Log} \mu_i} &= \alpha, \end{aligned}\right\} \tag{5.3}$$

where $P_i = \lambda_i^a \mu_i^b$. (In Lemma 5.3, we will determine the values of α for which this system has a solution.)

If $(i_1, \ldots, i_n)$ is a branch over x, let $(n_j)_{j=1,\ldots,d}$ be the proportion of $j's$ in the sequence $i_1, \ldots, i_n$ and let F be the subset of K composed of the points x such that

$$n_j \to P_j \tag{5.4}$$

(meaning here that $\forall \varepsilon > 0$, $\exists n : \forall m, \geq n$, if $(i_1, \ldots, i_m)$ is a branch over x, then $|n_j - P_j| \leq \varepsilon \; \forall j = 1, \ldots, d$ for this branch).

If $x \in F$, then

$$\liminf_{j \to \infty} \inf_{i \in B_j(x)} \frac{\mathrm{Log}\lambda_i}{\mathrm{Log}\mu_i} = \lim \frac{\mathrm{Log}\lambda_i}{\mathrm{Log}\mu_i} = \frac{\sum P_j \mathrm{Log}\lambda_j}{\sum P_j \mathrm{Log}\mu_j} = \alpha$$

so that $F \subset \Gamma^\alpha$.

Let ν be the corresponding probability defined by (5.2). We can associate with ν another probability P defined on $\{1, \ldots, d\}^{\mathbb{N}}$ as follows. If $i = (i_1, \ldots, i_n)$ and I_i is the subset of $\{1, \ldots, d\}^{\mathbb{N}}$ of all of the sequences starting with $(i_1, \ldots, i_n)$, then

$$P(I_i) = P_{i_1} \ldots P_{i_n}.$$

With probability P, the i_n's are a sequence of i.i.d. random variables. The strong law of large numbers implies that with probability 1, $n_j \to P_j$ for a sequence $i \in \{1, \ldots, d\}^{\mathbb{N}}$. Clearly, ν is the image of the probability P by the application $x(i)$. We want to show that on K, ν-almost everywhere $n_j \to P_j$. It would be obvious if $x(i)$ were one to one.

First, note that if $(i_1, \ldots, i_n)$ is a branch over x, so is $(i_1, \ldots, i_{n-1})$. Now suppose that (5.4) fails. For n arbitrarily large, we can find a branch over x such that

$$|n_j - P_j| \geq \varepsilon. \tag{5.5}$$

Consider such a sequence of branches over x for $n \to \infty$. Since at a scale r there are at most N branches over x, (following Lemma 2.5) such branches for which (5.4) fails can be grouped into at most N sets of increasing branches. Among these, at least one, $\tilde{b}_x$, has infinite length.

We call a branch of infinite length i such that $x = x(i)$ a *principal branch over* x. Because of Lemma 2.5, for each x, there are at most N such branches. Clearly, $\tilde{b}_x$ is a principal branch over x.

Consider the event $\{x$ is such that (5.4) fails$\}$. It is included in the event $\{\exists b$ principal branch over x such that (5.5) holds$\}$. Since the probability for one given branch is 0, the probability that (5.5) holds for at least one of the (at most) N principal branches over x is also 0 such that for probability ν, almost every point of K is such that (5.4) holds. Thus $\nu(F) = 1$, and since $F \subset \Gamma^\alpha, \nu(\Gamma^\alpha)1$. We can now apply Proposition 5.2. Hence we have the first part of Proposition 5.1.

LEMMA 5.3. *Suppose that* $\alpha_{\min} < \alpha_{\max}$. *System* (5.3) *has a solution if and only if*

$$\alpha_{\min} < \alpha < \alpha_{\max}. \tag{5.6}$$

If $\alpha_{\min} = \alpha_{\max}$, *the only solution is* $\alpha = \alpha_{\min} = \alpha_{\max}$.

Proof. One can easily check that if $a_1, \ldots, a_d,\ b_1, \ldots, b_d > 0$, and the P_i's are weights (i.e., $0 < P_i$ and $\sum P_i = 1$), then

$$\inf\left(\frac{a_i}{b_i}\right) \neq \sup\left(\frac{a_i}{b_i}\right) \implies \inf\left(\frac{a_i}{b_i}\right) < \frac{\sum P_i a_i}{\sum P_i b_i} < \sup\left(\frac{a_i}{b_i}\right)$$

so that (5.6) is necessary. Now suppose that this holds. Since $\sum \lambda_i^a \mu_i^{-\tau(a)} = 1$, $\forall i, \tau(a) \leq a \mathrm{Log}\lambda_i/\mathrm{Log}\mu_i$. If $a \to +\infty$, $\tau(a) \leq a\ \alpha_{\min}$ so that if j is such that $\mathrm{Log}\lambda_j/\mathrm{Log}\mu_j > \alpha_{\min}$, then $\lambda_j^a \mu_j^{-\tau(a)} \to 0$. Thus if $\alpha_{\min}$ is reached for i in a subset $J \subset \{1, \ldots, d\}$, then $\sum_{i \in J} \lambda_i^a \mu_i^{-\tau(a)} \to 1$, but $\sum_{i \in J} \lambda_i^a \mu_i^{-\tau(a)} = \sum_{i \in J} e^{\mathrm{Log}\mu_i(a\alpha_{\min} - \tau(a))}$ so that $\tau(a)/a \to \alpha_{\min}$. Thus all of the $P_i \to 0$ except for $i \in J$ so that $\sum P_i \mathrm{Log}\lambda_i / \sum P_i \mathrm{Log}\mu_i \to \alpha_{\min}$.

If $a \to -\infty$, $\tau(a) \leq a\alpha_{\max}$ and the same argument yields $\sum P_i \mathrm{Log}\lambda_i / \sum P_i \mathrm{Log}\mu_i \to \alpha_{\max}$. By continuity, $\sum P_i \mathrm{Log}\lambda_i / \sum P_i \mathrm{Log}\mu_i$ takes all values between $\alpha_{\min}$ and $\alpha_{\max}$.

Notice that if $\alpha_{\min} = \alpha_{\max} = \alpha_0$, then $\alpha = \alpha_0$ is the only possible value for which (5.3) has a solution.

6. Proof of the multifractal formalism. Now that we have determined the spectrum of a self-similar function, we will prove the multifractal formalism for these functions. First, we will do so for the wavelet-transform integral method. We recall the formulas that are used. We compute

$$\tilde{Z}(a, q) = \int_{\mathbb{R}^m} |C(a, b)|^q db.$$

Let

$$\eta(q) = \liminf \frac{\log \tilde{Z}(a, q)}{\log a}. \tag{6.1}$$

The Hölder spectrum is computed using the formula

$$d(\alpha) = \inf_q (q\alpha - \eta(q) + m). \tag{6.2}$$

In order to estimate $\tilde{Z}(a,q)$ for self-similar functions, we first have to estimate $C(a,b)$ everywhere. Let

$$\begin{cases} i &= (i_1, \dots, i_n), \\ \Omega_i &= S_{i_1} \cdots S_{i_n}(\Omega), \\ B_i &= \Omega_i + B(0,a), \\ C_i &= B_{(i_1,\dots,i_{n-1})} - B_{(i_1,\dots,i_n)}. \end{cases}$$

If $a \le \mu_i$, $\mathrm{Vol}(B_i) \sim (\mu_i)^m$ and $\mathrm{Vol}(C_i) \le C(\mu_i)^m$. Inequality (4.6) shows that there exists one point $b \in B_i$ and an a such that $C2^{-j} \le a \le 2^{-j}$ for which the order of magnitude of $C(a,b)$ is Λ_n. We show that this order of magnitude holds on a ball of size $\sim a$. To this end, we bound $C(a,b)$ in B_i (and also in C_i, which will be useful later).

LEMMA 6.1. *Let $a > 0$ and B_i be such that $a \sim \mu_i$. Then if $b \in B_i$,*

$$|C(a,b)| \le CL_i,$$

and if $a \le \mu_{i_1} \cdots \mu_{i_n}$, then if $b \in C_i$,

$$|C(a,b)| \le CL_i \left(\frac{a}{\mu_i}\right)^k.$$

Lemma 6.1 is derived from (2.8) exactly as in the beginning of the proof of Proposition 2.7. We leave the details to the reader.

We return to the estimation of $C(a,b)$. In order to prove that it keeps the same order of magnitude in a ball of size $\sim a$, we bound $\bigtriangledown_b C(a,b)$ and $\partial_a C(a,b)$. Let $\partial_b C(a,b)$ be a partial derivative of $C(a,b)$ in a certain direction $b_0 \in \mathbb{R}^m$. Clearly,

$$\partial_b C(a,b) = \frac{1}{a}\tilde{C}(a,b),$$

where $\tilde{C}(a,b)$ is a wavelet transform using the wavelet $\partial\psi$.

The bound given by Lemma 6.1 for $C(a,b)$ holds for $\tilde{C}(a,b)$ so that

$$|\partial_b C(a,b)| \le \frac{C}{a} L_i.$$

Since at a certain point of B_i, $C(a,b)$ is of the order of magnitude of L_i, this is also the case on a ball of size $\sim a$.

If we now differentiate the wavelet transform with respect to the variable a, the same procedure yields

$$\partial_a C(a,b) = \frac{1}{a}\tilde{C}(a,b),$$

where $\tilde{C}(a,b)$ is a wavelet transform using the wavelet $\psi(x) - x.\bigtriangledown\psi(x)$, so that $C(a,b)$ is of the order of magnitude of L_i on a ball of size $\sim a$ in the time-scale half-space.

Furthermore, on C_i,

$$|C(a,b)| \le C \frac{a^k \lambda_i}{(\mu_i)^k}. \tag{6.3}$$

Let A_j be the interval $[2^{-(j+A)}, 2^{-j}]$. For each branch i such that $\mu_i \sim 2^{-j}$, Proposition 4.1 shows that there exists a ball of radius at least $C2^{-j}$ in the time-scale half-space located near x_i and in scale in the interval A_j, where $|C(a,b)| \ge C\lambda_i$. Thus

$$\begin{aligned} C \sum_{\mu_i \sim 2^{-j}} 2^{-j(m+1)} \lambda_i^q &\le \int_{A_j \times \mathbb{R}^m} |C(a,b)|^q da\, db \\ &\le C' \sum_{\mu_i \sim 2^{-j}} 2^{-j(m+1)} \lambda_i^q \, + O\left(2^{-j} \sum_{\mu_i \ge 2^{-j}} \frac{2^{-kqj}\lambda_i^q}{\mu_i^{qk-m}} \right) \end{aligned}$$

so that

$$\begin{aligned} C \sum_{2^{-j} \le \mu_i < 2.2^{-j}} \mu_i^m \lambda_i^q &\le 2^j \int_{A_j \times \mathbb{R}^m} |C(a,b)|^q da\, db \\ &\le C' \left[\sum_{2^{-j} \le \mu_i < 2.2^{-j}} \mu_i^m \lambda_i^q + O\left(\sum_{\mu_i \ge 2^{-j}} \frac{2^{-kqj}\lambda_i^q}{\mu_i^{qk-m}} \right) \right]. \end{aligned} \tag{6.4}$$

We first estimate the term

$$\sum_{2^{-j} \le \mu_i < 2.2^{-j}} \mu_i^m \lambda_i^q. \tag{6.5}$$

The reader should notice that in the following estimation, we do not have to assume that q is positive. This remark will be useful in section 7.

Recall that $\tau(q)$ is such that

$$\sum_{j=1}^{d} \mu_j^{-\tau(q)} \lambda_j^q = 1. \tag{6.6}$$

Thus

$$\sum_{2^{-j} \le \mu_i < 2.2^{-j}} \mu_i^m \lambda_i^q \sim 2^{-(m+\tau(q))j} \sum_{2^{-j} \le \mu_i < 2.2^{-j}} \mu_i^{-\tau(q)} \lambda_i^q.$$

Let

$$F(j) = \sum_{2^{-j} \le \mu_i < 2.2^{-j}} \mu_i^{-\tau(q)} \lambda_i^q.$$

From (6.6), we get

$$\sum_{|i|=N} \mu_i^{-\tau(q)} \lambda_i^q = \left(\sum_{j=1}^{d} \mu_j^{-\tau(q)} \lambda_j^q \right)^N = 1$$

so that

$$\sum_{|i|\leq N_0} \mu_i^{-\tau(q)} \lambda_i^q = N_0. \tag{6.7}$$

Clearly,

$$\sum_{j=1}^{J} F(j) = \sum_{\mu_i \geq 2^{-J}} \mu_i^{-\tau(q)} \lambda_i^q. \tag{6.8}$$

After permuting the indexation of the S_i's, we can assume that $\mu_1 = \inf \mu_i$ and $\mu_d = \sup \mu_i$.

The right-hand side of (6.8) contains all of the terms of length N if $\mu_1^N \geq 2^{-J}$ and no terms of length M if $\mu_d^M \leq 2^{-J}$. Thus from (6.7) and (6.8), we get

$$N \leq \sum_{j=1}^{J} F(j) \leq M,$$

which can be written as

$$J \frac{\mathrm{Log} 2}{\mathrm{Log}(\frac{1}{\mu_1})} \leq \sum_{j=1}^{J} F(j) \leq J \frac{\mathrm{Log} 2}{\mathrm{Log}(\frac{1}{\mu_d})}$$

so that there exist $C_1, C_2 > 0$ such that

$$\left. \begin{array}{rcl} \limsup \dfrac{F(j)}{j} & \leq & C_1, \\ \\ \limsup F(j) & \geq & C_2. \end{array} \right\} \tag{6.9}$$

Thus

$$\limsup \frac{1}{j} 2^{(m+\tau(q))j} \sum_{2^{-j} \leq \mu_i < 2.2^{-j}} \mu_i^m \lambda_i^q \leq C$$

and

$$\limsup 2^{(m+\tau(q))j} \sum_{2^{-j} \leq \mu_i < 2.2^{-j}} \mu_i^m \lambda_i^q \geq C'.$$

Now consider the term

$$\sum_{\mu_i \geq 2^{-j}} \frac{2^{-kqj} \lambda_i^q}{\mu_i^{qk-m}}.$$

This is bounded by

$$C 2^{-kqj} \sum_{\mu_i \geq 2^{-j}} \lambda_i^q \mu_i^{m-qk}. \tag{6.10}$$

We split this sum into bands B_l defined by $2^{-l-1} \le \mu_i < 2^{-l}$. Using (6.7), we get

$$C2^{-kqj} \sum_{\mu_i \in B_l} \lambda_i^q \mu_i^{m-qk} \le C2^{-kqj} 2^{-l(m-kq+\tau(q))} \sum_{\mu_i \in B_l} \lambda_i^q \mu_i^{-\tau(q)}$$
$$\le C2^{-kqj} 2^{-l(m-kq+\tau(q))}.$$

Now suppose that q is such that $m - kq + \tau(q) \le 0$. Equation (6.10) is bounded by

$$C2^{-kqj} 2^{-(m-kq+\tau(q))j} \le C2^{-(m+\tau(q))j},$$

and using (6.4) and (6.9), we obtain the followingproposition.

PROPOSITION 6.2. *Suppose that F is self-similar and let q be such that*

$$\tau(q) \le kq - m. \tag{6.11}$$

Then

$$\limsup_{a \to 0} a^{-m-\tau(q)} \int |C(a,b)|^q db \ge C > 0 \tag{6.12}$$

$$\limsup_{a \to 0} \frac{a^{-m-\tau(q)}}{|\log a|} \int |C(a,b)|^q db \le C' < +\infty. \tag{6.13}$$

This result together with Proposition 2.7 proves the multifractal formalism for the wavelet-integral method.

The multifractal formalism is also valid for the structure function method since we showed in Part I that $\zeta(q) = \eta(q)$ for $q > 1$. However, the restriction $q > 1$ shows that it might not yield the whole left-hand side of the spectrum but a smaller part corresponding to the region where the infimum in the Legendre transform formula is attained for $q > 1$.

Note that if k can be chosen arbitrarily large (when g is C^∞), condition (6.11) reduces to $q \ge 0$. We consider the case of negative values of q in the next section.

7. The wavelet-box method. In this section, we first show some pitfalls of the wavelet-maxima method and then show that a slight modification allows us to obtain the spectrum even for its decreasing part.

Let us first briefly recall the principles of the wavelet-maxima method. Consider for a given $a' > 0$ the local maxima of the function $b \to C(a', b)$. Generically, they belong to a line of maxima $b = l(a)$ defined in a small left-neighborhood $[a'', a']$ of a' by the following condition: $b \to C(a, b)$ has a local maximum for $b = l(a)$. The wavelet-transform maxima method requires first the computation of

$$Z(a,q) = \sum_l \sup_{b=l(a)} |C(a,b)|^q,$$

where l is a line of maxima of the wavelet transform defined on $[a'', a']$, and the sum is taken on all lines of local maxima defined in a left-neighborhood $[a'', a']$ of a'. We then define

$$\theta(q) = \liminf \frac{\log Z(a,q)}{\log a}.$$

The counterexamples concerning the wavelet-maxima method that were given in Part I easily adapt to the self-similar case. Suppose, for instance, that g is one of these counterexamples supported by the interval $[3,4]$. (We can make this assumption because they are compactly supported and these properties still hold after a contraction and a translation.) Then

$$F(x) = \sum 2^{-\alpha j} F(2^j x),$$

where $\alpha > 0$ is self-similar and yields a similar counterexample. Nonetheless, we will see that we can adapt the wavelet-maxima method so that it yields results as good as and even better than the other methods. To this end, we introduce a slight variant, the *wavelet-box method*.

Let C be a parameter larger than 1. The wavelet-box method consists of dividing $\mathbb{R}^m$ into cubes of length C and, for each cube included in Ω, keeping only the largest local maximum (if there is one on each of these cubes). Clearly, this procedure has the advantage of not taking into account accumulations of lines of local maxima, on which the counterexamples of Part I were based. We still use the notation $\theta(q)$, which will avoid confusion with the wavelet-maxima method.

THEOREM 7.1. *Under the same hypotheses as Theorem* 2.2*, the wavelet-box method yields the increasing part of the spectrum of self-similar functions. Furthermore, if $d_{\max} = m$ or, equivalently, if $\overline{\cup S_i(\Omega)} = \overline{\Omega}$, the wavelet-box method yields the whole spectrum of self-similar functions, provided that we keep only the largest maximum of a box of size Ca for a constant C large enough, i.e., $d(\alpha)$ can be obtained by computing the Legendre transform of $\theta(q) - m$.*

With regard to the increasing part of the spectrum (corresponding to $p \geq 0$), the theorem is a consequence of the wavelet-integral method because of the following lemma.

LEMMA 7.2. *The two quantities*

$$\int |C(a,b)|^q db \quad \text{and} \quad a^m \sum_{\max} |C(a,b)|^q$$

are of the same order of magnitude if the sum is computed as in the wavlet box method.

This result is quite straightforward since we estimated $\int |C(a,b)|^q db$ precisely by computing its order of magnitude near the wavelet maxima. We showed that its value is about λ_i^q near the tree T and smaller far from the tree. This shows that there exists at least one maximum near each point of the tree. The estimation of $a^m \sum_{\max} |C(a,b)|^q$ then follows exactly the estimation performed in Proposition 6.2. Thus, in that case, the verification of the fractal formalism reduces to the verification for the integral formula, and the multifractal formalism holds when using the two-wavelet methods. Note that for positive q's we do not have to restrict the sum to cubes included in Ω, which is interesting if we do not have a priori knowledge on Ω.

Before proving Theorem 2.3 for the decreasing part of the spectrum, we make some general remarks.

Consider again the case $q < 0$ but for the quantity $\sum_{\max} |C(a,b)|^q$. An important difference from the exact computations of [2] appears. Recall that the authors of [2] were interested in the case where F is the indefinite integral of a multinomial measure supported by a Cantor set. In this case, the wavelet maxima are situated on the "tree over the Cantor set" since F is piecewise constant outside this set, so, for a small enough, the wavelet transform vanishes there. Thus in this case, the last term of (6.4) vanishes, and the same proof as above shows that Proposition 6.2 will hold for $q < 0$ (with the same restriction concerning the distance between the maxima).

In the general case that we consider in this paper, F is a C^k function outside K for which we do not have special information (since g is C^k but arbitrary). Thus there may be extremely small wavelet maxima "far away" (on the scale of a) from K (i.e., at a distance $\gg a$). Thus no formula involving negative values of q can work reasonably. We present an example of this phenomenon.

First, note that if $\Omega \neq K$, it is easy to construct examples where g and thus F will be locally constant so that $F(x+h) - F(x)$ will vanish on an open subset for h small enough, as does $C(a,b)$ for a small enough. Thus $\zeta(q)$ and $\eta(q)$ take the value $+\infty$ for negative values of p so that in all generality, computing $\zeta(q)$ and $\eta(q)$ for negative values of p does not make sense. The same problem appears for formulas involving wavelet maxima. Of course, in the regions where $C(a,b)$ vanishes, there are no more maxima. However, it is easy to construct g with lines of very small maxima as follows.

Let ψ be a C^{k+2} function with moments of order $k+1$ vanishing and supported by $[3/2, 2]$, and let

$$h(x) = \sum_{j \geq 0} 2^{-2kj} \psi(2^{2j} x).$$

This is supported by $[0, 2]$, and if ψ is the analyzing wavelet and $a = 2^{-j}$, $C(a,b)$ vanishes outside the interval $|b| \leq 2^{-j}/2$ (if $|b| \leq 2^{-j}/8$), but $C(a,0) = C 2^{-kj}$. Thus $C(a,b)$ has a line of maxima that goes through the interval $|b| \leq 2^{-j}/2$ and the supremum on this line is of the order of magnitude of 2^{-kj}. Now suppose that $F = g$ in a neighborhood of 0 (which we can always assume if $\Omega \neq K$). Then $Z(a,q)$ is larger than Ca^{-kq}. We see that no bound of $Z(a,q)$ can be found independent of k. If the multifractal formalism held, since $d(\alpha)$ is independent of k, the order of magnitude of $Z(a,q)$ would be as well. Hence we have a contradiction.

We make one final remark on Theorem 2. First, suppose that g is C^∞. If $\alpha \leq \alpha_0$, the infimum in the Legendre transform is obtained for $q > 0$ (because $\tau(0) = -d_{\max}$ and τ is convex and increasing). In that case, we cannot directly calculate $d(\alpha)$ up to α_0 since we cannot use a C^∞ wavelet with all vanishing moments, but following [1], this can be done using a sequence of increasingly smoother wavelets and determining increasingly larger parts of the spectrum. The case where g is C^k is still easier to check.

We now want to show that using the wavelet-box method, we can recover the left part of the spectrum corresponding to negative values of q when

$$\overline{\cup S^j(\Omega)} = \Omega.$$

(This implies that there exists no region where F is smooth.) The validity of this condition can actually be checked on the part of the spectrum computed for $q > 0$ since at the maximum (the case where $q = 0$) $d(\alpha) = d_{\max}$, which satisfies

$$\sum \mu_j^{d_{\max}} = 1.$$

However, the condition $\overline{\cup S^j(\Omega)} = \Omega$ can be rewritten $\sum \mu_j^m = 1$ since $\mathrm{Vol}(\Omega_j) = \mu_j^m \mathrm{Vol}(\Omega)$. Thus $\overline{\cup S^j(\Omega)} = \Omega$ is equivalent to $d_{\max} = m$, which is easy to check.

In this case, the tree T leaves no void in the region of the upper half-plane above Ω. (By this we mean that for any (a,b) in this region, we can find a point of the tree in a domain $[a/C, Ca] \times [b - Ca, b + Ca]$); however, after perhaps increasing the constant C, we can also find a point $(\mu_i, S^i(t))$ where the order of magnitude of the wavelet transform of F is λ_i. Thus if we modify the wavelet-maxima method by imposing the condition that we first take the largest local maximum on the box $[a/C, Ca] \times [b-Ca, b+Ca]$ (which amounts to considering a kind of maximal function), we see that for a given scale a,

$$\sum_{\max} |C(a,b)|^q \sim \sum_{a \sim \mu_i} |\lambda_i|^q.$$

Returning to the estimation of (6.5), we see that

$$\limsup_{a \to 0} a^{-\tau(q)} \sum_{\max} |C(a,b)|^q db \geq C > 0,$$
$$\limsup_{a \to 0} \frac{a^{-\tau(q)}}{|\log a|} \sum_{\max} |C(a,b)|^q db \leq C' < +\infty.$$

Hence we have the multifractal formalism in that case.

Note that the constant C in the definition of the wavelet-box method must be chosen "large enough" depending on the self-similar function that is analyzed.

8. Unbounded self-similar functions. Thus far, we made the assumption $\alpha_{\min} > 0$ (which is equivalent to $|\lambda_j| < 1\ \forall j = 1, \ldots, d$). This implied that $F \in C^{\alpha_{\min}}$. Thus we were interested only in Hölder exponents larger than $\alpha > 0$. However, we would like to consider negative exponents, which, as mentioned before, should be pertinent in some applications. We have already discussed the definition of negative exponents in Part I. We will now show that the multifractal formalism holds using a slightly different definition for the Hölder spectrum. We suppose that

$$\sum_{i=1}^{d} |\lambda_i||\mu_i|^m < 1 \tag{8.1}$$

so that a function F that satisfies (2.3) belongs to L^1. Condition (8.1) can clearly give rise to unbounded functions. In fact, the following result holds.

LEMMA 8.1. *Suppose that one of the λ_j's satisfies $|\lambda_j| > 1$ and that g does not vanish identically. Then $\forall x \in K$, F is unbounded in any neighborhood of x so that $d(\alpha) = 0\ \forall \alpha$.*

Proof. First, note that a straightforward estimation yields that if $F \in L^\infty$, then $|C(a,b)| \leq C$ so that if F is bounded in a neighborhood of x $|C(a,b)| \leq C$ if b is in a (perhaps) smaller neighborhood of x.

Let $x \in K$ and $i = (i_1, \ldots, i_n, \ldots)$ such that $x = x(i)$. Let j be such that $|\lambda_j| > 1$. Then we fix an n and define $i' = (i_1, \ldots, i_n, j, j, \ldots) = (i'_1, \ldots)$. Proposition 4.1 shows that there exists an x_0 such that if $b_l = S^{i'_1} \cdots S^{i'_l}(x_0)$ and $a_l = \mu_{i'_1} \cdots \mu_{i'_l}$, $|C(a_l, b_l)| \sim |\lambda_{i'_1} \ldots \lambda_{i'_l}|$ and is thus unbounded. If n is large enough and $l \geq n$, the points b_l are arbitrarily close to x so that F is unbounded in any neighborhood of x. Hence we have Lemma 8.1.

In this case, the Hölder spectrum of F for $\alpha > -n$ is trivial: $d(\alpha) = 0\ \forall \alpha > -n$.

Note that if $\sup |\lambda_j| = 1$, F can be either unbounded or bounded depending on the specific values taken by the function g, as shown by the following example.

Suppose that F satisfies

$$F(x) = F(2x) + g(x)$$

and suppose that g is Lipschitz. If g is positive and does not vanish at 0, F is unbounded, whereas if $g(0) = 0$, F is bounded.

Lemma 8.1 suggests that if F is unbounded, the spectrum defined by the Hausdorff dimension of α-singularities is not the right quantity to consider, but we should instead compute the packing dimension of strong singularities.

We now suppose that the separated open-set condition holds and that g and the λ_j's are positive. We prove Theorem 2.2 in that case.

Let $x \in K$ and $i = (i_1, \ldots, i_n, \ldots)$ be the (unique) branch over x. We define two subsets A_1 and A_2 of $B(x, \mu_i)$ as follows. First, let Ω' be a set where $g(x) \geq C > 0$. Suppose that $\lambda_1 > 1$ and let $i' = (i_1, \ldots, i_n, 1, \ldots, 1)$, where the number p of ones will be made precise later. Since

$$F(y) = \sum \lambda_i g(S_i^{-1}(y)),$$

if $y \in S^{i'}(\Omega)$, $F(y) \geq C_1 \lambda_i \lambda_1^p$. $S^{i'}(\Omega)$ is the set A_1. If $\operatorname{dist}(y, K) \geq \mu_i/2$, $F(y) \leq C_2 \lambda_i$. A_2 is the set of points of $B(x, \mu_i)$ such that $\operatorname{dist}(y, K) \geq \mu_i/2$.

We choose p such that $C_1 \lambda_1^p \geq 2C_2$. The volumes of A_1 and A_2 are $\sim \mu_i^m$ (because of the separated open-set condition). Thus if

$$\beta(x) = \limsup \frac{\log |\lambda_{i_1}(x)| \cdots |\lambda_{i_N}(x)|}{\log \mu_{i_1}(x) \cdots \mu_{i_N}(x)}, \tag{8.2}$$

F has a strong singularity of order $\beta(x)$ at x. Furthermore, the order of magnitude of F on a subset of $B(x, C\mu_{i_1}(x) \cdots \mu_{i_N}(x))$ of size comparable to the size of this ball is exactly $\mu_{i_1}(x) \cdots \mu_{i_N}(x)$ so that if $\beta(x)$ is negative, the order of the strong singularity at x is not higher than $\beta(x)$. Hence we have the first part of the following proposition. (The last part is a direct consequence of Proposition 4.1.)

PROPOSITION 8.2. *If the separated open-set condition holds, if g and the λ_j's are positive, and if* (8.1) *holds, F has a "strong singularity" of order $\beta(x)$ at x but no strong singularity of higher order. Furthermore, without any assumption on g, the S_i's or the λ_i's, F has a "wavelet singularity" of order $\beta(x)$ at x.*

It would be interesting to prove this proposition without the assumption of the separated open-set condition. It clearly holds if $K \neq \Omega$ (i.e., if $\sum \mu_i^m \neq 1$) because in that case the choice that we made for the y's is still possible.

The case where $K = \Omega$ is perhaps less interesting for applications since f is then nowhere locally bounded (by Lemma 7.2), which is usually not realistic for "physical" functions. Actually, in the case of three-dimensional turbulence, the singularities seem to concentrate on a set of dimension < 3 (see [1] or [23]).

We now determine the packing dimension of the strong α-singularities of F.

PROPOSITION 8.3. *Under the same assumptions, the packing dimension of the strong α-singularities of f is given by $D(\alpha) = \inf(a\alpha - \tau(a))$.*

We already know that $D(\alpha) \leq \inf(a\alpha - \tau(a))$, so we have to prove only the lower bound. We will use the following result (see [14] or [25]).

PROPOSITION 8.4. *Let H^s be the packing measure of dimension s. Let μ be a probability measure on $\mathbb{R}^m$, $F \in \mathbb{R}^m$, and C be such that $0 < C < +\infty$.*

If $\liminf_{r \to 0} \mu(B_r(x))/r^s < C \ \forall\, x \in F$, $H^s(F) \geq \mu(F)/C$.

The proof of Proposition 8.3 follows the proof of Proposition 5.1 since we can take exactly the same measure μ and the same set F. It is now easy to check that the multifractal formalism holds in this setting because the proof of Proposition 6.2 actually holds without any change.

We conclude this paper with the proof of the following corollary, which shows that the multifractal formalism holds in a more general setting.

COROLLARY 8.5. *Let A be a pseudodifferential elliptic operator of order s whose symbol σ satisfies*

$$\forall\alpha,\ \beta,\quad |\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \le C(\alpha,\beta)(1+|\xi|)^{s-\beta}.$$

Suppose that h is a self-similar function that satisfies

$$h(x)=\sum_{i=1}^{d}\lambda_i h(S_i^{-1}(x))+g(x),$$

and if s is negative, suppose further that the moments of g of order at most $|s|$ vanish.

Let $F = A(h)$. If $F \in C^\epsilon(\mathbb{R}^m)$ for an $\epsilon > 0$, then its spectrum is a concave function whose increasing part is given by

$$d(\alpha)=\inf_q(\alpha q-\eta(q)+m).$$

Proof. Following a result of Calderón and Zygmund [7], A can be written $\tilde{A}(-\Delta)^{m/2}$ (up to a regularizing operator), where $\tilde{A}$ and $\tilde{A}^{-1}$ are Calderón–Zygmund operators. Clearly, $(-\Delta)^{m/2}h$ is a self-similar function if $m \ge 0$ and also if $m < 0$ and g has the corresponding number of vanishing moments. Thus the multifractal formalism holds for F because Calderón–Zygmund operators are continuous on the Besov spaces so that $(-\Delta)^{m/2}h$ and F share the same function η. Since these operators are also continuous on the two-microlocal spaces (see [18]), $(-\Delta)^{m/2}h$ and F have the same Hölder regularity at each point (perhaps up to a logarithmic correction).

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