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On iterations of $1 - ax^2$ on $(-1, 1)$

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Chapter I

1. Introduction

We study iterations of $F(x; a) = 1 - ax^2$, $-1 < x < 1$, where a is a parameter in the interval $(0, 2)$. Let the ν^{th} iteration of F be denoted by $F^\nu(x; a)$ and in particular set $\xi_\nu(a) = F^\nu(0; a)$. We shall consider two related problems.

1) Does F have an attractive cycle? It is known, see Section 4 below, that this is not the case if $\overline{\lim}_{\nu \to \infty} |\partial_x F^\nu(1; a)| = \infty$.

2) Suppose that for a certain value of a there are no attractive cycles. In this case, the question arises as to what can be said of the distribution of $\{\xi_\nu\}_{\nu=1}^\infty$ and in particular of what type is the corresponding invariant measure of the map $x \to F(x; a)$.

The last problem was treated by Jakobson [3]. For the first problem and related results see the book [1] and the paper [2] by Collet-Eckmann.

In Part II of our paper we prove that for a set of a-values Δ_∞ of positive Lebesgue measure $x \to F(x; a)$ has no attractive cycles, and in Part III it is proved that for almost all $a \in \Delta_\infty$, $x \to F(x; a)$ has an absolutely continuous invariant measure. We even prove that the density function of the invariant measure belongs to L^p for $p < 2$, which is best possible in general. The proof also shows that for $2 - a_0$ small enough the set of $a \in (a_0, 2)$ for which $F(x; a)$ has an attractive cycle is open and dense in Δ_∞.

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2. Notation

We shall study how $\xi_\nu(a) = F^\nu(0; a)$ returns to a (fixed) short interval $I^* = (-\delta, \delta)$, $\delta = e^{-\sqrt{\alpha}}$, and in particular how close they are to the origin. We divide $I^* = \bigcup_{|\mu| > \alpha} I_\mu$ so that $I_\mu = (c_\mu, c_{\mu-1})$ where $c_\mu = \exp\{-\sqrt{\mu}\}$ for $\mu > 0$ and $I_\mu = -I_{-\mu}$, $\mu < 0$.

We shall consider three types of returns to $(-\delta, \delta)$, called *free*, *bound* and *inessential* and denoted by x, y and z respectively. If the free returns are $x_1, x_2, \dots,$ each x_i is followed by bound returns y_{ij}, $j = 1, \dots, Y_i$ and these *could* be followed by inessential returns z_{ij}, $j = 1, \dots, Z_i$. The sequence $\{m_i\}_{i=1}^{\infty}$ consists of the indices of the free returns as iterations, i.e. $x_i(a) = \xi_{m_i}(a)$.

3. Some lemmas

For the proof we need some initial information about the behaviour of iterations of F. Here this is obtained by a perturbation calculation from the case $F(x; 2) = 1 - 2x^2$, which is formulated in the following sequence of lemmas.

LEMMA 1. *For sufficiently small $\delta > 0$, there exists $a_0 < 2$ such that if $a \in [a_0, 2]$ and if $\eta_0 \in [-1, 1]$ and k are such that*

$$\left|F^j(\eta_0; a)\right| \geq \delta, \qquad j = 0, 1, 2, \dots, k-1,$$

and

$$\left|F^k(\eta_0; a)\right| \leq \delta,$$

then

$$\left|\partial_x F^k(\eta_0; a)\right| \geq (1.9)^k. \tag{3.1}$$

Proof. We make the standard change of variables $x = \varphi(\theta) = \sin(\pi/2)\theta$. Let $\tilde{F}(\theta; a)$ denote the transformation expressed in the new variables; i.e.,

$$\tilde{F}(\theta; a) = \frac{2}{\pi}\arcsin\left(1 - a\sin^2\frac{\pi}{2}\theta\right).$$

Its θ-derivative is

$$\partial_\theta \tilde{F}(\theta; a) = -\sqrt{2a}\,\operatorname{sgn}(\theta)\frac{\cos\frac{\pi}{2}\theta}{\sqrt{\cos^2\frac{\pi}{2}\theta + \left(\sin^2\frac{\pi}{2}\theta\right)\left(1 - \frac{a}{2}\right)}}.$$

Let $\eta_\nu = F^\nu(\eta_0; a)$ and assume that

$$|\eta_\nu| \geq 1 - 2\delta^2, \qquad \nu = 0, 1, 2, \dots, j-1,$$

but $|\eta_j| \leq 1 - 2\delta^2$ and write $F^k = \varphi \circ \tilde{F}^{k-j} \circ \varphi^{-1} \circ F^j$. We have

$$\partial_x F^k = \varphi'(\theta_k)\prod_{\nu=j}^{k-1}\partial_\theta \tilde{F}(\theta_\nu; a)\frac{d}{dx}\varphi^{-1}(\eta_j)\prod_{\nu=0}^{j-1}(-2a\eta_\nu). \tag{3.2}$$

Since $|\eta_{\nu-1}| \geq \delta$, $\nu = 1, 2, \dots, j$, it follows inductively that $|\eta_\nu| \leq 1 - a\delta^2$,

$\nu = j, \ldots, k$ and from (3.2) and $|\eta_\nu| \geq 1 - 2\delta^2$, $\nu = 0, \ldots, j-1$, (3.1) is easy to establish.

LEMMA 2. *There are constants* $\gamma > 0$ *and* $K = K(\delta)$ *such that when* $a_0 \in (0,2)$ *is sufficiently close to* 2, *for all* $a \in (a_0, 2]$ *and all* $|x| > c_{\alpha+1}$, *there exists* $l = l(x,a) \leq K$ *such that*

(i) $|F^i(x;a)| \geq \delta, \qquad i = 1, \ldots, l-1,$

(ii) $\log|\partial_x F^l(x;a)| \geq \gamma l.$

Proof. Let $\eta_\nu = \mathrm{F}^\nu(x;a)$. When $|\eta_0| \geq \frac{1}{2}$, $|\partial_\eta F(\eta_0)| = |2a\eta_0| \geq a$. We observe that $\xi = -1 + \varepsilon$, $\sqrt{(2-a)/a} < \varepsilon < \frac{1}{8}$, implies $\xi < F(\xi) \leq -1 + 4\varepsilon$. We chose l so that $2^{-l+1} \geq |\eta_0| \geq 2^{-l}$, $l \geq 2$, and note that

$$\begin{aligned} \left|\partial_x F^l(\eta_0; a)\right| &\geq |2a\eta_0|\,|2a\eta_1| \cdots |2a\eta_{l-1}| \\ &\geq (2a)\cdot 2^{-l}\cdot(2a)(1 - 2\cdot 2^{-2l+2}) \\ &\quad \cdots (2a)(1 - 4\cdot 2\cdot 2^{-2l+2}) \cdots (2a)\cdot\left(1 - \tfrac{1}{2}\right) \\ &= (a)^l\cdot\left(1 - \tfrac{1}{2}\right)\cdots\left(1 - 2^{-2l+3}\right). \end{aligned}$$

The lemma is therefore proved with $\gamma = \frac{1}{2}\log 2$ and $K = 2 + [{}^2\log(1/\delta)]$.

LEMMA 3. *For any sufficiently small* $\delta > 0$, *any* $a_0 < 2$ *and any positive integer* N, *there is an integer* $m_1 \geq N$ *and an interval* $\Delta_0 \subset (a_0, 2)$ *such that*:

(i) *For any* $a \in \Delta_0$, $\xi_\nu(a) = F^\nu(0;a) \leq -\frac{1}{2}$ *for* $2 \leq \nu \leq m_1 - 1$.

(ii) $a \mapsto F^{m_1}(0;a)$ *is a one-to-one map of* Δ_0 *onto* $I^* = (-\delta, \delta)$.

(iii) $|\partial_x F^j(1;a)| \geq (1.9)^{j-1}; \qquad j = 1, 2, \ldots, m_1 - 1.$

Proof. Since

$$\begin{aligned} \xi_{\nu+1} - \xi_\nu &= 1 - a\xi_\nu^2 - \xi_\nu \\ &= 2 - a + a(\xi_\nu + 1)\left[1 - \frac{1}{a} - \xi_\nu\right], \end{aligned}$$

it follows that while $\xi_\nu \leq -\frac{1}{2}$, $\{1 + \xi_\nu\}$ is exponentially increasing. Furthermore it follows inductively that if $\xi_\nu \leq 0$ for $2 \leq \nu \leq j$ then $a \to \xi_{j+1}(a)$ is decreasing since

$$\frac{d\xi_{j+1}}{da} = -\xi_j^2 - 2a\xi_j\frac{d\xi_j}{da} \leq -\xi_j^2.$$

(i) and (ii) of the lemma are therefore easy to establish and since

$$\left|\partial_x F^j(1;a)\right| = \prod_{\nu=1}^{j}(-2a\xi_\nu),$$

where $\xi_1 = 1$ and $-1 \le \xi_\nu \le -\frac{1}{2}$, $\nu = 2, \dots, j$, $j \le m_1 - 1$. (iii) is also evident.

We next turn to an important general principle, which will be used throughout this paper: Under suitable assumptions the x- and a-derivatives of F^k are comparable.

For the derivatives ∂_x and ∂_a we have in our case the two recursion formulas

$$\partial_x F^{\nu+1} = -2aF^\nu \partial_x F^\nu, \qquad \partial_x F^0 = 1$$

and

$$\partial_a F^{\nu+1} = -2aF^\nu \partial_a F^\nu - (F^\nu)^2, \qquad \partial_a F^0 = 0.$$

This gives

$$\partial_x F^\nu = \prod_{i=0}^{\nu-1} (-2aF^i), \qquad \nu = 1, 2, \dots, \tag{3.3}$$

and

$$\partial_a F^\nu = \partial_x F^\nu \frac{x}{2a} \prod_{i=1}^{\nu-1} \left(1 + \frac{F^i}{2a\,\partial_a F^i}\right), \qquad \nu = 2, 3, \dots,$$

$$\partial_a F = -x^2. \tag{3.4}$$

These formulas show that $\partial_x F^\nu$ and $\partial_a F^\nu$ grow at the same rate. The following version of this principle will be used in the sequel.

LEMMA 4. *There exist* $\delta > 0$ *and* $a_0 < 2$ *such that, if*
1. $1 - 2\delta^2 \le \eta_1 \le 1$
2. $a_0 \le a \le 2$
3. $|\partial_x F^{j-1}(\eta_1; a)| \ge e^{j^{2/3}}$ *for* $j = 8, 9, \dots, k$, *then*

$$\frac{1}{16} \le \frac{|\partial_a F^{k-1}(\eta_1; a)|}{|\partial_x F^{k-1}(\eta_1; a)|} \le 16. \tag{3.5}$$

Proof. First it follows by continuity and a simple numerical calculation that if δ is sufficiently small, if $1 - \delta^2/10 \le a_0 \le a \le 2$ and $1 - 2\delta^2 \le \eta_1 \le 1$, then

$$\frac{\eta_1}{2a} \prod_{i=1}^{7} \left(1 - \frac{|F^i(\eta_1; a)|}{|2a\,\partial_a F^i(\eta_1; a)|}\right) \ge \frac{1}{8}.$$

Furthermore we verify that

$$\prod_{i=8}^{\infty}\left(1 - \frac{8}{e^{i^{2/3}}}\right) \geq \frac{1}{2}.$$

When these estimates are inserted in (3.4), then (3.5) readily follows.

Remark. With computer calculations similar lemmas could be proved for a in small intervals Δ_0 in a more general position. Certain modifications are then also needed later in the proof. Such a systematic investigation would seem to be of interest.

Chapter II

4. The non-existence of attractive cycles

Our first main result is

THEOREM 1. *There is a set $\Delta_\infty \subset (0,2)$ of positive Lebesgue measure, such that for all $a \in \Delta_\infty$ the map $F(\cdot; a): x \mapsto 1 - ax^2$ from $[-1,1]$ into $[-1,1]$ has no attractive cycles (stable periodic orbits).*

From Corollary II.4.2 of Collet and Eckmann [1] it follows that if for a certain a the point $x = 0$ is not attracted to a stable periodic orbit then $F(\cdot; a)$ has no stable periodic orbits.

Thus Theorem 1 follows from:

THEOREM 2. *There is a set $\Delta_\infty \subset (0,2)$ of positive Lebesgue measure and an integer ν_0, such that for all $a \in \Delta_\infty$,*

$$|\partial_x F^\nu(1; a)| \geq e^{\nu^{2/3}}, \qquad \nu \geq \nu_0.$$

Remark. By a minor modification of the proof we can replace $e^{\nu^{2/3}}$ by e^{ν^σ} for any $\sigma < 1$. What one expects to be true is however that $|\partial_x F^\nu(1; a)|$ grows exponentially almost everywhere on Δ_∞.

This is indicated by the following argument. In the second part we prove the existence of a "good" invariant measure μ. It follows that for almost all a

$$\log|\partial_x F^\nu(1, a)| \sim \nu \int (\log|2ax|)\, d\mu(x)$$

and

$$\int (\log|2ax|)\, d\mu(x) \geq 0,$$

and one can therefore expect exponential increase in general. However, we do not have uniform exponential increase for all a considered, and the introduction of $\sigma < 1$ has allowed us to make very simple rules for the definition of Δ_∞.

The rest of the first part of the paper is devoted to the proof of Theorem 2. The set Δ_∞ is constructed as $\bigcap_{k=0}^{\infty}\Delta_k$, $\Delta_0 \supset \Delta_1 \supset \cdots \searrow \Delta_\infty$, where Δ_k is defined at the k^{th} induction step.

5. The partition

We wish to define recursively the concept of a *free return* to I^* and the *partition associated with a free return*.

The index of a free return is denoted by k.

By Lemma 3 there is an integer m_1 such that $a \overset{f_1}{\mapsto} F^{m_1}(0;a)$ is a one-to-one map of Δ_0 onto I^*. Also m_1 is the index of the first free return. The corresponding partition is

$$\left\{ f_1^{-1}(I_{\alpha+1}), f_1^{-1}\left(I_{-(\alpha+1)}\right)\right\} \quad \text{and} \quad \Delta_1 = f_1^{-1}\left(I_{-(\alpha+1)}\right) \cup f_1^{-1}(I_{\alpha+1}).$$

The set $E_1 = \Delta_0 \setminus \Delta_1$ is excluded from further consideration.

Let now ω be an interval of the k^{th} partition, $k \geq 1$. We intend to define the $(k+1)^{\text{st}}$ free return and the corresponding partition.

Suppose that $a \to F^{m_k}(0;a)$ maps $\omega = [a_1, a_2]$ onto I_μ. For convenience we assume that $\mu > 0$. We wish to study the further evolution of ω under mappings by F. The integer $p = p(\omega)$ is defined by the following stopping rule:

$$\left|\xi_j(a) - F^j(\eta;a)\right| \leq \frac{\left|\xi_j(a)\right|}{10 j^2} \tag{5.1}$$

holds for all $a \in \omega$, and all η, $0 < \eta < c_{\mu-1}$, for $j = 1, 2, \ldots, p$ but

$$\left|\xi_{p+1}(a) - F^{p+1}(\eta;a)\right| > \frac{\left|\xi_{p+1}(a)\right|}{10(p+1)^2} \tag{5.2}$$

for some $a \in \omega$ and some $\eta, 0 < \eta < c_{\mu-1}$. For certain j's, $1 \leq j \leq p$, $\xi_{m_k+j}(a)$ may return to I^*. Those are the *bound returns* $\{y_{ki}\}_{i=1}^{Y_k}$.

Let i be the smallest integer $i \geq p+1$, such that $F^{m_k+i}(0;\omega) \cap I^* = J_i \neq \varnothing$. Let S_i be the set of ν so that (i) $I_\nu \subset J_i$ and (ii) $|\nu| \leq \alpha + k$. The sets of $a \in \Delta_k$ for which $\xi_{m_k+i}(a) \in I_\nu$ with $|\nu| \geq \alpha + k + 1$ are now excluded and are denoted E_{ki}.

(a) If $S_i = \varnothing$ we do not make any construction of a partition but go on to consider the next integer $i+1$. Then $n_{k1} = m_k + i$ is the index of the first *inessential return*. Now "bound" inessential returns defined by the relations (5.1) and (5.2) with $\mu = \nu_0$, where $|\nu_0| = \min_{\nu \in S_i}|\nu|$ may follow. At the return thereafter, case (a) may occur again. In that case this "free" inessential return is $\xi_{n_{k2}}$, which in its turn may be followed by "bound" inessential returns and then a "free" inessential return $\xi_{n_{k3}}$ etc. The set of *all* inessential returns following x_k up to x_{k+1} is denoted $\{z_{ki}\}_{i=1}^{Z_k}$. These are thus all returns between n_{k1} and m_{k+1}.

(b) If $S_i \neq \varnothing$, let $\omega_{i\nu}$ be the subinterval of ω mapped to I_ν, $\nu \in S_i$. Then $|\nu| > \alpha$. Let K be one of the possibly 0, 1 or 2 intervals of the set

$$F^{m_k+i}(0;\omega)\setminus\left(\left(\bigcup_{\nu\in S_i} I_\nu\right)\cup(-c_{\alpha+k+1}, c_{\alpha+k+1})\right).$$

If $K \subset I_{\nu'}$ for some $I_{\nu'}$, which is then adjacent to one of the selected I_ν's we expand the corresponding $\omega_{i\nu}$ so that $F^{m_k+i}(0;\omega_{i\nu}) = I_{\nu'\pm 1} \cup K$. Observe that if $\nu = \pm\alpha$ this means that we go slightly outside $(-\delta, \delta)$.

The construction means that

$$I_\nu \subset F^{m_k+i}(0;\omega_{i\nu}) \subset I_\nu \cup I_{\nu\pm 1} \tag{5.3}$$

for suitable ν and suitable sign $\pm$. For $a \in \omega_{i\nu}$ the next free return $m_{k+1}(a) = m_k + i$ is defined.

We have now constructed $\omega_{i\nu}$, $\nu \in S$, in our partition P and have obtained 0, 1 or 2 remaining intervals of ω. On each of these we repeat the construction by considering $i+1, i+2, \ldots$ until we get an intersection with I^* as above. We continue the construction and get a partition $P(\omega; m_k) = \{\omega_{i\nu}\}$. The set ω may be expressed as the disjoint union

$$\omega = \left(\bigcup_{i,\nu}\omega_{i\nu}\right)\cup R\cup\left(\bigcup_{i,\nu}E_{i\nu}\right)$$

where R consists of those a which never return to I^* inside $[-c_{\alpha+1}, c_{\alpha+1}]$.

By Lemma 2 and Lemma 3, (iii),

$$|\partial_x F^\nu(1;a)| \geq e^{\gamma\nu}, \qquad a \in R, \qquad \gamma > 0, \qquad \nu \geq \nu_0,$$

and hence Theorem 2 holds for $a \in R$.

The set Δ_{k+1} is now defined as

$$\Delta_{k+1} = \Delta_k \setminus \bigcup_{\omega\subset\Delta_k}\ \bigcup_{i,\nu} E_{i\nu}.$$

We make two comments. It follows from Section 12 that the measure of R is zero. From the present point of view this is irrelevant since it is obvious that attractive cycles do not exist for $a \in R$. The second comment concerns the partition P. For the intervals ω, the associated intervals $\subset I^*$ are precisely intervals I_μ except at the ends where we can have the situation (5.3). In what follows we are going to ignore this possibility, in order to avoid cumbersome notation.

6. The main induction step

We assume that for all free returns of order k and for all $a \in \Delta_k$ the following are true:

(i) If $m_k = m_k(a)$ is an index of a free return of order $k \geq 1$,

$$\left|\partial_x F^{m_k - 1}(1; a)\right| \geq e^{2m_k^{2/3}}.$$

(ii) If $4 \leq i \leq m_k(a)$, then

$$\left|\partial_x F^{i-1}(1; a)\right| \geq e^{i^{2/3}}.$$

(iii) $|\xi_j(a)| \geq e^{-\sqrt{j}}, \qquad 1 \leq j \leq m_k.$

Note first that Lemma 3 implies that (i), (ii) and (iii) hold for $k = 1$.

We shall now turn to the proof of the main induction step and intend to prove (i), (ii) and (iii) for index $k + 1$.

Let ω be an interval of Δ_k and let $p = p(\omega)$, m_k and μ be as defined in Section 5. We also assume that the induction step is proved in the case where ω is mapped onto an interval I_ν with $|\nu| < \mu$ if such ν exist. We first use the induction assumption (i) and (5.1) to give an upper bound for p in terms of μ.

By the mean value theorem

$$\xi_p(a) - F^p(\eta; a) = F^{p-1}(1; a) - F^{p-1}(1 - a\eta^2; a) \tag{6.1}$$

$$= a\eta^2 \partial_x F^{p-1}(1 - a\eta'^2; a),$$

where $0 < \eta' < \eta < c_{\mu-1}$.

From (3.3) (the product formula for $\partial_x F$) and (5.1), it follows that

$$2^{-1} \leq C^{-1} \leq \left|\frac{\partial_x F^\nu(1 - a\eta^2; a)}{\partial_x F^\nu(1; a)}\right| \leq C \leq 2, \tag{6.2}$$

$\nu = 1, 2, \ldots, p - 1$, since C may be chosen as

$$\exp\left\{\frac{1}{10}\sum_1^\infty \frac{1}{n^2}\right\} \leq 2$$

and the same estimate holds for all η', $0 < \eta' < \eta$.

Hence it follows from (5.1) and (6.1) that

$$\left|\partial_x F^{j-1}(1; a)\right| \leq 2\frac{1}{10 j^2 a} e^{2\sqrt{\mu}} \leq e^{2\sqrt{\mu}}.$$

As long as $j - 1 \leq m_k$, the induction hypothesis (ii) implies that

$$e^{j^{2/3}} \leq e^{2\sqrt{\mu}}.$$

By (5.1) and Lemma 3,

$$j \leq 2^{3/2}\mu^{3/4} \leq 2^{3/2}(\alpha + k)^{3/4} \leq 2^{3/2}(\alpha^{3/4} + k^{3/4}) \tag{6.3}$$
$$\leq 15^{-3}\left(\tfrac{1}{2}m_1 + \tfrac{1}{2}m_k\right) \leq 15^{-3}m_k < m_k$$

provided that δ is small enough and a_0 is close enough to 2.

Hence the inequality $j \leq 2^{3/2}\mu^{3/4}$ will be violated before the inequality $j \leq m_k$ as j increases and it follows that

$$p \leq 2^{3/2}\mu^{3/4}. \tag{6.4}$$

Let $\omega = (a, b)$. Next we give a lower bound for the length of $\Omega = F^{m_k+p+1}(0; \omega)$ which may be expressed as

$$|\Omega| = \left|F^{m_k+p+1}(0; a) - F^{m_k+p+1}(0; b)\right| \tag{6.5}$$
$$= \left|F^{p+1}\left(\xi_{m_k}(a); a\right) - F^{p+1}\left(\xi_{m_k}(b); b\right)\right|.$$

Since

$$\left|\xi_{m_k}(a) - \xi_{m_k}(b)\right| = |a - b|\,\left|\partial_a F^{m_k}(0; a')\right|$$
$$\geq |a - b| \cdot \tfrac{1}{16}e^{m_k^{2/3}}, \qquad a < a' < b,$$

the explicit dependence on the parameters a and b in (6.5) is inessential.

From the mean value theorem it follows that for some $\theta, 1 - a\delta^2 \leq \theta \leq 1$,

$$|\Omega| \geq \tfrac{2}{3}2a\left|\xi_{m_k}(a)\right|\left|\xi_{m_k}(a) - \xi_{m_k}(b)\right|\left|\partial_x F^p(\theta; a)\right| \tag{6.6}$$
$$\geq \tfrac{4}{3}ac_\mu|I_\mu|\,\left|\partial_x F^p(\theta; a)\right|.$$

From (5.2) and (6.1) with p replaced by $p + 1$, it follows that for some η and $\eta'', 0 < \eta'' < \eta < c_{\mu-1}$,

$$a\eta^2\left|\partial_x F^p\left(1 - a(\eta'')^2; a\right)\right| \geq \frac{1}{10}\,\frac{\left|\xi_{p+1}(a)\right|}{(p+1)^2}. \tag{6.7}$$

From (6.4), (6.5) and the uniformity estimate (6.2) we conclude that

$$|\Omega| \geq \frac{4}{3} \cdot \frac{1}{2} \cdot \frac{c_\mu|I_\mu|}{c_{\mu-1}^2}\,\frac{1}{10} \cdot \frac{\left|\xi_{p+1}(a)\right|}{(p+1)^2}. \tag{6.8}$$

By the induction assumption (iii), $|\xi_{p+1}| \geq e^{-\sqrt{p+1}}$. It follows from (6.4), (6.8) and $\mu \geq \alpha = (\log 1/\delta)^2$ that

$$|\Omega| \geq \frac{1}{80\mu^{1/2}} e^{-(2+\mu^{3/4})^{1/2}} \geq e^{-2\mu^{3/8}}, \tag{6.9}$$

provided that δ is small enough.

Proof of (i) *at the next free return.* We first prove:

$$\left|\partial_x F^{m_k+p-1}(1;a)\right| \geq e^{2(m_k+p)^{2/3}+(1/10)p^{2/3}}, \qquad a \in \Delta_k. \tag{6.10}$$

By the chain rule

$$\partial_x F^{m_k+p-1}(1;a) = \partial_x F^{m_k-1}(1;a) 2a\xi_{m_k}(a)\,\partial_x F^{p-1}\big(\xi_{m_k+1}(a);a\big).$$

By (6.7), the uniformity estimate (6.2) and (5.2),

$$\begin{aligned} &\left|2a\xi_{m_k}(a)\,\partial_x F^{p-1}\big(\xi_{m_k+1}(a);a\big)\right| \\ &\quad\geq 2ae^{-\sqrt{\mu}} \cdot \frac{1}{2} e^{2\sqrt{\mu-1}} \cdot \frac{1}{40} \frac{1}{(p+1)^2} |\xi_{p+1}| \\ &\quad\geq \frac{1}{40} \frac{1}{(p+1)^2} \exp\left\{\tfrac{1}{2}p^{2/3} - \sqrt{p+1}\right\} \geq e^{(1/4)p^{2/3}}, \end{aligned} \tag{6.11}$$

if δ is made sufficiently small. (Note that (6.4) implies $\sqrt{\mu} \geq \frac{1}{2}p^{2/3}$ and that (5.2) implies that there is a lower bound for p of the type $\mathcal{O}(\log 1/\delta)$ so that p is large for small δ.)

Hence, since by (6.3) $m_k \geq 15^3 p$,

$$\left|\partial_x F^{m_k+p-1}(1;a)\right| \geq e^{2(m_k+p)^{2/3}+(1/10)p^{2/3}}$$

for $a \in \omega$, and (6.10) is verified.

According to the rules of the definition of the partition we now follow the image of ω until $F^{m_k+p+i}(0;\omega)$ hits I^* for $i = i_1$. From Lemma 1 and (6.10) it follows that

$$\left|\partial_x F^{m_k+p+i_1-1}(1;a)\right| \geq e^{2(m_k+p+i_1)^{2/3}}.$$

If the return $\xi_{m_k+p+i_1}(a)$ is free, the proof of (i) is complete. If the return is inessential it follows from (8.9) that $F^{m_k+p+i_1}(0;\omega)$ hits I_ν, with $\nu < \mu$. The preceding argument for I_ν may be used to conclude that at the next return

$$\left|\partial_x F^{n_{k2}-1}(1;a)\right| \geq e^{2n_{k2}^{2/3}}.$$

We keep on until the return $\xi_{n_{ks}}$ is free. This happens eventually since the lengths of the images of ω at the returns are rapidly increasing.

Note that we have also proved that

$$(6.12) \qquad |\partial_x F^{n_{kj}-1}(1;a)| \geq e^{2n_{kj}^{2/3}}$$

holds for the inessential returns $\xi_{n_{kj}}$.

Proof that (ii) *holds for* $m_k \leq i < m_{k+1}$. We give the proof in the case $m_k \leq i < n_{k1}$. The case $n_{kj} \leq i < n_{k,j+1}$ is similar. Assume first that $i - m_k \leq p_k < m_k$, where $p = p_k$ is defined by (5.1) and (5.2). From the induction assumption (i) combined with (ii) and the chain rule it follows that

$$\begin{aligned} |\partial_x F^{i-1}(1;a)| &\geq e^{2m_k^{2/3}} e^{-\sqrt{\mu}} e^{(i-1-m_k)^{2/3}} \\ &\geq e^{m_k^{2/3}+(i-1-m_k)^{2/3}} e^{m_k^{2/3}-\sqrt{\mu}} \\ &\geq e^{i^{2/3}}. \end{aligned}$$

In the last inequality we have used the fact that $m_k \geq \mu$. For $i > p_k + m_k$ we use (6.10) and Lemma 2 and note that

$$2(m_k + p_k)^{2/3} + \gamma(i - 1 - p_k - m_k) - K \log 4 \geq i^{2/3}.$$

This completes the proof of (ii).

Proof of (iii). We have to prove $|\xi_j| \geq e^{-\sqrt{j}}$ for $m_k \leq j \leq m_{k+1}$. Note that by Lemma 3, m_1 may be chosen $\geq 2\alpha$. It follows that

$$m_k \geq \tfrac{1}{2}m_k + \tfrac{1}{2}m_1 \geq k + 1 + \alpha$$

for $2 - a_0$ sufficiently small, since $m_k \geq m_1$ and $m_1 \to \infty$, as $a_0 \to 2$. Hence

$$|\xi_j| \geq e^{-\sqrt{\alpha+k+1}} \geq e^{-\sqrt{m_k}} \geq e^{-\sqrt{j}},$$

for $m_k \leq j \leq m_{k+1}$.

Estimate of the excluded set. To complete the proof of Theorem 2 we have to show that the set Δ_{k+1} defined in Section 5 satisfies $|\Delta_{k+1}| \geq |\Delta_k|(1 - \alpha_k)$, where $0 \leq \alpha_k < 1$ and $\prod_{k=1}^{\infty}(1 - \alpha_k) > 0$. We assume temporarily that

$$C^{-1} \leq \frac{|\partial_a F^{m_{k+1}}(0; a_1)|}{|\partial_a F^{m_{k+1}}(0; a_2)|} \leq C$$

for $a_1, a_2 \in \omega$, where the constant C is independent of k. This will be proved in the next section.

It follows from (6.9) that the portion of ω excluded when Δ_{k+1} is formed is less than

$$\text{Const} \frac{e^{-\sqrt{\alpha+k+1}}}{e^{-2\mu^{3/8}}}.$$

From this we see that

$$\alpha_k \le \text{Const}\, e^{(2(k+\alpha)^{3/8} - \sqrt{\alpha+k+1})},$$

and $\prod_{k=1}^{\infty}(1-\alpha_k) > 0$. Hence $|\Delta_\infty| = |\bigcup_{k=1}^{\infty}\Delta_k| > 0$.

7. A uniform estimate for the *a*-derivative

To finish the proof of Theorem 2 we must prove:

LEMMA 5. *Suppose that $x_k(a) \in I_\mu$ is a free return after m_k iterations and that $F^{m_k}(0;\omega) \subseteq I_\mu$. Then there are constants C and C' such that for $a, b \in \omega \subset \Delta_k$ and $j < m_{k+1}$*

$$\frac{|\partial_x F^j(1;b)|}{|\partial_x F^j(1;a)|} \le C \quad \textit{and} \tag{7.1}$$

$$\frac{|\partial_a F^j(1;b)|}{|\partial_a F^j(1;a)|} \le C'. \tag{7.2}$$

Proof. It is enough to prove the estimate (7.1), since (7.2) follows from (7.1) and Lemma 2. Note that (7.1) is equivalent to

$$\left(\frac{b}{a}\right)^j \prod_{\nu=1}^{j} \frac{|\xi_\nu(b)|}{|\xi_\nu(a)|} \le C.$$

First, by the induction assumption (i) and Lemma 4

$$|\omega| \le Ce^{-m_k^{2/3}}.$$

Thus the first factor is inessential. It is sufficient to estimate

$$S = \sum_{\nu=1}^{m_k-1} \frac{|\xi_\nu(a) - \xi_\nu(b)|}{|\xi_\nu(a)|}.$$

Let $\{t_j\}_{j=1}^{N_k}$ be the indices of the free and the free inessential returns to I^* arranged in increasing order. We shall subdivide the sum

$$S = \sum_{j=1}^{N_k-1} \sum_{\nu=t_j}^{t_{j+1}-1} = \sum_{j=1}^{N_k-1} S_j.$$

At time t_j, $\xi_{t_j} \in I_{\mu_j}$, and we denote $\sigma_j = (\xi_{t_j}(a), \xi_{t_j}(b)) \subset I_{\mu_j}$. We shall first estimate the sum S_j. The contribution from the bound part $t_j \le i \le t_j + p_j$ can be estimated by an absolute constant times

$$\frac{|\sigma_j|}{c_{\mu_j}} + \frac{|\sigma_j|}{|I_{\mu_j}|} \sum_{s=1}^{p_j} \frac{c_{\mu_j}|I_{\mu_j}|\,|\partial_x F^{s-1}(1;a)|}{|\xi_s(a)|}. \tag{7.3}$$

(We have used (6.2) and the mean value theorem; compare (6.6).)

The sum is split into two parts

$$\sum_{s=1}^{p_j} = \sum_{s=1}^{p_j'} + \sum_{s=p_j'+1}^{p_j},$$

where $p_j' = \frac{1}{4}\sqrt{\mu_j}$. In the first we invoke the elementary estimates

$$\begin{cases} |\partial_x F^s| \le 4^s \\ |\xi_s(a)| \ge e^{-\sqrt{s}} \end{cases}$$

and in the second we use the stopping rule (5.1) and (6.1). The result is that the term (7.3) may be estimated by

$$\text{Const}\frac{|\sigma_j|}{|I_{\mu_j}|} \cdot \frac{1}{\sqrt{\mu_j}}.$$

After time $t_j + p_j$ the interval moves up to time t_{j+1} outside I^*. Note that by Lemma 1 and (3.2),

$$|\xi_\nu(a) - \xi_\nu(b)| \le \left[\sqrt{2a}\,|\xi_{t_{j+1}-1}|\,\left|1 - \frac{a}{2}\xi^2_{t_{j+1}-1}\right|^{1/2}\right]^{-1} \times \left(\frac{19}{10}\right)^{-t_{j+1}+\nu+1} |\xi_{t_{j+1}}(a) - \xi_{t_{j+1}}(b)|,$$

where $\xi_{t_{j+1}-1}$ is very close to $1/\sqrt{a}$. Therefore

$$\sum_{\nu=t_j+p_j+1}^{t_j+1} \frac{|\xi_\nu(a) - \xi_\nu(b)|}{|\xi_\nu(a)|} \le \text{Const}\frac{|\xi_{t_{j+1}}(a) - \xi_{t_{j+1}}(b)|}{|\xi_{t_{j+1}}(a)|};$$

so this contribution may be absorbed in $\sum_{t_{j+1}}^{t_{j+2}-1}$. Therefore

$$S \le \sum_{j=1}^{N_k} \frac{1}{\sqrt{\mu_j}} \frac{|\sigma_j|}{|I_{\mu_j}|}.$$

This sum is estimated as follows. By an argument equivalent to that in Section 6 (cf. (6.9)) it follows that

$$\left|F^{p_j}(I_{\mu_j}; a')\right| \ge e^{-2\mu_j^{3/8}}, \qquad a' \in [a,b]. \tag{7.4}$$

During the bound period $1 \le i \le p_j$, consider

$$\left|\frac{\partial_x F^i(x_1; a_1)}{\partial_x F^i(x_2; a_2)}\right| = \left(\frac{a_1}{a_2}\right)^i \frac{\prod_{\nu=1}^{i} F^\nu(x_1; a_1)}{\prod_{\nu=1}^{i} F^\nu(x_2; a_2)}$$

$$\le \left(\frac{a_1}{a_2}\right)^i \exp\left(\sum_{\nu=1}^{i} \frac{|F^\nu(x_1; a_1) - F^\nu(x_2; a_2)|}{|F^\nu(x_2; a_2)|}\right)$$

for $x_1, x_2 \in I_{\mu_j}, a_1, a_2 \in \omega$. Since

$$|F^\nu(1 - ax^2; a) - F^\nu(1; a)| \le \frac{1}{10} \frac{|\xi_{\nu+1}(a)|}{(\nu + 1)^2}$$

$x \in I_{\mu j}, a \in \omega$, this quotient is uniformly bounded by an absolute constant C_2. From this uniformity and Lemma 1 it follows that

$$|\sigma_{j+1}| \ge \left|\xi_{t_j+p_j}(a) - \xi_{t_j+p_j}(b)\right| \ge C_2^{-1} \frac{\left|F^{p_j}(I_{\mu_j}; a)\right|}{|I_{\mu_j}|} |\sigma_j| \tag{7.4}$$

$$\ge C_2^{-1} e^{\sqrt{\mu_j}} e^{-2\mu_j^{3/8}} |\sigma_j|.$$

Hence, $|\sigma_{j+1}| \ge 5|\sigma_j|$ for sufficiently small δ. (Note that $\mu_j \ge \alpha = (\log 1/\delta)^2$.) Because of this exponential increase

$$\sum_{j=1}^{N_k} \frac{1}{\sqrt{\mu_j}} \frac{|\sigma_j|}{|I_{\mu_j}|} = \sum_{s \in \{\mu_j\}} \sum_{\mu_j = s} \frac{1}{\sqrt{\mu_j}} \frac{|\sigma_j|}{|I_{\mu_j}|}$$

$$\le \text{Const} \sum_{s \in \{\mu_j\}} \frac{1}{\sqrt{s}} \frac{|\sigma_{J(s)}|}{|I_s|},$$

where $J(s)$ is the last j for which $\mu_j = s$.

This amounts to the fact that the μ_j's can be assumed to be distinct. The sum

$$\sum_{\mu_j} \frac{1}{\sqrt{\mu_j}} \frac{|\sigma_j|}{|I_{\mu_j}|}$$

is then split into two parts. Let J_1 be the set of indices such that $(1/\sqrt{\mu_j})(|\sigma_j|/|I_{\mu_j}|) \le \mu_j^{-2}$ and J_2 is the set of remaining indices. The sum

$$\sum_{j \in J_1} \frac{1}{\sqrt{\mu_j}} \frac{|\sigma_j|}{|I_{\mu_j}|} \le \sum_{\nu=1}^{\infty} \frac{1}{\nu^2}$$

and is therefore uniformly bounded. If $j \in J_2$

$$\frac{1}{\sqrt{\mu_j}} \frac{|\sigma_j|}{|I_{\mu_j}|} \geq \frac{1}{\mu_j^2}$$

and by (7.4) it follows for $k > j$ that

$$|\sigma_k| \geq C_2^{-1} 2e^{\sqrt{\mu_j}} e^{-2\mu_j^{3/8}} e^{-\sqrt{\mu_j}} \frac{1}{\mu_j^2} \geq e^{-3\mu_j^{3/8}}$$

if δ is sufficiently small. Since $I_{\mu_k} \supseteq \sigma_k$, $\mu_k \leq 9\mu_j^{3/4}$. The set $\{\mu_j\}_{j \in J_2}$ is therefore very lacunary and hence

$$\sum_{j \in J_2} \frac{1}{\sqrt{\mu_j}} \frac{|\sigma_j|}{|I_{\mu_j}|} \leq \sum_{j \in J_2} \frac{1}{\sqrt{\mu_j}}$$

is uniformly bounded. The proof of Lemma 5 is complete.

Chapter III

8. The distribution problem

Let again $\xi_n(a)$ be the images of 0 under the n^{th} iteration. Let

$$\mu_n = \frac{1}{n} \sum_{\nu=1}^{n} \delta_{\xi_\nu}. \tag{8.1}$$

We shall study the asymptotic behaviour of μ_n and more precisely prove the following theorem.

THEOREM 3. *For almost all $a \in \Delta_\infty$, defined in Section 6, the weak-* limit μ of $\{\mu_n\}_{n=1}^{\infty}$ is absolutely continuous, $d\mu = g\,dx$, and $g \in L^p$ for all $p < 2$.*

The strategy of the proof is as follows. We first study the distribution of free returns x_n on I^*.

It turns out that x_n behaves in a pseudo-random way as a "pseudo-Markov-process" and from this we deduce in Section 11 that the x_n have a distribution in I^* with a bounded density.

We next prove in Section 10 that the *inessential* returns to I^* are so few that–although we have no control of the location of these returns–they still give rise to a bounded density.

The *bound* returns, finally, reflect the tendency of the process to repeat its early history and can therefore be described in terms of the preceding free returns. These returns produce singularities s in g, which have the form

$$s = \begin{cases} |x-u|^{-1/2}, & x > u \text{ (resp. } x < u) \\ 1 & x < u \text{ (resp. } x > u), \end{cases}$$

where the u are the free returns to I^*. From this it follows that $g \notin L^2$. These results are proved in Section 11. When the distribution is known in I^* it can be transferred to $(-1, 1)$ (Section 12).

9. Distribution of the free returns

In the previous section we defined a set $A = \Delta_\infty \subset \Delta_0$, where the mapping is aperiodic. Let us renormalize the Lebesgue measure so that $mA = 1$ and consider it as a probability measure. We can then speak of expectations E and conditional probability in the usual way. We are going to study returns to some fixed interval $\omega \not\ni 0$. In doing that we can disregard the restrictions on returns, which were the basis of the definition of A.

Let $\mathscr{A}$ be a subset of Δ_0 which is defined by conditions on the free returns $x_1, \dots, x_n$. More precisely, let $1 \le \nu_1 \le \nu_2 \le \cdots \le \nu_l \le n$ and $\mu_1, \mu_2, \dots, \mu_l$ be given. Then $\mathscr{A} = \{a \in \Delta_n | x_{\nu_j} \in I_{\mu_j}, j = 1, \dots, l\}$, or $\mathscr{A} = A$. Sets of this type can be subdivided into subsets $\mathscr{A}(i)$ corresponding to the different itineraries i going through these different intervals I_{μ_j} at the ν_j^{th} free return.

We define the sets

$$A_\nu = \{a \in \mathscr{A} | x_n(a) \in I_\nu\},$$
$$A_{\nu\mu}(i) = \{a \in \mathscr{A} | x_n(a) \in I_\nu, \quad x_{n+1}(a) \in I_\mu, \quad \text{itin. } i\}$$
$$A'_\mu = \bigcup_{\nu, i} A_{\nu\mu}(i).$$

The only information available (see (6.9)) is that for each i, I_ν is increased to an interval $\Omega_\nu(i)$, where

$$|\Omega_\nu(i)| = L(\nu, i) \ge \varphi(\nu) = e^{-2\nu^{3/8}}, \tag{9.1}$$

which is then subdivided into intervals I_μ. This is the significance of the definition of the partition P (Section 5). With regard to the uniformity estimates in Section 7 this means that

$$mA_{\nu\mu}(i) \le \text{Const} \frac{|I_\mu|}{L(\nu, i)} \sum_k mA_{\nu k}(i). \tag{9.2}$$

We have the following lemma.

LEMMA 6. *Let $\mathscr{A} = \bigcup_\nu A_\nu$ be given $\subset A$ or $\mathscr{A} \subset \Delta_n$ as above. There is a constant C_0, independent of $\mathscr{A}$, so that if for some $Q > C_0$*

$$mA_\nu \le 2Q|I_\nu| m\mathscr{A} \quad \textit{for all } \nu,$$

then

$$mA'_\mu \le Q|I_\mu| m\mathscr{A}$$

for all μ.

Remark. The statement clearly means that we have an estimate of "conditional probability" of transition from ν to μ.

Proof. From (9.2) we get

$$mA'_\mu = \sum_{\nu, i} mA_{\nu\mu}(i) \le \sum_{\substack{1 \le \nu \le q \\ i}} mA_{\nu\mu}(i) + C|I_\mu| \sum_{\substack{\nu > q \\ i}} \frac{1}{L(\nu, i)} \sum_k mA_{\nu k}(i),$$

and by (11.1) the second sum may be estimated by

$$\sum_{\nu > q} \frac{1}{\varphi(\nu)} \sum_{i,k} mA_{\nu k}(i) \le 2Q \sum_{\nu > q} \frac{|I_\nu| m\mathscr{A}}{\varphi(\nu)} \le \frac{Q}{2C} m\mathscr{A}$$

if q is suitably chosen since $\sum |I_\nu|/\varphi(\nu) < \infty$. For $\nu \le q$ we have $L(\nu, i) \ge L_0$, so that

$$\sum_{\substack{\nu \le q \\ i}} mA_{\nu\mu}(i) \le \frac{C}{L_0} |I_\mu| \sum_{\substack{\nu, k \\ i}} mA_{\nu k}(i) \le \frac{C_0}{2} |I_\mu| m\mathscr{A}$$

if $C_0 \ge (2C/L_0)$. This implies

$$mA'_\mu \le \left(\frac{Q}{2} |I_\mu| + \frac{C_0}{2} |I_\mu| \right) m\mathscr{A} \le Q|I_\mu| m\mathscr{A},$$

which completes the proof.

We formulate a special case of this in

LEMMA 7. *If* $A_{\nu n} = \{a \in \Delta_n | x_n(a) \in I_\nu\}, n = 1, 2, \ldots,$ *then for all* n

$$mA_{\nu n} \le C_0 |I_\nu|.$$

Proof. For $n = 1$ this holds by Lemma 3. We then use Lemma 6 inductively.

With these lemmas it is now easy to estimate the distribution function of the x_j's. Let ω be an interval of length ε not containing 0, and assume that $\omega \subset I_\nu$ for some ν. Let χ_ω be the characteristic function of ω. Then with the notation above with $\mathscr{A} = A$

$$E(\chi_\omega(x_n)) \stackrel{\text{def}}{=} \int_{A_\nu} \chi_\omega(x_n(a))\, da.$$

A_ν can be decomposed, $A_\nu = \bigcup_i A_\nu(i)$ for different itineraries i. For each i the distribution of $x_n(a)$ in I_ν has a uniform density by Lemma 5. Hence

$$E(\chi_\omega(x_n)) \leq \text{Const}\, mA_\nu \cdot \frac{\varepsilon}{|I_\nu|} \leq \text{Const}\,\varepsilon$$

where the last inequality follows from Lemma 7. Hence if

$$F_N(a) = \frac{1}{N}\sum_{j=1}^{N} \chi_\omega(x_j)$$

then

$$E(F_N) \leq \text{Const}\,\varepsilon. \tag{9.3}$$

We now choose h as a large integer (fixed as $N \to \infty$).

$$E(F_N^h) = \sum_{j_1,\ldots,j_h \leq N} N^{-h}\int_A \chi_\omega(x_{j_1}) \cdots \chi_\omega(x_{j_h})\, da. \tag{9.4}$$

The number of h-tuples $(j_1,\ldots,j_h)$ for which $\min_{k,l}|j_k - j_l| < \sqrt{N}$ is bounded by $C(h)N^{h-(1/2)}$. The contribution from these terms to (9.4) is therefore $\mathcal{O}(N^{-1/2})$. Take $j_1 < j_2 < \cdots < j_h \leq N$ so that $j_{k+1} - j_k \geq \sqrt{N}, k = 1,\ldots, h-1$, and consider an interval $\omega \subset I_\nu$. Suppose that $x_{j_s} \in \omega$ for $s \leq l$. We choose, in Lemma 6, $n = j_l$ and $\mathscr{A} = \mathscr{A}_l = \{a \in A | x_{j_s} \in I_\nu, s \leq l\}$.

The assumptions of Lemma 6 are satisfied with $2Q = |I_\nu|^{-1}$ and $A_\mu = \{a \in \mathscr{A}_l | x_{j_l}(a) \in I_\mu\}$. Let $A_\mu^{(k)} = \{a \in \mathscr{A}_l | x_{j_l+k}(a) \in I_\mu\}$. It follows that

$$mA_\mu^{(1)} \leq Qm\mathscr{A}_l|I_\mu|,$$

and after $k = j_{l+1} - j_l \geq \sqrt{N}$ iterations we have

$$mA_\mu^{(k)} \leq 2^{-k+1} \cdot Q \cdot m\mathscr{A}_l \cdot |I_\mu| < C_0|I_\mu| \cdot m\mathscr{A}_l$$

provided that $\sqrt{N} \geq {}^2\log(1/|I_\mu|C_0)$; i.e. $|I_\mu| > C_0^{-1}2^{-\sqrt{N}}$. Hence,

$$m\mathscr{A}_{l+1} \leq C_0|I_\nu|m\mathscr{A}_l;$$

i.e.

$$m\mathscr{A}_h \leq (C_0|I_\nu|)^h$$

for N large enough. Now, if for each j_s we require that $x_{j_s} \in \omega \subset I_\nu$, it follows from Lemma 5 in Section 7 that the measure is decreased by

$$C^h\left(\frac{|\omega|}{|I_\nu|}\right)^h.$$

(We have used the fact that x_{j_k} has a probability density bounded above.) Hence

$$E(F_N^h) \leq (C_0\varepsilon)^h, \qquad N \geq N_0(h,\omega),$$

so that

$$\overline{\lim_{N\to\infty}} \|F_N\|_\infty \le C_0\varepsilon$$

with probability 1. It follows that the limit distribution has a density bounded by C_0 outside the point $x = 0$. The remaining possibility of a pointmass at $x = 0$ is excluded by Lemma 6, which holds uniformly in n and ν, and the fact that $x_n \neq 0$, for all n.

10. The inessential returns

As described above, a free return x_k to I_μ is followed by a sequence y_{kj}, $j = 1, 2, \ldots, Y_k$, of bound returns and, possibly, a sequence of inessential returns z_{ki}, $i = 1, \ldots, Z_k$. For the sequence $\{y_{kj}\}$, we know by Theorem 2 that when x_k describes I_μ, y_{kj} describes an interval J so that

$$e^{N_{kj}^{2/3}}|I_\mu| \le |J| \le 4^{N_{kj}}|I_\mu|$$

where $y_{kj}(a) = \xi_{m_k+N_{kj}}(a)$. Remember that $x_k(a) = \xi_{m_k}(a)$. By (8.4) it follows that $p \le 2^{3/2}\mu^{3/4}$ and furthermore we know from (6.8) that

$$|\Omega| \ge \frac{1}{20}\frac{1}{p^2}e^{-\sqrt{p}}.$$

Let us generically denote this interval Ω by I'_μ. When the first inessential return z_{k1} occurs, I'_μ is mapped into an interval σ_1 such that $\sigma_1 \subset I_{\mu_1} \cup I_{\mu_1+1}$. It must then hold that $\mu_1 \le 4\mu^{3/4}$ (cf. (6.9)). I_{μ_1} (together with I_{μ_1+1}) may now have a bound evolution and σ_1 mapped into $\sigma_{1\nu}$, $\nu = 1, 2, \ldots, Z_k^{(1)}$. After this time our interval σ_1 may be mapped onto σ_2, which intersects I_{μ_2}. The results about the uniformity of the derivatives in Section 7 give

$$|\sigma_2| \ge C|\sigma_1|\frac{|I'_{\mu_1}|}{|I_{\mu_1}|} \ge C|\sigma_1|e^{-2\mu_1^{3/8}+\sqrt{\mu_1}} \ge A|\sigma_1|$$

for some fixed $A \ge 2$ (say). This process is continued until the intervals σ_j or $\sigma_{j\nu}$ are so long that a free return takes place. Observe that inessential returns occur for all σ_j and $\sigma_{j\nu}$.

Let again ω be a fixed interval and consider

$$G(a) = \frac{1}{N}\sum_{\substack{1\le k\le N\\ j}} \chi_\omega(z_{kj}(a)) = \frac{1}{N}\sum_{k=1}^{N} G_k(a), \tag{10.1}$$

where G_k are the sums corresponding to free returns x_k, $k = 1, 2, \ldots, N$. We shall estimate $E(G(a))$. Let us fix μ and consider an itinerary i so that $x_k \in I_\mu$. E is the sum of expectations E_i for different i's, and x_k is distributed on I_μ with a bounded density (Section 11). For those z_{kj} which return to ω, the corresponding density is bounded from above and below by fixed positive constants. We shall make the worst possible assumption and assume that all z_{kj} return to ω. As long as $|\sigma_{j\nu}| \leq |\omega|$ we get 1 as a contribution to G. When $|\sigma_{j\nu}| > |\omega|$ we get a relative contribution $\leq C|\omega|/|\sigma_{j\nu}|$. The conditional expectation $E_i(G_k(a))$ can therefore be estimated by a constant times

$$\left(\sum_{|\sigma_{j\nu}| \leq |\omega|} 1\right) + |\omega| \sum_{|\sigma_{j\nu}| > |\omega|} \frac{1}{|\sigma_{j\nu}|} \leq |\omega| \sum_{j,\nu} |\sigma_{j\nu}|^{-1} \tag{10.2}$$

and we claim that

$$\sum_{j,\nu} \frac{1}{|\sigma_{j\nu}|} \leq \text{Const } \mu^{(1/2)+(3/4)} e^{4\mu^{3/8}}. \tag{10.3}$$

Observe first that by the mean value theorem (cf. (6.9), (6.2) and (ii) of the induction step)

$$|\sigma_{j\nu}| \geq \tfrac{1}{2} e^{-4\mu^{3/8}} e^{\nu^{2/3}} \geq \tfrac{1}{2} e^{-4\mu^{3/8}},$$

the length of each bound evolution of some σ_j is less or equal to $C\mu^{3/4}$. Since $|\sigma_j|$ increases exponentially, the number of σ_j's $\leq$ Const $\sqrt{\mu}$ and (10.3) follows.

To complete the proof we must again consider high powers h of $G(a)$:

$$G(a)^h = \frac{1}{N^h} \sum G_{j_1}(a) \cdots G_{j_h}(a).$$

By (10.2) and (10.3), the argument of Section 9 can be repeated when $|j_k - j_l| \geq \sqrt{N}$.

The number of remaining terms is $\leq C(h) N^{h-(1/2)}$. For such a choice of $\{j_k\}$ observe that

$$G_{j_1} \cdots G_{j_h} \leq \frac{1}{h}\left(G_{j_1}^h + \cdots + G_{j_h}^h\right).$$

Note that the number of terms in G_k for $x_k \in I_\mu$ is less than $C\mu^{5/4}$. Therefore

$$\int_A G_k^h(a)\, da \leq C|\omega| \sum_\mu \mu^{(5/4)h} e^{-\sqrt{\mu}} < \infty.$$

We note that the normalization of $G(a)$ is made by dividing by N, i.e. the number of free returns. The normalization of μ_n in (8.1) is therefore smaller.

We conclude that the limit distributions of the inessential returns also have a bounded density.

11. Distribution of bound returns

We are now going to study the distributions of the bound returns $\{y_{nj}\}$, $n = 1, \ldots, N$, $j = 1, 2, \ldots,$. We first study the distribution for fixed $j \leq J$ and then show that the total mass corresponding to $j > J$ is small.

Take $j \leq J$ and let the j^{th} return of 0 to I^* occur at time q_j and set $u_j(a) = \xi_{q_j}$. We call the return odd (even) if the corresponding kneading sequence is odd (even). Let ρ and $l > 0$ be given and let $\omega(a)$ be the interval

$$\omega(a) = \begin{cases} (u_j - \rho - l, & u_j - \rho), \quad \text{odd returns} \\ (u_j + \rho, & u_j + \rho + l), \quad \text{even returns.} \end{cases}$$

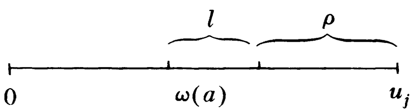
We consider

$$H_N^{(j)} = H_N(a) = \frac{1}{N} \sum_{n=1}^{N} \chi_{\omega(a)}\big(y_{nj}(a)\big).$$

Let U be an interval in $\{a \in \Delta_0 | x_n(a) \in I_\mu\}$. Suppose that $u_j(a)$ is the q_j^{th} iteration of 0. For $a \in U$, consider the images of $\sigma = (0, \pm c_{|\mu|-1}) \supset I_\mu$ under F, i.e. $\sigma_j(a) = F^{q_j}(\sigma; a)$. It follows by (6.2), Theorem 2 and the mean value theorem that

$$|\sigma_j(a)| \geq \text{Const}|\sigma(a)|^2 e^{q_j^{2/3}} \geq \text{Const}|\sigma(a)|^2 e^{j^{2/3}}. \tag{11.1}$$

Note that the right (left) endpoint of σ_j is u_j if $\{\xi_\nu\}_{\nu=1}^{q_j}$ has an odd (even) kneading sequence.



It follows that the "conditional probability" p, that a point x_n from I_μ at the j^{th} bound return belongs to $\omega(a)$, satisfies

$$p \leq \begin{cases} \text{Const} \dfrac{l}{|\sigma_j|} & \text{if } |\sigma_j| \geq \rho \\ 0 & \text{if } |\sigma_j| < \rho. \end{cases}$$

By conditioning with respect to $x_n \in I_\mu$ and by (11.1), we conclude that

$$E\Big(\chi_{\omega(a)}\big(y_{nj}(a)\big)\Big) \leq \text{Const}\, le^{-j^{2/3}} \sum_{\mu \in M} \frac{|I_\mu|}{c_\mu^2},$$

where

$$M = \Big\{\mu | c_\mu^2 \geq \text{Const}\, \rho e^{-j^{2/3}}\Big\}.$$

(The set M contains $\{\mu|\ |\sigma_j| \geq \rho\}$.) This gives

$$E\big(H_N^{(j)}(a)\big) \leq \text{Const}\, le^{(-1/2)j^{2/3}} \rho^{-1/2}.$$

We consider again high powers. The sum

$$N^{-h} \sum_{n_1, n_2, \ldots, n_h \leq N} E\Big(\chi_{\omega(a)}\big(y_{n_1 j}(a)\big) \cdots \chi_{\omega(a)}\big(y_{n_h j}(a)\big)\Big)$$

is estimated by splitting into the cases $\min_s|h_{s+1} - h_s| \le \sqrt{N}$ and $|n_{s+1} - n_s| \ge \sqrt{N}$ as in Sections 11 and 12. The result is

$$\left[E\left(\frac{1}{N}\sum_{n=1}\chi_{\omega(a)}(y_{nj}(a))\right)^h\right]^{1/h} \le \text{Const}\, le^{(-1/2)j^{2/3}}\rho^{-1/2} + \left(C_1(h)N^{-1/2}\right)^{1/h}.$$

As before it now follows that the contribution of the y_{nj}'s to the limit distribution has for almost all $a \in A$, a density g_j so that

$$\int_{\omega(a)} g_j(x)\,dx \le \text{Const}\, le^{(-1/2)j^{2/3}}\rho^{-1/2}$$

for every choice of $\rho, l > 0$, and $j = 1, 2, \ldots, J$.

We next turn to the total mass corresponding to returns with $j > J$. Let $\theta_{nj}(a) = 1$ if the return y_{nj} exists and set $\theta_{nj} = 0$ otherwise. We define $\theta_n = \sum_{j=J+1}^{\infty}\theta_{nj}$, and have to estimate

$$T_N = \frac{1}{N}\sum_{n=1}^{N}\theta_n.$$

If $x_n \in I_\mu$, $\mu \ge \alpha$, it follows from (6.4) that $\sum_{j=1}^{\infty}\theta_{nj} \le 2^{3/2}|\mu|^{3/4} \le |\mu|$. Therefore $\theta_n \le |\mu|$ and $\theta_n \ne 0$ only when $|\mu| \ge J + 1$. We therefore conclude that

$$E(\theta_n) \le \text{Const} \sum_{\mu=J+1}^{\infty} \mu e^{-\sqrt{\mu}} \le e^{(-1/2)\sqrt{J}}$$

and the same estimate holds for $E(T_N)$. As before we estimate $E(T_N^h)$ and since

$$E(\theta_n^h) \le C \cdot \sum_{\mu} \mu^h e^{-\sqrt{\mu}}$$

the argument is the same as in Section 10. We conclude that

$$\varlimsup_{N\to\infty} T_N(a) \le e^{(-1/2)\sqrt{J}}$$

for almost all $a \in A$.

We can summarize the results so far in the following theorem.

THEOREM 4. *For almost all $a \in A$, any limit distribution of free and of inessential returns to I^* has a uniformly bounded density. A limit distribution of bound returns has a density $g(x)$ and*

$$g(x) \le \text{Const}\left[\sum_{u_j \text{ odd}} e^{(-1/2)j^{2/3}}\varphi(u_j - x) + \sum_{u_j \text{ even}} e^{(-1/2)j^{2/3}}\varphi(x - u_j)\right]$$

where

$$\varphi(x) = \begin{cases} \dfrac{1}{\sqrt{x}} & x > 0 \\ \infty & x = 0 \\ 1 & x < 0. \end{cases}$$

12. Iterated points outside I^*

In the previous discussion we have only described the returns to I^*. We now keep $a \in A$ fixed and study the orbit more in detail. Let

$$\mu_n = \frac{1}{n} \sum_{\nu=1}^{n} \delta_{\xi_\nu(a)}.$$

For $\{\mu_n\}_{n=1}^{\infty}$ the following result holds.

THEOREM 5. *For almost all $a \in A$ every weak-* convergent subsequence of $\{\mu_n\}_{n=1}^{\infty}$ has an absolutely continuous limit gdx and $g \in L^p(-1,1)$ for every $p < 2$.*

Proof. As before, let $\{u_\nu(a)\}_{\nu=1}^{M}$ be the returns to I^* up to time n. Then

$$\begin{aligned}\mu_n &= \frac{1}{n} \sum_{\nu=1}^{M} \delta_{u_\nu} + \sum_{k=1}^{n} \frac{1}{n} \sum_{\nu}{}' \delta_{F^k(u_\nu)} \\ &= \sum_{k=0}^{M} \mu_n^{(k)}.\end{aligned}$$

The prime in the summation sign indicates that $\delta_{F^k(u_\nu)}$ is included in the sum if and only if $F^j(u_\nu) \notin I^*$, $j = 1, 2, \dots, k$.

Let μ^* denote the weak-* limit of $\{\mu_{n_j}\}_{j=1}^{\infty}$. We have already proved that

$$\lim_{j \to \infty} \mu_{n_j}^{(0)} = \mu^*\Big|_{I^*} \in L^p$$

for $p < 2$ (Theorem 4), and we turn to the $\{\mu_n^{(1)}\}_{n=1}^{\infty}$, which are supported on $[1 - a\delta^2, 1]$. Let ω be a subinterval of $[1 - a\delta^2, 1]$. Now $F^{-1}\omega$ consists of two symmetric intervals Ω and $-\Omega$ in I^*. We have

$$\mu_n(\omega) = \mu_n^{(1)}(\omega) = \mu_n^{(0)}(\Omega \cup -\Omega).$$

By Theorem 4,

$$\mu^*(\omega) \le \int_{\Omega \cup -\Omega} h(x)\, dx,$$

where

$$h(x) = \text{Const} \sum_{\nu=1}^{\infty} e^{(-1/2)\nu^{2/3}} \frac{1}{|u_\nu - x|^{1/2}}.$$

By a change of variables

$$\int_{\Omega \cup -\Omega} h(x)\, dx = \int_{\omega} \sum_{i=1}^{2} h\big(F_i^{-1}(x)\big)\big(F_i^{-1}\big)' dx,$$

where $\{F_i^{-1}\}_{i=1}^{2}$ are the two branches of F^{-1}. We claim that the density

$\sum_{i=1}^{2} h(F_i^{-1}(x))(F_i^{-1})'(x) \in L^p$ for $p < 2$. We have

$$\text{(12.1)} \qquad \int_{1-a\delta^2}^{1} \left|h(F^{-1}(x))\right|^p \left|(F^{-1}(x))'\right|^p dx$$

$$= \int_0^{\delta} |h(x)|^p \frac{1}{|F'(x)|^{p-1}} dx$$

$$\leq \text{Const} \int_0^{\delta} |h(x)|^p \frac{1}{x^{p-1}} dx$$

$$\leq \text{Const} \sum_{\mu=\alpha}^{\infty} \int_{I_\mu} |h(x)|^p \frac{1}{x^{p-1}} dx$$

$$\leq \text{Const} \sum_{\mu=\alpha}^{\infty} e^{(p-1)\sqrt{\mu}} \int_{I_\mu} |h(x)|^p dx.$$

But

$$\left(\int_{I_\mu} |h(x)|^p dx\right)^{1/p}$$

$$\leq \text{Const} \sum_{\nu=1}^{[\mu^{7/8}]} \left(\int_{I_\mu} \frac{e^{(-p/2)\nu^{2/3}}}{|x-u_\nu|^{p/2}} dx\right)^{1/p} + \sum_{\nu=[\mu^{7/8}]+1}^{\infty} = S_1 + S_2.$$

To estimate S_1 we use the relation $|u_\nu| \geq e^{-\sqrt{\nu}}$, which implies $|x - u_\nu| \geq \frac{1}{2} e^{-\mu^{7/16}}$, $\nu \leq [\mu^{7/8}]$, $x \in I_\mu$. Hence

$$S_1 \leq \text{Const}\left(\sum_{\nu=1}^{[\mu^{7/8}]} e^{(-1/2)\nu^{2/3}}\right) \cdot |I_\mu|^{1/p} \cdot 2^{1/2} e^{(1/2)\mu^{7/16}}$$

$$\leq \text{Const} \exp\left(\mu^{7/16} - (1/p)\sqrt{\mu}\right).$$

The second sum is estimated as follows:

$$S_2 \leq \text{Const} \sum_{\nu=[\mu^{7/8}]+1}^{\infty} e^{(-1/2)\nu^{2/3}} \left(\int_{-1}^{1} \frac{dx}{|x-u_\nu|^{p/2}}\right)^{1/p}$$

$$\leq \text{Const}\, e^{(-1/2)\mu^{7/12}},$$

and the convergence of (12.1) follows. Hence

$$\text{(12.2)} \qquad \mu^*|_{(1-a\delta^2,1)} \in L^p \quad \text{for } p < 2.$$

Let now ω be an interval contained in $(-1, 1 - a\delta^2) \setminus I^*$. From Lemma 2 and Lemma 3, (iii), we conclude that for $x \in (1 - a\delta^2, 1)$ such that $F^\nu(x) \notin I^*$, $\nu = 1, 2, \ldots, k-1$,

$$\text{(12.3)} \qquad \left|\partial_x F^k(x)\right| \geq \lambda^k,$$

where $\lambda > 1$ is a uniform constant. From (12.3) it now follows that

$$\lim_{j\to\infty} \mu_{n_j}^{(k)}(\omega) \le \int_\omega g_k(x)\,dx, \quad k = 2,3,\dots,$$

where $\|g_k\|_p \le \text{Const}\,\lambda^{-k}$, $p < 2$.

Let $E_k = \{x \in I^* | F^\nu(x) \notin I^*, \quad \nu = 1,2,\dots,k\}$. By (12.3), cf. [1], Theorem II.5.2, part 3, there is some $\eta < 1$ such that $|E_k| = \mathcal{O}(\eta^k)$. By this fact and the absolute continuity of $\mu^*|_{(1-a\delta^2,1)}$, it is clear that for every $\varepsilon > 0$ there is a $q = q(\varepsilon)$ such that

$$\frac{1}{n}\sum_{k=q+1}^{n} \|\mu_n^{(k)}\| \le \varepsilon.$$

We finally conclude that

$$\mu^*(\omega) \le \int_\omega \left\{ \sum_{k=2}^{\infty} g_k(x) \right\} dx,$$

where $\sum_{k=2}^{\infty} g_k \in L^p$ and Theorem 5 is proved.

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REFERENCES

[1] P. COLLET AND J.-P. ECKMANN, *Iterated Maps on the Interval as Dynamical Systems*, Birkhäuser, 1980.

[2] ———, On the abundance of aperiodic behaviour for maps on the interval, Comm. Math. Phys. **73** (1980), 115–160.

[3] M. JAKOBSON, Absolutely continuous invariant measures for one-parameter families of one-dimensional maps. Comm. Math. Phys. **81** (1981), 39–88.

[4] S. M. ULAM AND J. VON NEUMANN, On combinations of stochastic and deterministic processes, Bull. A. M. S. **53** (1947), 1120.

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