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Physics

Self-similarity of the "1/*f* noise" called music

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ABSTRACT **Suggestions have been made that computer musicians should attempt to compose fractal music, and questions have been raised whether there is such a thing as fractal music. Voss and Clark observed that music is scaling, or 1/*f* noise, as analyzed on the basis of the amplitude (loudness) of the audio signals; they failed to find a fractal distribution of acoustic frequencies (music notes) in music. Analyzing Bach's and Mozart's compositions, we have shown that the incidence of the frequency intervals, or of the changes of acoustic frequency, has a fractal geometry. Fractal phenomena are characterized by scale-independency. The purpose of this investigation is to demonstrate the self-similarity of music and to explore its implications.**

An anecdote has been told about the dialogue between Emperor Joseph and Mozart after the first performance of his *Abduction from the Seraglio*: The emperor complimented the composer for the heavenly music but complained that it had too many notes. Mozart's reply was, of course, that not one note was dispensable. We agree that we cannot arbitrarily subtract a single note from the master's composition. Yet could we eliminate half of the notes and still consider the music to be by Mozart?

The question of whether music has a fractal geometry has been debated (1–4). We have presented an analysis and given a positive answer (5). If this is indeed the case, the implication is self-similarity and scale-independency. In other words, a musical composition could be represented by a music score of a different scale, using a half, a quarter, or twice as many notes as were written by the composer; Emperor Joseph could have an *Abduction*, at least a music score of the *Abduction*, that contains half as many notes as it is written by Mozart.

Mandlebrot (6) explained this phenomenon with the paradox of indeterminate length of state boundaries: L. F. Richardson, a British physicist, discovered to his astonishment that the lengths of the common frontier between Spain and Portugal, as reported in his neighbors' encyclopedias, differed by 20%. This so-called Richardson Effect is now expressed in terms of fractal geometry, relating the yardstick length ε and the total measured length $L(\varepsilon)$, which could be a state boundary, coastline, or any other linear dimension. This relation is shown by Fig. 1*A*.

Could we compare the fractal geometry of music to that of a coastline? If a coastline has no definite length, could we state that Mozart's music has no definite number of notes or note intervals? If the total length of coastline $L(\varepsilon)$ is a function of ε, could we find a similar expression for music?

The total length of a music score $L(|i|)$ is the sum of all note intervals i; each note interval i is defined by the relation (5)

$$f_{1+i}/f_1 = (15.9/15)^i,$$

where f is the acoustic frequency (hertz) of two successive musical notes separated by interval i. The melody of music

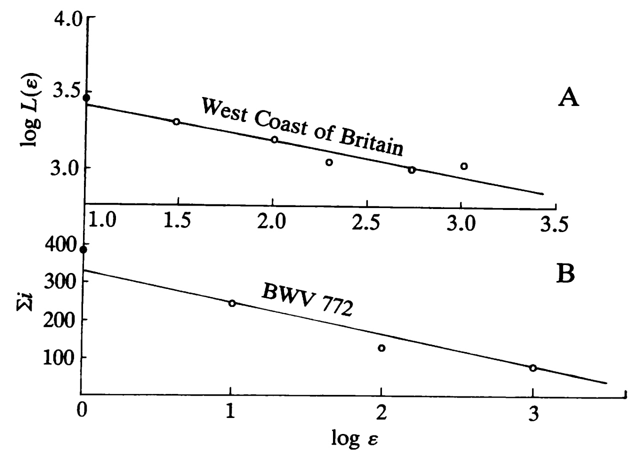

FIG. 1. Rate of decrease of coastline length $L(\varepsilon)$ (*A*) and decrease of music note intervals i (*B*) as a function of the "yardstick" length (see text for detailed explanation).

consists of combinations of intervals of frequency change, such as a second (= full note), small and large thirds, fourth, fifth, octave, etc., with values of i equal to 2, 3, 4, 5, 7, 12, etc. The fractal frequency distribution is commonly valid between $i = 1$ and $i = 7$ (ref. 5).

What is the yardstick to measure the length of a music score? The total note intervals should be drastically reduced if the number of notes is reduced by half; i.e., if we measure the music interval i by choosing to measure the note interval between every other note. The yardstick ε could thus be expressed as a power function $(1/2)^\varepsilon$, with $\varepsilon = 0, 1, 2, 3, \ldots$ for zero, half, quarter, eighth . . . reduction of the total notes.

We chose to work again on J. S. Bach's Invention no. 1 in C Major, BWV 772, because it has been shown to have a fractal geometry (5). To transform the audio into visual signals, we have digitized the notes: their frequencies (as expressed by note-interval difference j from a standard) are plotted against the successive number of notes in that composition, where j is defined by

$$(f_j/f_0) = (15.9/15.0)^j,$$

where f_0 is the frequency of deepest C, or 60 Hz. The value j for the middle A standardized for baroque music is, for example, 415 Hz.

Music notes, represented by their j values, go up and down, like the peaks and valleys of the profile of a mountain, or the zigzagging promontories and bays of a coastline (Fig. 2*A*). Digitized notes for Bach's Invention no. 10 in G Major are shown graphically by Fig. 2*B*. By plotting the scores for the right and for the left hand on the same graph, an approximate symmetry of Bach's music can be discerned. The symmetry is triclinic, with centers of symmetry.

The "irregularities" of music landscape could be smoothed out by using a "broad brush." Take, for example, the first five notes of BWV 772 for the right hand (Fig. 2): the $L(|i|)_{\varepsilon=0}$ value is $1 + 1 + 1 + 2 = 5$; the $L(|i|)$ value for $\varepsilon = 1$, however, is $2 + 1 = 3$. For the straight part of the note variation, like

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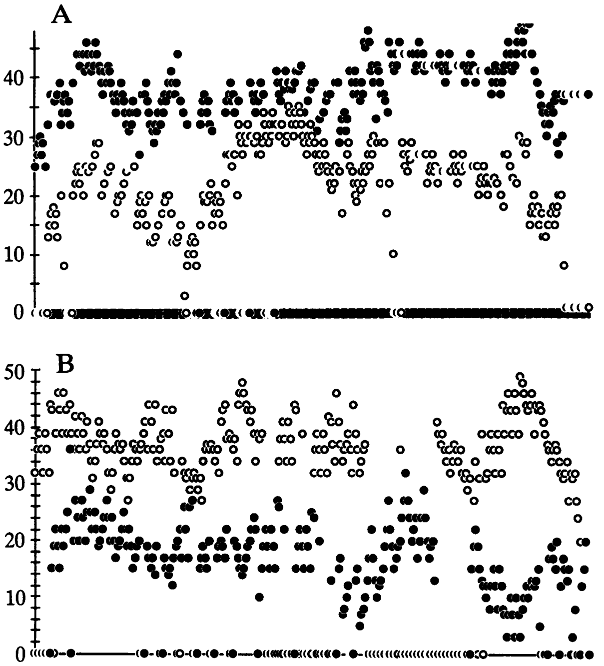

FIG. 2. Digitized scores of Bach's Inventions no. 1 (*A*) and no. 10 (*B*). ○, right hand; ●, left hand.

a straight coastline, the measured length $L(|i|)$ is independent of the length of the yardstick (ε). For the irregular part, the measured length is smaller for larger ε values. The sum of the note intervals $L(|i|)$ of this score is 391 intervals ($\varepsilon = 0$). It is 243 intervals if only every other note is considered ($\varepsilon = 1$), and 121 and 77 intervals for $\varepsilon = 2$ and 3, respectively. The Richardson Effect of BWV 772 is demonstrated by the relation

$$L(|i|) = c/\varepsilon^D$$

as shown by Fig. 1*B*, where *D* is a fractal dimension and *c* an empirical number.

Other aspects of fractal geometry have been called Noah's Effect and Joseph's Effect. Referencing the Bible, Mandlebrot used the term Noah's Effect to denote that an event of very great magnitude is extremely rare. We have shown this fractal relation in our last article (5). Joseph's Effect refers to clustering: "There came seven years of great plenty throughout the land of Egypt. And there shall arise after them seven years of famine." (Gen. 41:29–30). Mandlebrot has shown on the basis of historical data that successive yearly discharges and flood levels of the Nile and many other rivers are extraordinarily persistent (6). The clustering of notes in music as shown by Fig. 2 is thus also an expression of its fractal geometry.

The theoretical basis for the postulate that a music score could be reduced into 1/2, 1/4, 1/8, . . . is the scale independency or self-similarity of a fractal landscape. The British Isles, as defined by their coastlines, look pretty much the same whether the map is printed on a scale of 1:1,000,000, 1/50,000, or 1:25,000. Music with self-similarity should be susceptible to analogous scale reduction.

Plotted in Fig. 3 are the shapes of the "reduced landscapes" of BWV 772, for $\varepsilon = 0$ (original composition) 1, 2, 3, 4, 5, and 6. The visual self-similarity is shown clearly by Fig. 3 *A–C*. Fig. 4 is a magnification of a part of Fig. 3 *A* and *B*, showing that the reduction tends to smooth out the irregularity of music topography, just like the irregularities of a topographical profile tend to be subdued in a diagram of reduced scale.

A half or quarter reduction of notes seems to give an outline of the music as it was written by Bach. The reader can find

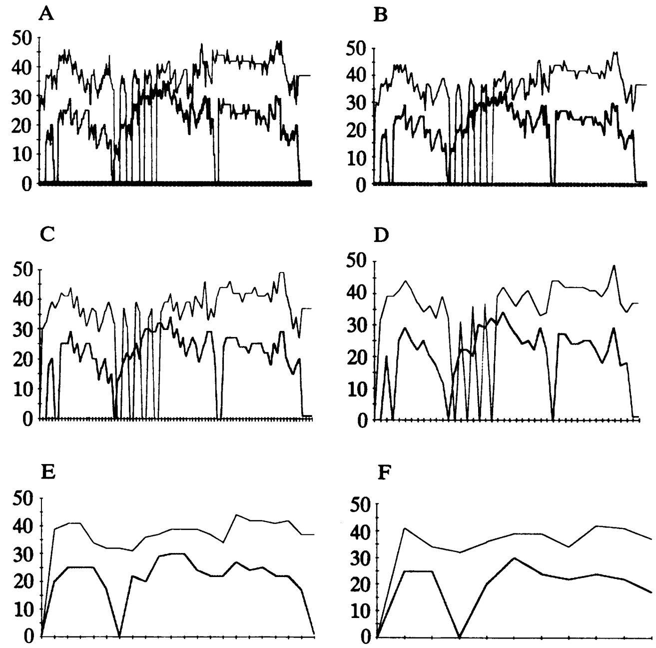

FIG. 3. Fractal reductions of Bach's Invention no. 1, BW 772. (*A*) The original. (*B–F*) The 1/2, 1/4, 1/8, 1/16, 1/32 reductions of the score, respectively.

the audio self-similarity by playing the reduced notes on a piano, as we did. To a novice, the half- or quarter-Bach sounds like Bach, although he gains the impression that the composition has an economy of thrills and ornamentations. Further reductions to 1/8, 1/16, and 1/32 tend to eliminate more of the "irregularity of the silhouette," yet the distinguishing overall line of the music landscape is preserved. The final reduction to 1/64 gives only three notes; they are the three key notes, the foundation upon which the whole composition is built.

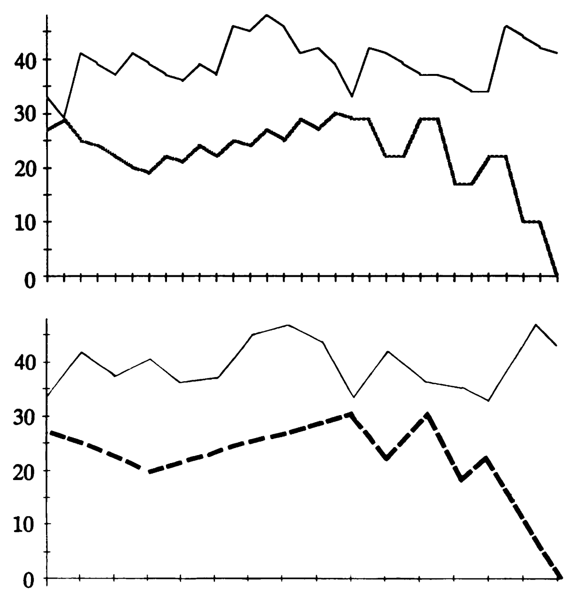

FIG. 4. A detailed expression of fractal reduction of Bach's Invention no. 1, showing the similarity after 30 notes were reduced to 15. (*Upper*) "Silhouette" of Bach's original composition. (*Lower*) The half reduction. The ordinate gives the numerical value of *i* (acoustic frequency), and the abscissa is a parameter of time ("beat"). The scores for the right hand are shown by the upper lines, and the scores for the left are shown by the lower lines.

In reversing the procedure to compare a higher-order reduction to a lower-order fractal or to the original composition, we find an analogy in chess games. To start from higher-order reductions could be compared to the opening moves, the progression to lower-order fractals the middle game, and the completion the endgame of a master's play. The comparison serves to analyze the procedure of structural changes from the key notes to the melodies and to the whole, even though Bach must not have gone through such a step when he composed intuitively. Such an analysis does not help a creative composer but could yield rules for software programming. Like a chess game with a classic opening, we could start with a 1/32 reduction of Bach and start to build the music up into an alternative score, according to the theory of music harmony. The "endgame" will not be that of Bach's, but it could be comparable in quality if it has been constructed on the basis of a correct understanding of the mathematic structure of Bach's music.

The graphs of digitalized notes, as shown in Fig. 3, are also reminiscent of a mountain silhouette; the pointed peaks and the V-shaped valleys find their analogues in the Alps. We venture to suggest from such a comparison that Bach's music could be considered a three-dimensional manifestation of a music-truth in four dimensions. The reduction or abstraction of BWV 772 into reduced scores could be viewed as manifestations on reduced scales. If such an analogy is appropriate, could we say that there could be deeper manifestations of BWV 772? Could we not have a music landscape of BWV 772 mapped on a scale twice the original? Theoretically, it is possible with landscape-enhancement techniques used by specialists to refine the resolution of satellite photographs of planets.

We have so far chosen to make the music "yardstick" a power of 1/2. For music such as waltzes with 3/4 rhythm, a one-third reduction is more appropriate than a half. We have found this to be so while experimenting with some of Chopin's etudes.

The discovery of the self-similarity of music has opened a new world of sound. We may not see the day when "music digests" of "half-Mozarts," "third-Chopins", or "quarter-Bachs" are produced for novices like Emperor Joseph, who enjoy music but have too little understanding or too little time to listen to the original scores. On the other hand, reduced music gives the skeleton and enhances our understanding of the great music. We might also inquire whether music could be composed by computers like computer chess. Could we start with a "Bach Opening," namely a 1/16 or 1/32 reduction of a Bach composition, and experiment with software programs to produce a new composition in the whole sound range?

We emphasize that this is a methods paper. Analyses of many compositions are needed before we can begin to find the rules of the mathematic structures of music. Great music should have a structure like a great edifice. Folk songs are short melodies. Does the integration of many thousands of notes into a symphonic composition imply a sound structuring? This is a mystery of music. Is fractal geometry an adequate language to express this mystery?

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