Bounds for the representation of quadratic forms.

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Milnor's conjecture for the Witt ring of quadratic forms over a field (characteristic $\neq 2$) says, in a precise way, that quadratic form theory over that field is essentially determined by the so-called Pfister forms; these quadratics forms have a particularly pleasant behaviour and generalize to arbitrary dimension such classic objects as the norm forms of quadratic extensions and quaternion algebras over the given field. Milnor's conjecture was proved true, in characteristic 0, by Voevodsky in 1996. His work raises, among many others, a number of questions as to the way in which a given quadratic form may be represented as a linear combination of Pfister forms. This is the problem we address, in the cases of Pythagorean fields (more generally, of fields with Pythagoras number bounded by a fixed integer) and of preordered fields.

Our solution of Marshall's signature conjecture (Inventiones Math. 133 (1998), 243-278) —and, more generally Lam's conjecture (Algebra Colloq. 10 (2003), 149-176)— combined with the compactness theorem of first-order logic, yields the existence, for fixed integers n, $m \geq 1$, of a *uniform* and *recursive* bound on the number of Pfister forms of degree n over any Pythagorean (resp., bounded Pythagoras number, preordered) field F which suffice to represent (in the Witt ring of F) any form of dimension m as a linear combination of such (Pfister) forms with non-zero coefficients in F. "Uniform" means here that the bound does not depend either on the coefficients of the form or on the field F; it is given by a recursive function of n and m.

It is an open and, perhaps, difficult problem to determine whether "reasonable" bounds for this representation problem exist for arbitrary fields of the types considered in the preceding general result. However, we single out a fairly large class of Pythagorean fields and, more generally, of reduced special groups, for which a simply exponential bound of the form cm^{n-1} (c a constant) exists for the representation problem. Such a class (actually a countable hierarchy) is closed under certain —possibly infinitary— operations which preserve Marshall's signature conjecture. In the case of groups of finite stability index s, we obtain an upper bound which is quadratic on $[m/2^n]$, where the constant c depends on s. In the case of function fields over real closed fields, the stability index is just the transcendence degree.

[Joint work with F. Miraglia, Univ. of São Paulo, S. Paulo, Brazil. Published in J. Algebra 268 (2003), 209-251.]