and, by multiplying these two equations together,

$$zA\bar{A}'\bar{z}' = \lambda\bar{\lambda}(z\bar{z}') = \lambda\bar{\lambda}.$$

But $zA\bar{A}'\bar{z}'$ is less than or equal to the greatest latent root of $A\bar{A}'$, which by Theorem II is less than or equal to σ; hence*

$$\lambda\bar{\lambda} \leqslant \sigma,$$

$$|\lambda| \leqslant \sqrt{\sigma} = \sqrt{(h_{a_1}+h_{a_2}+\ldots+h_{a_s})}.$$

Finally, we mention the corollary:

If A is a square matrix whose rows are normalized ($h_\nu = 1$) and none of whose columns possesses more than s non-zero components, then every latent root of A is less than or equal to $\sqrt{s}$.

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SETS OF FRACTIONAL DIMENSIONS (V): ON DIMENSIONAL NUMBERS OF SOME CONTINUOUS CURVES

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1. We first recall the numerical definition of d-dimensional measure of a "measurable set"‡.

Definition. Given a plane set of points E, denote by C_ρ any set of circles of radii less than or equal to ρ, covering all the points of E. Given $0 < d < 2$, d-measure of E is defined by the equation

$$d-mE = \varliminf_{\rho \to 0} \sum_{C_\rho} (2r)^d,$$

* *Cf.* E. T. Browne, *Bull. American Math. Soc.*, 34 (1928), 363–368.

† Received 3 July, 1936, read 14 January, 1937.

‡ F. Hausdorff, "Dimension und äusseres Mass", *Math. Annalen*, 79 (1918), 157–179; A. S. Besicovitch, "On linear sets of fractional dimensions", *Math. Annalen*, 101 (1929), 161–193; "Sets of fractional dimensions (II)", *Math. Annalen*, 110 (1934), 321–329; "Sets of fractional dimensions (III)", *Math. Annalen*, 110 (1934), 331–335; "Sets of fractional dimensions (IV)", *Journal London Math. Soc.*, 9 (1934), 126–131.

where r is the radius of the general circle of C_ρ and $\sum_{C_\rho}$ denotes the summation extended over all circles of C_ρ.

To every plane set E corresponds one of the three possibilities:

(i) There exists a number $0 < d < 2$ such that, for any $d' > d$, $d'-mE = 0$, and for any $d' < d$, $d'-mE = \infty$. Then we say that E is a d-dimensional set and we call d the dimensional number of E.

(ii) $d-mE = \infty$ for any $d < 2$. Then we say that E is a 2-dimensional set.

(iii) $d-mE = 0$ for any $d > 0$. Then we say that E is a 0-dimensional set.

2. THEOREM. *The dimensional number d of the curve $y = f(x)$, where $f(x)$ belongs to the Lipschitz δ-class* (Lip^δ), *satisfies the inequalities*

$$1 \leqslant d \leqslant 2-\delta.$$

Consider the curve for $0 \leqslant x \leqslant 1$. Suppose first that the coefficient in the Lipschitz inequality can be taken equal to 1, so that to any x corresponds an interval $(x-k,\ x+k)$ such that, for any $x+h$ of this interval,

$$|f(x+h)-f(x)| < |h|^\delta. \tag{1}$$

By the Heine-Borel theorem, there exists a finite set

$$(0,\ k_0),\quad (x_1-k_1,\ x_1+k_1),\quad \ldots,\quad (x_{n-1}-k_{n-1},\ x_{n-1}+k_{n-1}),\quad (1-k_n,\ 1)$$

of overlapping intervals of the above kind covering the whole of (0, 1). Denoting by c_i an arbitrary point between x_{i-1} and x_i belonging to both of the intervals

$$(x_{i-1}-k_{i-1},\ x_{i-1}+k_{i-1}),\quad (x_i-k_i,\ x_i+k_i),$$

we have

$$0 < c_1 < x_1 < c_2 < x_2 < \ldots < x_{n-1} < c_n < 1.$$

The oscillation of $f(x)$ in the interval (c_{i-1}, c_i) is less than $2|c_i-c_{i-1}|^\delta$, and thus the part of the curve corresponding to the interval $(c_{i-1},\ c_i)$ can be enclosed in a rectangle of height $2|c_i-c_{i-1}|^\delta$ and of base c_i-c_{i-1}, and consequently in $[2(c_i-c_{i-1})^{\delta-1}]+1$ squares of side c_i-c_{i-1} or in the same number of circles of radius $(c_i-c_{i-1})/\sqrt{2}$ circumscribed about each of these squares.

Given an arbitrary $\rho > 0$, we can always assume all $c_i - c_{i-1} < \rho$. Denote by C_ρ the set of all the above circles and consider

$$\sum_{C_\rho} (2r)^{2-\delta}.$$

The sum of the terms corresponding to the interval (c_{i-1}, c_i) is

$$\{[2(c_i - c_{i-1})^{\delta-1}]+1\}\{(c_i - c_{i-1})\sqrt{2}\}^{2-\delta} < 6(c_i - c_{i-1}),$$

and thus
$$\sum_{C_\rho} (2r)^{2-\delta} < 6\Sigma(c_i - c_{i-1}) < 6,$$

which shows that the $(2-\delta)$-dimensional measure of the curve is finite and hence that the dimensional number of the curve is less than or equal to $2-\delta$.

If now $f(x)$ satisfies the Lipschitz condition with a variable coefficient

$$| f(x+h) - f(x) | < C |h|^\delta,$$

and C is not bounded, then, for any $\epsilon > 0$, it satisfies the condition

$$|f(x+h) - f(x)| < |h|^{\delta-\epsilon}$$

for sufficiently small h, and thus the dimensional number of the curve is less than or equal to $2-\delta+\epsilon$, *i.e.* less than or equal to $2-\delta$.

This completes the proof.

COROLLARY. *The dimensional number of the curve $y = f(x)$, where $f(x)$ has a finite derivative at all points, is* 1.

This follows at once from the fact that $f(x)$ belongs to Lip^δ for $\delta = 1$.

3. A curve of class Lip^δ may have any dimension number in the range $1 \leqslant d \leqslant 2-\delta$, as we now show by examples. We write $\phi(x)$ for the function equal to $2x$ in $0 \leqslant x \leqslant \frac{1}{2}$ and defined elsewhere by the relations

$$\phi(x) = \phi(-x) = \phi(x+1).$$

We consider curves

$$y = f(x) = \Sigma\, a_n \phi(b_n x),$$

where
$$a_n = b_n^{-\delta} \quad (0 < \delta < 1),$$

and we write
$$s_\nu(x) = \sum_{n<\nu} a_n \phi(b_n x).$$

I. *If* $b_{n+1} \geqslant Bb_n$, *where* $B > 1$, *then* $f(x)$ *is of class* Lip^{δ}

We have $$0 \leqslant \phi \leqslant 1, \quad |\phi'| = 2,$$

and hence $$|\phi(b_n x+b_n h)-\phi(b_n x)| \leqslant 1,$$
$$|\phi(b_n x+b_n h)-\phi(b_n x)| \leqslant 2b_n h.$$

Hence

$$|f(x+h)-f(x)| \leqslant \Sigma a_n |\phi(b_n x+b_n h)-\phi(b_n x)|$$
$$\leqslant \sum_{n \leqslant \nu} 2a_n b_n h + \sum_{n>\nu} a_n$$
$$\leqslant 2h \,.\, b_\nu^{1-\delta}[1+B^{\delta-1}+B^{2(\delta-1)}+\ldots+B^{\nu(\delta-1)}]$$
$$+b_{\nu+1}^{-\delta}[1+B^{-\delta}+B^{-2\delta}+\ldots]$$
$$= K_1 b_\nu^{1-\delta} h + K_2 b_{\nu+1}^{-\delta},$$

the numbers K_1, K_2 depending only on B and δ. Now choose $\nu = \nu(h)$ so that

$$\frac{1}{b_\nu} > h \geqslant \frac{1}{b_{\nu+1}}.$$

Then we get at once

$$|\Delta f| \leqslant (K_1+K_2) h^{\delta}.$$

II. $f(x)$ *is not of any higher Lipschitz class.*

For, taking $x = 0$ and $h = 1/2b_\nu$, we get

$$\Delta f = f(h) > b_\nu^{-\delta} \phi(\tfrac{1}{2}) = K_3 h^{\delta},$$

and h here is arbitrarily small.

III. *If* $1 < d < 2-\delta$, $b_1 > 1$, $b_{n+1} = b_n^{\mu_n}$, *where* $\mu_n \geqslant \frac{1-\delta}{\delta}\frac{2-d}{d-1}$, *then the d-measure of the part of the curve arising from a finite range of x is finite or zero: and the curve is of dimension less than or equal to d.*

As in I we get

$$|\Delta y| \leqslant K_1 b_\nu^{1-\delta} h + K_2 b_{\nu+1}^{-\delta} = H, \quad \text{say.}$$

By dividing the range of x into intervals of length h we are able to cover the curve with rectangles of width h and height H, and these in turn can be covered with squares of side h. Choose $h = h_\nu$ so that

$$b_\nu^{1-\delta} h_\nu = b_{\nu+1}^{-\delta},$$

whence $$h_\nu = b_\nu^{-\delta\mu_\nu - 1 + \delta} \leqslant b_\nu^{-(1-\delta)/(d-1)} \to 0 \quad \text{as} \quad \nu \to \infty.$$

The number of squares of side h_ν required is less than

$$\left(\frac{l}{h_\nu}+1\right)\left(\frac{H_\nu}{h_\nu}+1\right),$$

l being the length of the range of x. For $h_\nu < l$ this is less than

$$\frac{2l}{h_\nu}\left[(K_1+K_2)\, b_\nu^{1-\delta}+1\right] < K_4\, b_\nu^{1-\delta}\, h_\nu^{-1},$$

and the corresponding approximation to the d-measure is less than

$$K_5\, b_\nu^{1-\delta} h_\nu^{d-1} \leqslant K_5.$$

Since h_ν is arbitrarily small, this proves the result stated.

Note that it is sufficient to have

$$\mu_n \geqslant \frac{1-\delta}{\delta}\,\frac{2-d}{d-1} \quad \text{for} \quad n \geqslant n_0,$$

instead of for all n, provided that $b_{n_0} > 1$. If we now construct an example in which $b_n \to \infty$, $\mu_n \to \infty$, then we can take d arbitrarily near to 1 in the above argument and hence the curve is *of dimension precisely* 1. This will be true, for instance, if

$$b_n = 2^{2^{2^n}}.$$

IV. *If* $1 < d < 2-\delta$, $b_1 > 1$, $b_{n+1} = b_n^{\mu}$, *where* $\mu = \dfrac{1-\delta}{\delta}\,\dfrac{2-d}{d-1}$, *then the d-measure of the curve* $y = f(x)$ *is greater than zero.*

Let ϕ be a square of side h with its sides parallel to the axes of coordinates. The points of the curve $y = f(x)$ which lie in ϕ give by orthogonal projection on the x-axis a set which we denote by E_ϕ. We show that the linear measure of E_ϕ is less than Kh^d. The desired result then follows immediately.

The gradient $s_\nu'(x)$ is dominated by its final term $\pm 2b_\nu^{1-\delta}$, and the remainder $f - s_\nu$ is dominated by its first term when ν is large. We suppose that

$$|s'_{\nu-1}(x)| < b_\nu^{1-\delta}, \quad \text{so that} \quad |s_\nu'(x)| > b_\nu^{1-\delta},$$

and

$$0 \leqslant f - s_\nu < 2a_{\nu+1}$$

for all the relevant values of ν.

We choose $\nu = \nu(h)$, as in I. We have, of course, $a_\nu > b_\nu^{-1} > h$: choose κ so that

$$a_{\nu+\kappa-1} > h \geqslant a_{\nu+\kappa}.$$

Then

$$b_{\nu+\kappa-1}^{-\delta} > h \geqslant b_{\nu+1}^{-1},$$

$$(b_1^{\mu^{\nu+\kappa-2}})^{-\delta} > b_1^{-\mu^\nu},$$

$$\delta\mu^{\nu+\kappa-2} < \mu^\nu,$$

$$\delta_\mu^{\kappa-2} < 1,$$

$$\kappa < \kappa_0 = 2\log_\mu \delta.$$

We distinguish two cases: (i) $h \geqslant h_\nu$, (ii) $h < h_\nu$.

(i) Suppose first that $\kappa = 1$ or $h \geqslant a_{\nu+1}$. If (x, f) lies in ϕ, then (x, s_ν) lies in a rectangle ϕ' obtained by prolonging ϕ downwards a distance $2a_{\nu+1} \leqslant 2h$. The range h of x we divide now into subintervals in which s_ν' is of constant sign: since $h < b_\nu^{-1}$, there are at most three of them. In any one of them the curve $y = s_\nu(x)$ can lie in ϕ' only in an interval of x of length at most $3h\, b_\nu^{\delta-1}$. For the height of ϕ' is at most $3h$ and the gradient of the curve at least $b_\nu^{1-\delta}$. Hence

$$\text{measure of } E_\phi < 9hb_\nu^{\delta-1} < 9h^d h_\nu^{1-d} b_\nu^{\delta-1} = 9h^d.$$

If $\kappa > 1$ we first divide the range h of x into at most three parts in which $s_\nu'(x)$ is of constant sign. The rectangle ϕ' is of height $h + 2a_{\nu+1} \leqslant 3a_{\nu+1}$, and hence in each of these parts we have only to consider a subinterval of length at most $3a_{\nu+1} b_\nu^{\delta-1}$. These we now further subdivide into intervals in which $s'_{\nu+1}(x)$ is also of constant sign. The number of new intervals obtained from each of the old is less than

$$3a_{\nu+1} b_\nu^{\delta-1} 2b_{\nu+1} + 1 < 7\left(\frac{b_{\nu+1}}{b_\nu}\right)^{1-\delta}$$

If $\kappa > 2$, we repeat the construction, obtaining from each of the intervals last constructed a set of intervals in which $s'_{\nu+2}(x)$ is also of constant sign, in number less than

$$7\left(\frac{b_{\nu+2}}{b_{\nu+1}}\right)^{1-\delta}.$$

Finally we get E_ϕ covered by a set of intervals in which $s'_{\nu+\kappa-1}$ is of constant sign. The number of these intervals is less than

$$3 \,.\, 7^{\kappa-1}\left(\frac{b_{\nu+\kappa-1}}{b_\nu}\right)^{1-\delta}.$$

In each of them (x, f) lies in ϕ only if $(x, s_{\nu+\kappa-1})$ lies in a rectangle $\phi^{(\kappa)}$ of height $h+2a_{\nu+\kappa} \leqslant 3h$, and hence only in an interval of length less than $3h\, b_{\nu+\kappa-1}^{\delta-1}$. Thus

$$\text{measure of } E_\phi < 9 \,.\, 7^{\kappa-1}\, hb_\nu^{\delta-1} < K_6 h^d.$$

(ii) In this case $a_{\nu+1} > h_\nu > h$, $\kappa > 1$. If $\kappa = 2$, we divide the range h of x into intervals in which $s'_{\nu+1}$ is of constant sign. The number of these is less than

$$2b_{\nu+1} h+1 < 3b_{\nu+1} h.$$

In each such interval (x, f) lies in ϕ only if $(x, s_{\nu+1})$ lies in a rectangle ϕ'' of height $h+2a_{\nu+2} \leqslant 3h$, and hence only in an interval of x of length less than $3h\, b_{\nu+1}^{\delta-1}$. Thus

$$\text{measure of } E_\phi < 9h^2 b_{\nu+1}^{\delta} < 9h^d h_\nu^{2-d} b_{\nu+1}^{\delta} = 9h^d.$$

If $\kappa > 2$, we construct a covering of E_ϕ as in (i) by intervals in which $s'_{\nu+\kappa-1}$ is of constant sign: only the first step in the construction is changed as above. Thus

$$\text{measure of } E_\phi < 9 \,.\, 7^{\kappa-2}\, h^2 b_{\nu+1}^{\delta} < K_6 h^d.$$

The curve considered in IV, in which

$$b_n = b_1^{\nu^{n-1}},$$

is of dimension precisely d. We cannot take $d = 2-\delta$ in the above work, since this gives $\mu = 1$ and every term will be the same. However, the same argument can be used to establish an example of dimension $2-\delta$, as follows.

V. *If the ratio b_{n+1}/b_n increases to infinity, and if also $\mu_n \to 1$, then the curve $y = f(x)$ is of dimension precisely $2-\delta$.*

We show that, for any fixed positive α, the linear measure of E_ϕ is less than $K_\alpha h^{2-\delta-\alpha}$, indeed is $o(h^{2-\delta-\alpha})$ as $h \to 0$, and hence that the dimension of the curve is greater than or equal to $2-\delta-\alpha$.

Write
$$\frac{b_{n+1}}{b_n} = \beta_{n+1}$$
and define κ as before. It is no longer bounded, but
$$b_{\nu+\kappa-1}^{-\delta} > b_{\nu+1}^{-1}$$
gives
$$(\beta_{\nu+2}\beta_{\nu+3}\dots\beta_{\nu+\kappa-1})^{-\delta} > b_{\nu+1}^{-1+\delta},$$
$$\beta_{\nu+1}^{\delta(\kappa-2)} < b_{\nu+1}^{1-\delta} < (b_1\beta_{\nu+1}^{\nu})^{1-\delta},$$
$$\delta(\kappa-2) < (1-\delta)(\nu+\log_{\beta_{\nu+1}} b_1),$$
$$\kappa < \frac{1-\delta}{\delta}\nu+3 < A\nu \quad \text{when } \nu \text{ is large,}$$
where A depends only on δ. Hence, as in IV, we get
$$\text{measure of } E_\phi \text{ (i)} < 9\,.\,7^{A\nu} h b_\nu^{\delta-1} \quad \text{if} \quad h \geqslant h_\nu,$$
$$\text{(ii)} < 9\,.\,7^{A\nu} h^2 b_{\nu+1}^{\delta} \quad \text{if} \quad h < h_\nu.$$
Now
$$b_\nu > \beta_{[\frac{1}{2}\nu]}^{\frac{1}{2}\nu} \succ K^\nu \quad \text{for any fixed } K,$$
$$b_\nu^{\frac{1}{2}\alpha} \succ 9\,.\,7^{A\nu}.$$
So also
$$\left(\frac{b_{\nu+1}}{b_\nu}\right)^{\delta(1-\delta-\alpha)} = b_\nu^{\delta(1-\delta-\alpha)(\mu_\nu-1)} = b_\nu^{o(1)} \prec b_\nu^{\frac{1}{2}\alpha}.$$
Hence we get
$$\text{measure of } E_\phi \prec h^{2-\delta-\alpha},$$
and it follows that the $(2-\delta-\alpha)$-measure of the curve is infinite.

We get an example of this type if we take
$$b_1 = 1, \quad \mu_n = 1+n^{-\frac{1}{3}}.$$

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SUMMATION OF SOME INFINITE SERIES OF WEBER'S PARABOLIC CYLINDER FUNCTIONS

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The object of this note is to show how an operational image of a parabolic cylinder function of negative order can be used to effect the summation of some infinite series involving the function.

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