La Escuela TERA y el Problema 17 de Smale*

(Bézout $5\frac{1}{2}$)

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November 9, 2005

*TERA'2005 por los 60 Años de un Amigo.

Algo en lo que creíamos

Creer que la Eliminación Geométrica Eficaz tenía algo que aportar a la resolución de problemas en la práctica.

Creer que la simple reescritura no era la respuesta. Los trabajos [Giusti-Heintz] [Krick–P.] ya habían demostrado estar por delante de la mera reescritura.

Creer que la semántica condicionaba la complejidad. Ambos éramos conscientes de que los invariantes semánticos de la geometría constituían obstrucciones a la complejidad en forma de cotas inferiores...Para cuándo las cotas superiores?.

Creer en la Matemática como creación colectiva, raramente individual. cada vez menos " à la mode".

Más en lo que creer.

Creer que la Matemática del final del Milenio no era un instrumento inútil. Ambos creíamos, sinceramente, que la matemática aún podía aportar algo a la tecnología, a pesar de los respectivos entornos de mentes abstrusas.

Creer que la Algorítmica debe ser adaptable a clases particulares de inputs. Algo así como un algoritmo que debía aceptar como buenos todos los tipos de datos que se usasen para representar los polinomios dados como inputs (SLP's sólo representaban esa idea) .

Creer en una nueva filosofía de programación. Que, al final, no fué tan necesaria.

PERO, SOBRE TODO, CREER QUE.

Y, SOBRE TODO, CREER QUE LA COMPLEJIDAD ALGORITMICA, POR DURA QUE SEA, PRECEDE A LA PROGRAMACION. Aunque parezca de perogrullo, sólo decimos aquello de:

- saber qué es un ordenador,

- saber cómo se comportará el algoritmo en el ordenador

- y, finalmente, decidir si vale la pena programarlo.

Por increíble que parezca, casi nada de lo que circula por ahí como matemática computacional o, simplemente como informática, reúne estas cualidades.

The TERA Experience 1993–1997

Elimination Theory = Computational Algebraic Geometry

Main Goal: *Algorithmic Treatment of Problems Defined by*
Polynomial Equations and Inequalities

Many (?) (Potential) **Applications:**

Central Open Problem: Efficiency

Efficiency (In a Broad Sense)

Time Complexity : Function that relates input length to running time of best algorithm.

Tractable algorithms :† Time Function bounded by a polynomial $N^{O(1)}$.

Exponential algorithms : Exponential Time Function $2^{O(N)}$.

Rk. *Most algorithms for Elimination Problems run in worse than exponential time in the number of variables:*

Intractable for Practical Applications.

†Applicable for practical purposes

Solving

INPUT: A list of multivariate polynomial equations: $f_1, \ldots, f_s \in \mathbb{C}[X_1, \ldots, X_n]$.

OUTPUT: A description of the solution variety
$V(f_1, \ldots, f_s) := \{x \in \mathbb{C}^n \ : \ f_i(x) = 0 \ldots\}$.

Description: The kind of description determines the kind of problems/questions you may answer about $V(f_1, \ldots, f_s)$

Example:
Symbolic/algebraic Computing $\longrightarrow$ questions involving quantifiers

Hilbert's Nulltellensatz (HN)

Teams Involded

Teams :

* Cantabria (P., Morais, Montaña, Hägele,...)
* Polytechnique (Giusti, Lecerf, Schost, Bostan, Salvy...)
* Buenos Aires (Heintz, Krick, Matera, Solerno, ...)
* Humboldt (Bank, Mbakop,Lehmann)

Elimination deals with computing information about algebraic varieties: (Quantifier Elimination, Dimension, Singularities...)

Algebraic Varieties have **Intrinsic/Semantic Invariants** that surely **dominate Complexity**.‡

‡Well–known fact in lower bound studies in **Algebraic Complexity Theory**

Examples

$$X_1^2 - X_1 = 0, \ldots, X_n^2 - X_n = 0, k - \sum_{i=1}^{n} m_i X_i = 0.$$

$$X_1^2 - X_1 = 0, \ldots, X_n^2 - X_n = 0, k - \sum_{i=1}^{n} 2^{i-1} X_i = 0.$$

$$X_1^2 - X_1 = 0, \ldots, X_n^2 - X_n = 0, 512 - \sum_{i=1}^{n} 2^{i-1} X_i = 0.$$

$$X_2^2 - X_1 = 0, X_3^2 - X_2 = 0 \ldots, X_n^2 - X_{n-1} = 0, k - X_n = 0.$$

Attacks based on **Semantical Invariants:**

Degree of V ([Heintz,83], [Vogel, 83], [Fulton, 81]) :$\sharp$ of intersection with generic linear mandifolds.

Height of V:
Bit length of the coefficients CHOW FORM

* *Geometric Degree* of a Sequence:

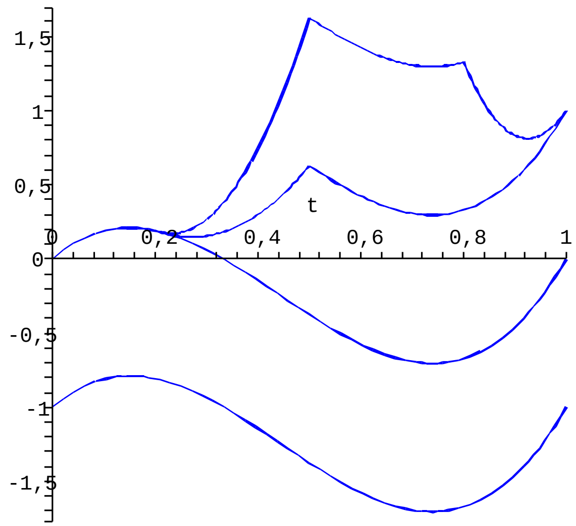
$$\delta(V_1, \ldots, V_r) := max\{deg(V_i) \ : \ 1 \leq i \leq r\}.$$

Theorem 1 **[Giusti-Heintz-Morais-P., 94-97]** *There is a bounded error probability Turing machine that answers* **HN** *in time polynomial in*

$$L\ \delta\ H,$$

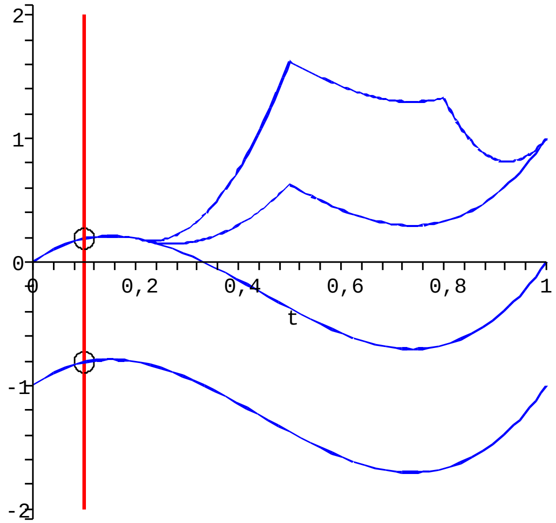
where
L *is the input length (whatever usual data structure),*
δ *is the geometric degree of a deformation sequence (Kronecker's deformation) and*
H *is the height of the last equi-dimensional variety computed.*

Remarks
* The quantities that dominate complexity are δ and H.
* Meaningful Technical Improv. by Lecerf, Matera, Schost, Bostan..2000–
* Applications to real solving by Bank, Mbakop, Lehmann... 2000–2005
* Implemented by Salvy, Lecerf, (2001).



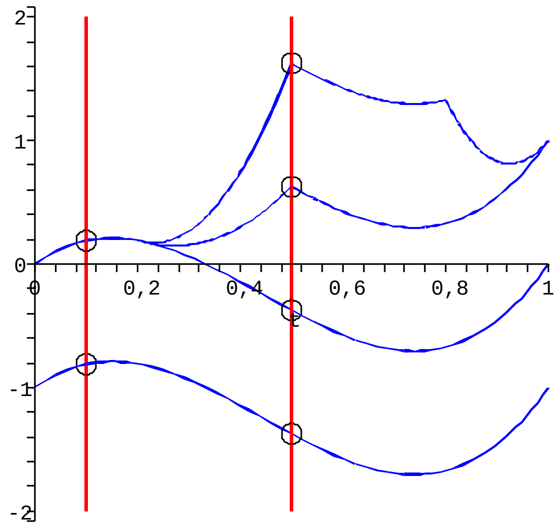
A deformation inside the variety of zero–dimensional varieties (Chow form).

Kronecker's Deformation (I)



Unramified fiber points *(Lifting Points)*

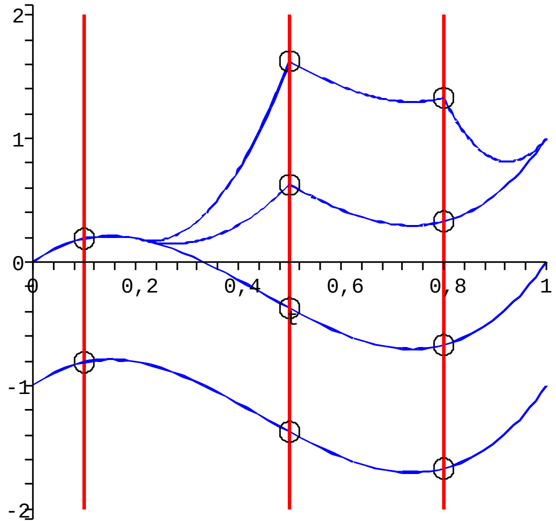
Kronecker's Deformation (II)



$\circ :=$ Intermediate zero-dimensional Varieties to be solved (V_i)

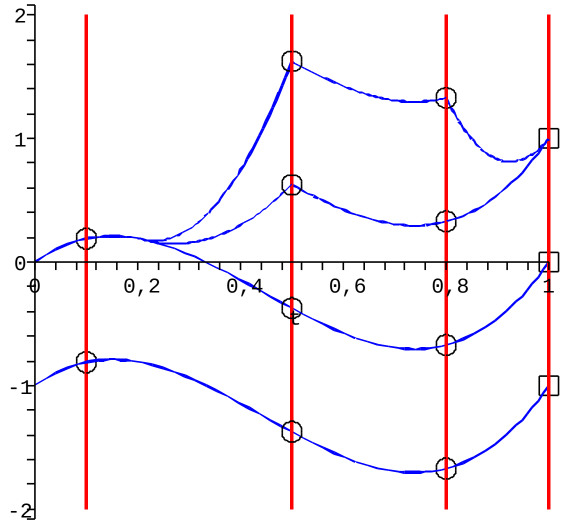
From a description (solving) of V_i compute a description (solving) of V_{i+1}.

Kronecker's Deformation (III)



And so on....

Kronecker's Deformation (IV)



Until you compute a **Description** of the variety defined by the input system.

But...

Is that Efficiently Optimal as Algorithm?

Universal Solving ?

A TERA notion

Algorithms based on **deformation scheme***:*
A sequence $V_1, \ldots, V_n$ of intermediate varieties that are solved before "eliminating"

Universal Solving
An algorithm that is called **Universal** if its output contains information enough to answer **all** elimination questions.

Remark 2 *All known symbolic algorithms in Computational Algebraic Geometry are Universal.*

Theorem [Castro-Giusti-Heintz-Matera-P.,2003]
All Universal Solving Procedures require exponential running time. Namely, the (geometric) degree of the output is the lower bound for output length (of any reasonable encoding) and hence for the time.

Main Idea of the Proof:

* Inputs in Elimination are usually given as unirational (parametrized) families of problems.

** The embedding dimension of these unirational families is a lower bound for the output length and, hence, a lower bound for the running time.*

* It is easy to exhibit simple families with exponential embedding dimension...

First Conclusions

** The TERA algorithm is* **essentially optimal.**

The running time is greater than the* **Bézout Number :

$$\prod_{i=1}^{n} deg(f_i) \geq 2^n.$$

***No other symbolic computation algorithm can improve this exponential lower bound.**

But...

Applications demand a "causal" algorithm.

Main Conclusion

An "efficient" Polynomial Equation Solver must satisfy:

It is Non- Universal:

Its complexity will depend on intrinsic/semantic quantities.

It should run efficiently on "most" input systems[§]

[§] We give up to solve efficiently some of the input systems.

Is there anything like this?

Shub & Smale Approximate Zero Theory

Approximate Zeros

* Input: *A System of Polynomial Equations*

$$F := [f_1, \ldots, f_n] \in \mathcal{H}_{(d)},$$

$deg(f_i) = d_i$, $(d) := (d_1, \ldots, d_n)$.

A zero $\zeta \in V(F)$

Approximate Zero(Smale'81) a point z where Newton operator N_F converges very fast to the zero.

$$d_T(N_F^k(z), \zeta) \leq \frac{1}{2^{2^{k-1}}}.$$

$d_T :=$ some distance function

* OUTPUT:

UNIVERSAL SOLUTION: **One** *Approximate Zero* z **for each** zero $\zeta \in V(F)$.

Running time of Universal Numerical Analysis solving is greater than number of solutions

Bézout's number $\geq \prod_{i=1}^{n} d_i \Rightarrow$ **Intractable**

From now on:

NON-UNIVERSAL SOLUTION: **One** *Approximate Zero* z **for some** *zero* $\zeta \in V(F)$.

Complexity of Non–Universal Numerical Analysis Pol's Solver ?

INPUT SPACE $\mathcal{H}_{(d)}$ space of systems of polynomial equations, dense encoding, degree list $(d) = (d_1, \ldots, d_n)$

$$N := \sum_{i=1}^{n} \binom{d_i + n}{n} = dim\mathcal{H}_{(d)}.$$

Deep advances in Shub & Smale (1987-95):**[Bézout I to V].**

See also contributions by Kim, Dedieu, Malajovich, Renegar, Yakoubsohn,...

AN ALGORITHMIC SCHEME

INPUT $F \in \mathcal{H}_{(d)}$

Newton's Homotopic Deformation Scheme (NHD) *on initial pair*

$$(G, \zeta) \in \mathcal{H}_{(d)} \times \mathbb{P}_n(\mathbb{C})$$

Following a line of systems of equations:

$$F_t := (1-t)F + tG, \quad t \in [0,1].$$

(A curve of equations/solutions $\{(F_t, \zeta_t) \, : \, \zeta_t \in V(F_t), t \in [0,1]\}$)

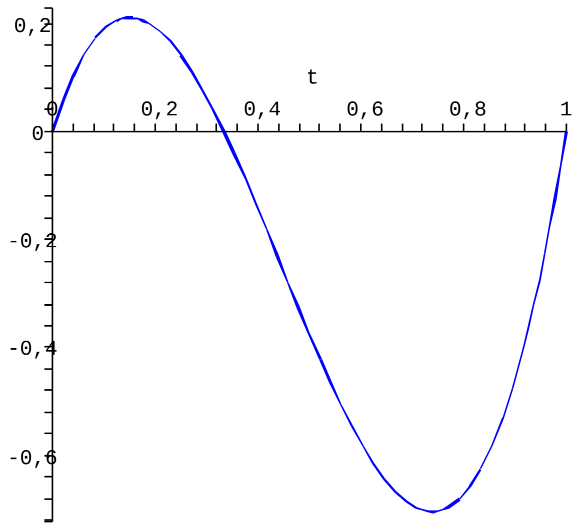
From some Appr. Zero of (G, ζ) $(t = 1)$, compute some Appr. Zero of F $(t = 0)$

OUTPUT:

– either failure

– or an approximate zero of F.

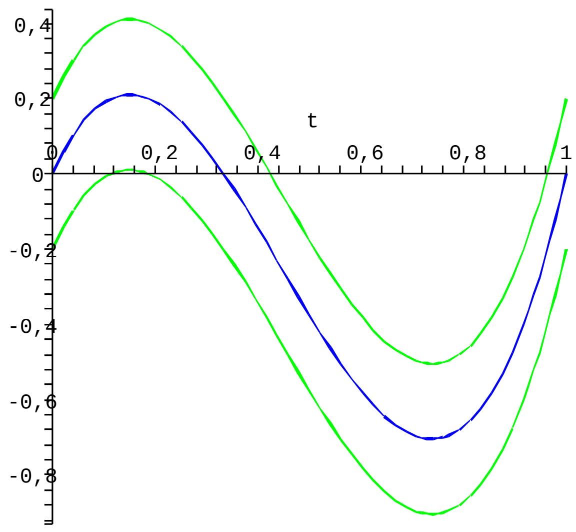
Homotopic Deformation



Fixing the initial pair (G, ζ) you fix a smooth curve inside the incidence variety:

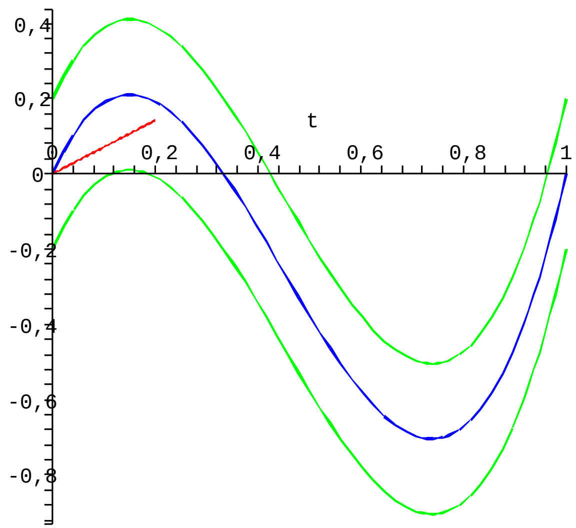
$\{(F_t, \zeta_t) \ : \ t \in [0, 1], F_t(\zeta_t) = 0\}$.

Homotopic Deformation (II)



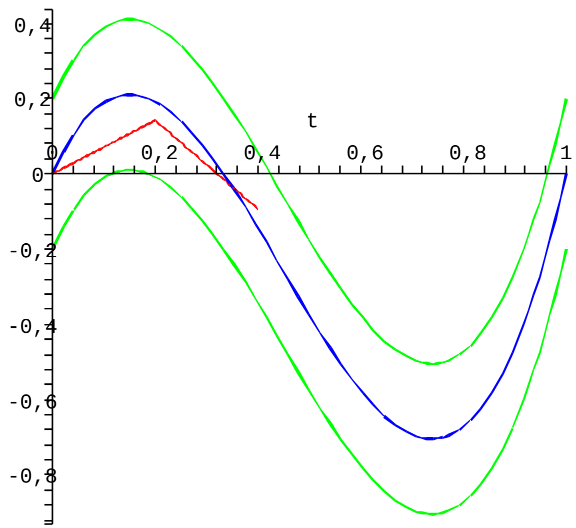
A tube about (F_t, ζ_t) of approximate zeros.

Homotopic Deformation (III)



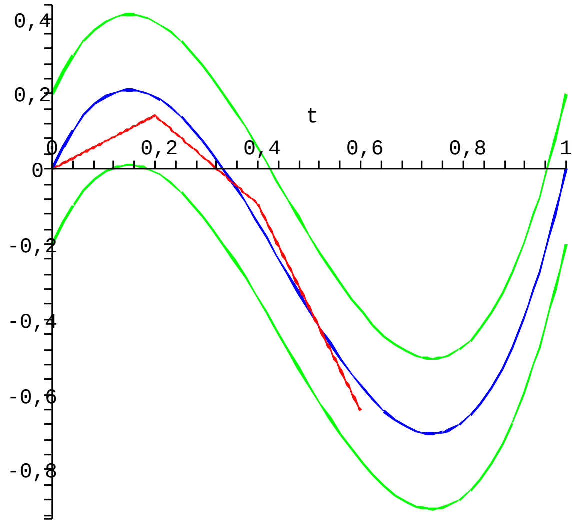
Starting at $(G, \zeta) = (F_0, \zeta_0)$ perform "Newton homotopic steps" to some approximate zero of F_{t_0}

Homotopic Deformation (IV)



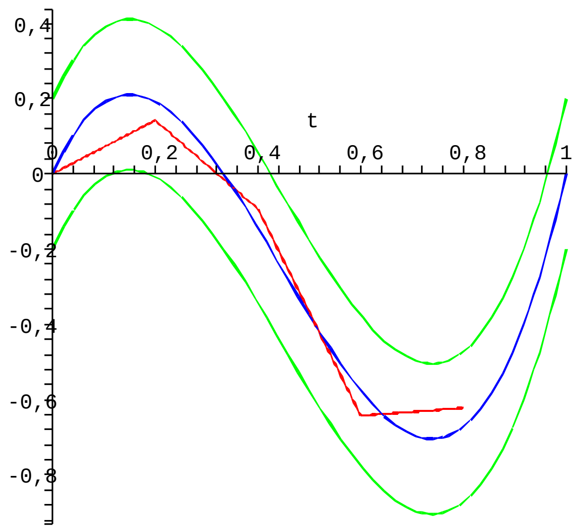
It goes on..

Homotopic Deformation (V)



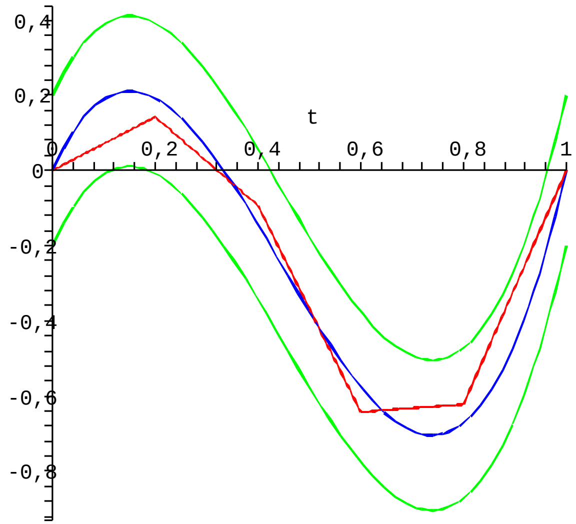
Still going on..

Homotopic Deformation (VI)



Still going on..

Homotopic Deformation (VII)



A Polygonal from one Approx. Zero of $G = F_0$ to one Approx. Zero of $F = F_1$

Complexity is dominated by number of homotopy steps.

Everything depends on where you start (G, ζ).

The scheme again

Input: A System of Equations $F \in \mathcal{H}_{(d)}$

Guess some initial pair (G, ζ)

Apply Newton's Homotopic Deformation to G and F, starting at ζ.

Output:

– either failure

– or an approximate zero of F.

Where to Begin the Homotoy?

Efficient Initial Systems

$\varepsilon > 0$ a positive real number.

A data $(G, \zeta) \in \mathcal{H}_{(d)} \times \mathbb{P}_n(\mathbb{C})$ is ε−efficient for NHD if

– *The running time is polynomial in* $\varepsilon^{-1}, n, N, d$.

– *NHD with initial data* (G, ζ) *solves "most" input systems* $F \in \mathcal{H}_{(d)}$*:*

$$Probability[\, F \in \mathcal{H}_{(d)} \; : \; NHD \text{ solves } F \text{ from } (G, \zeta)] \geq 1 - \varepsilon.$$

The probability that the scheme outputs "failure" is less than ε.

Theorem 3 (Shub-Smale, Béz. V, 95) *For every $\varepsilon > 0$ there is a $\varepsilon-$efficient initial pair $(G_\varepsilon, \zeta_\varepsilon) \in \mathcal{H}_{(d)} \times \mathbb{P}_n(\mathbb{C})$, where $\zeta \in V(G)$. The running time is polynomial in the combinatorial number*

$$N.$$

Namely, for every $\varepsilon > 0$, there is some initial pair for NHD that solves "most" systems, but it depends on ε.

Does it suffices?

Algorithmic Scheme:

INPUT $F \in \mathcal{H}_{(d)}, \varepsilon > 0$

Construct $(G_\varepsilon, \zeta_\varepsilon)$ an $\varepsilon-$efficient Initial System

Apply Newton's Homotopic Deformation to G_ε and F, starting at ζ_ε.

OUTPUT:
– either failure (with probability at most ε)
– or an approximate zero of F (with probability at least $1 - \varepsilon$).

The Number of steps is polynomial in $\varepsilon^{-1}N$.

[Shub& Smale, Bez. 5] "In Theorem 7.4 we employ a quasi-algorithm. This constructions fails to be an algorithm because it employs an infinite sequence $(g_i, \zeta_i) \in V, i = 1, 2, 3, ...$ without exhibiting them."

A Problem

No idea on how to construct $\varepsilon-$efficient initial pairs.

Implementation available?[¶]

[‖]

Open Problem: Smale's Conjecture, Smale's 17th Problem.

[¶] Of a proven efficient algorithm.

[‖] Efficient initial pairs in Verschelde–Sommesse–Wampler programs never proved...?

A recent small step forwards

Solving Smal'es 17th Problem (Bezout $5\frac{1}{2}$)

Joint work with C. Beltrán.

CORRECT TEST CLASSES (A QUESTOR SET) FOR NHD

[Beltrán- P., 2005]

A class $\mathcal{G} \subseteq \mathcal{H}_{(d)} \times \mathbb{P}_n(\mathbb{C})$ is a questor **set for NHD if the following holds:

For every $\varepsilon > 0$ the probability that a randomly chosen data $(G, \zeta) \in \mathcal{G}$ is $\varepsilon-$efficient for NHD is greater than

$$1 - (nNd)^{O(1)}\varepsilon.$$

**Note that the questor class is independent of ε. In particular, an algorithm can be defined from a fixed questor class.

Algorithmic Scheme

Input $F \in \mathcal{H}_{(d)}, \varepsilon > 0$

Choose at random $(G, \zeta) \in \mathcal{G}$

Apply Newton's Homotopic Deformation to G and F, starting at (G, ζ).

Output:

– either failure (with probability at most ε)

– or an approximate zero of F (with probability at least $1 - \varepsilon$).

Problems

Minor: **It's a Probabilistic Algorithm**

Relevant: **The class $\mathcal{G}$ should be constructible (and easy-to-compute).**

Bézout $5\frac{1}{2}$

Theorem[Beltrán, P. 2005] *We succeeded to exhibit a constructible and easy-to-compue questor class for efficient initial pais in the homogeneous case*

$e := (1 : 0 : \ldots : 0) \in \mathbb{P}_n(\mathbb{C})$ a "pole" in the complex sphere, representing the origin.

$V_e := \{F \in \mathcal{H}_{(d)} \ : \ F(e) = 0\}$. Systems that vanishes at the pole e.

$F \in V_e \longmapsto F : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^n$.

Tangent Mapping $T_eF := DF(e)$ restricted to the orthogonal complement $e^{\perp} = \mathbb{C}^n \subseteq \mathbb{C}^{n+1}$.:

$$T_eF := T_e\mathbb{P}_n(\mathbb{C}) = \mathbb{C}^n \longrightarrow \mathbb{C}^n.$$

Uncomplete Idea

$L_e := \{F \in V_e \ : \ T_e F = F\}$. "linear part" of systems in V_e.

$L_e^{\perp} :=$ Systems in V_e of order at least 2 on e.

Remark.- $V_e, L_e, L_e^{\perp}$ a linear subspaces of $\mathcal{H}_{(d)}$ given by coefficient lists.

Idea: Use

$$\mathcal{G} := \{(G, e) \ : \ G \in V_e = L_e^{\perp} \bigoplus L_e\}.(?)$$

Towards a Questor Set II (L_e)

$\mathcal{U}(n+1)$:= unitary matrices on $\mathbb{C}^{n+1}$.

$\mathcal{H}_{(1)} := \mathcal{M}_{n \times n+1}(\mathbb{C})$ space of complex $n \times (n+1)$ matrices.

$$X^{(d)} := \begin{pmatrix} X_0^{d_1 - 1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_0^{d_n - 1} \end{pmatrix}.$$

$$V_e^{(1)} := \{(M, U) \ : \ M \in \mathcal{H}_{(1)}, U \in \mathcal{U}, U Ker(M) = e\}.$$

THE LINEAR PART L_e

$$\psi_e : V_e^{(1)} \longrightarrow L_e$$

$$\psi_e(M, U) := X^{(d)}(MU) \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}.$$

$$L_e := Im(\psi_e(M, U)).$$

A useful constant

$$T := \left(\frac{n^2+n}{N}\right)^{n^2+n} \in \mathbb{R}, \qquad t \in [0, T].$$

Towards a Questor Class III

$$\mathbb{G} := [0, T] \times L_e^{\perp} \times V_e^{(1)}.$$

$$G : \mathbb{G} \longrightarrow V_e,$$

$$(t, L, M, U) \in \mathbb{G} \longmapsto G(t, L, M, U) \in V_e$$

$$G(t, L, M, U) := (1 - t^{\frac{1}{n^2+n}})^{1/2} L + t^{\frac{1}{n^2+n}} \psi_e(M, U) \in V_e,$$

Theorem 4 (Beltrán-P., 2005a) *For every degree list* $(d) := (d_1, \ldots, d_n)$, *the class*

$$\mathcal{G}_{(d)} := IM(G).$$

is a questor class of efficient initial pairs for NHD. Namely,

A randomly chosen system $(G, e) \in \mathcal{G}_{(d)}$ is $\varepsilon-$efficient for NHD with probability greater than

$$1 - (ndN)^5 \varepsilon.$$

The algorithm

Input: $f \in \mathcal{H}_{(d)}$, $\varepsilon > 0$.

Guess at random $(G, e) \in \mathcal{G}_{(d)}$ *(Guess* (t, L, M)*...)*

Apply Projective Newton's Deformation $O(\varepsilon^{-2}d^2)$ times

Output: either "failure" or an approximate zero z of f.

Meaning (I)

Corollary 5 *There is a bounded error probability (Monte–Carlo) numerical analysis algorithm that verifies :*

- *The running time is at most* $O(\varepsilon^{-2}d^2)$.

- *The probability that a randomly chosen* $(G, e) \in \mathcal{G}$ *is* $\varepsilon-$*efficient is at least*

$$1 - (ndN)^5\varepsilon.$$

- *The probability that an input system is solved is at least:*

$$1 - \varepsilon.$$

Corollary 6 *Assume* $(d) := (3, 3, \ldots, 3)$. *There is a bounded error probability numerical analysis algorithm that verifies :*

- *The running time is at most* $O(n^{60})$.
- *The probability that a randomly chosen* $(G, e) \in \mathcal{G}$ *is* $\varepsilon-$*efficient is at least*

$$1 - \frac{1}{(n)^5}.$$

- *The probability that an input system is solved is at least:*

$$1 - \frac{1}{n^{30}}.$$

Turing Machine Complexity

Using the transfer methods of [Castro, Montaña, San Martin, P., 2002] and [Castro, San Martin, P. 2003], we also have a "translation to the Turing machine world": **(i.e. real life computing world)**

Corollary 7 (Beltrán–P., 2005) *There is a bounded error probability Turing machine that solves "most" systems of multivariate polynomial equations in time which depends polynomially on the dense encoding of multivariate polynomials.*

Corollary 8 (Beltrán–P., 2005) **Computing a zero of homogeneous polynomial equations is in BPP (dense encoding of inputs).**

Lines of the Proof

Too Technical for a talk like this one!

A series of Geometric Reductions From great circles to pairs, from pairs to incidence variety, from incidence variety to tangent mapping,...

Integral Geometry at every geometric reduction... Federer's Coarea Formula, Santaló,..., some "tricks" when manipulating integrals...

Smale's Conjecture still open :
Is there any deterministic procedure to find $\varepsilon-$efficient data?.

Replace questor classes by deterministically defined efficient initial systems.

Behaviour of the condition number μ_{norm} on smaller (not necessarily linear) data?.

Input systems (in real life) are not given by their dense encoding
Search for a treatment adapted to special classes (also different data structures) of inputs. Interesting results on the sparse case by [Malajovich-Rojas]

Una pregunta TERA.

A Technical Example of treatment of special data:

Estimates on the Probability Distribution of Condition Numbers of SINGULAR matrices

Joint work work C. Beltrán, 2005

Condition Number of complex matrices

$\odot$Two Condition Numbers: $\mathcal{M}_n(\mathbb{C})$ complex space of all $n \times n$ matrices.

* Turing, von Neumann, Wilkinson...

$$\kappa(A) := ||A||_2 ||A^{-1}||_2.$$

* Demmel, Smale, Edelmann

$$\kappa_D(A) := ||A||_F ||A^{-1}||_2.$$

$|| \cdot ||_2 :=$ norm as linear operator
$||A||_F := \left(\sum_{i,j} |a_{i,j}|^2\right)^{1/2} :=$ Frobenius norm.

Rk Both condition numbers are (essent. equiv.) homogeneous (and hence projective) functions, i.e. $\kappa(A) \leq \kappa_D(A) \leq \sqrt{n}\kappa_A$ and

$$\kappa, \kappa_D : \mathbb{P}(\mathcal{M}_n(\mathbb{C})) \longrightarrow \mathbb{R} \cup \{\infty\}.$$

⊙ $\mathbb{P}(\mathcal{M}_n(\mathbb{C}))$ is a complex Riemannian manifold with a Riemannian metric (Fubini-Study) and a volume form $d\nu_{\mathbb{P}}$ which agrees with the standard Gaussian distribution in the affine space.

Riemannian (Fubini-Study) :

$$d_R(\pi(x), \pi(x')) := arccos\left(\frac{|\langle x, x'\rangle|}{\|x\|\|x'\|}\right).$$

Projective :

$$d_P(\pi(x), \pi(x')) := sen\ d_R(\pi(x), \pi(x')).$$

Theorem 9 (Eckart-Young-Schmidt-Mirsky) $\Sigma \subseteq \mathbb{P}(\mathcal{M}_n(\mathbb{C}))$ the algebraic variety of singular matrices. Then

$$\kappa_D(A) := \frac{1}{d_{\mathbb{P}}(A, \Sigma)}.$$

⊙ Some Notations:

$*\pi : \mathbb{C}^{n^2} \setminus \{0\} \longrightarrow \mathbb{P}(\mathcal{M}_n(\mathbb{C}))$:= the canonical projection.

$* \ A \subseteq \mathbb{P}(\mathcal{M}_n(\mathbb{C})), \widetilde{A} := \pi^{-1}(A) \cup \{0\}$:= the affine cone over A.

$$\gamma(\widetilde{A}) = \frac{\nu_{\mathbb{P}}(A)}{\nu_{\mathbb{P}}[\mathbb{P}(\mathbb{C}^{n^2})]}.$$

γ:= the gaussian volume in $\mathbb{C}^{n^2}$.

$\nu_{\mathbb{P}}$:= the Riemannian volume in $\mathbb{P}(\mathcal{M}_n(\mathbb{C}))$.

⊙ Probability in $\mathbb{P}(\mathcal{M}_n(\mathbb{C}))$

$$\frac{\nu_{\mathbb{P}}[\Sigma_\varepsilon]}{\nu_{\mathbb{P}}[\mathbb{P}(\mathcal{M}_n(\mathbb{C}))]},$$

$$\Sigma_\varepsilon := \{A \ : \ \kappa_D(A) > \frac{1}{\varepsilon}\} = \{A \ : \ d_{\mathbb{P}}(A, \Sigma) < \varepsilon\}.$$

Σ_ϵ:= Tube of radius ε about Σ.

* Smale, Edelmann,..

Theorem 10 (Edelman) *Same Notations as above,*

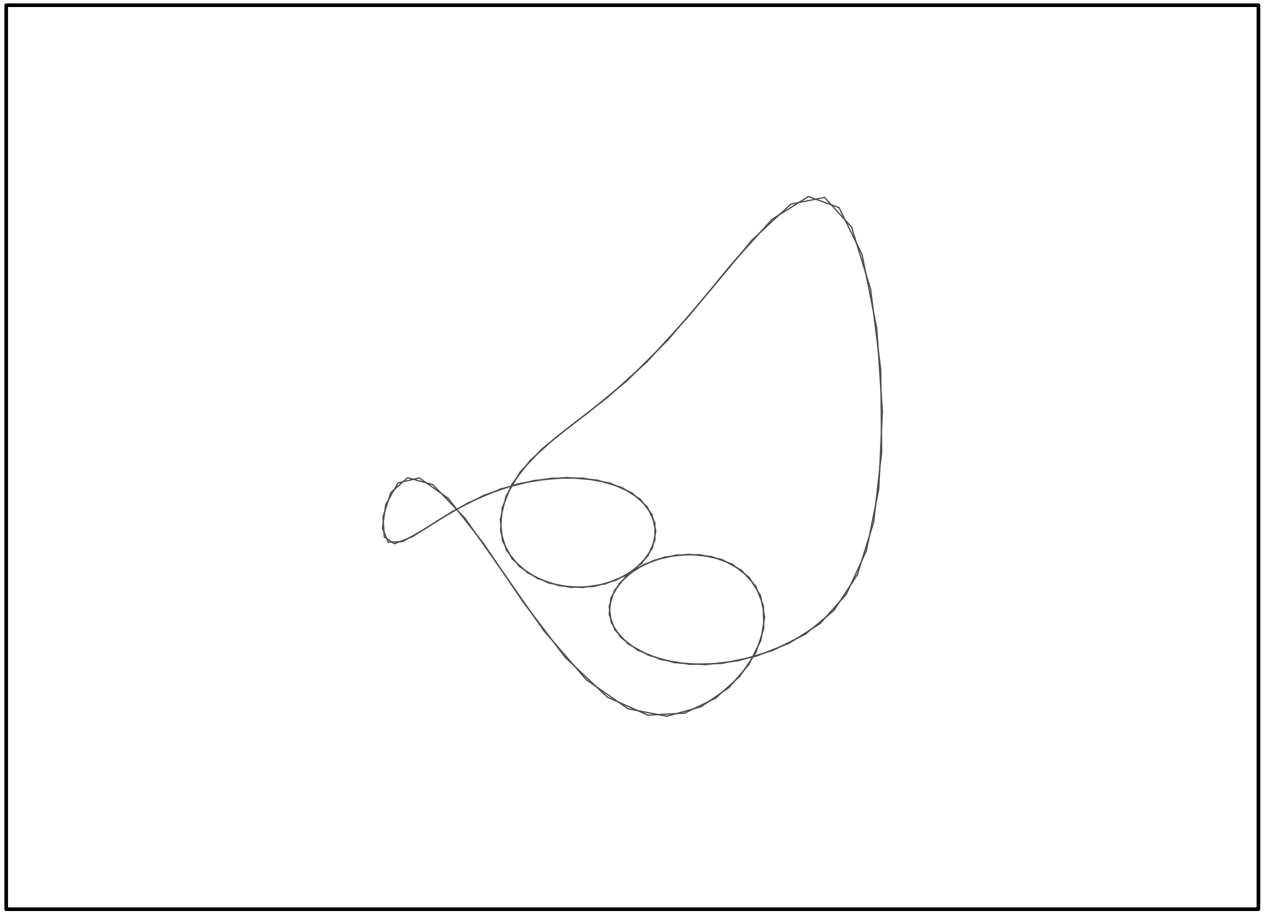
$$\frac{\nu_{\mathbb{P}}[\Sigma_{\varepsilon})]}{\nu_{\mathbb{P}}[\mathbb{P}(\mathcal{M}_n(\mathbb{C}))]} = 1 - (1 - n\varepsilon^2)^{n^2-1}.$$

⊙ **General Question:** Is this an specific bound or there are:

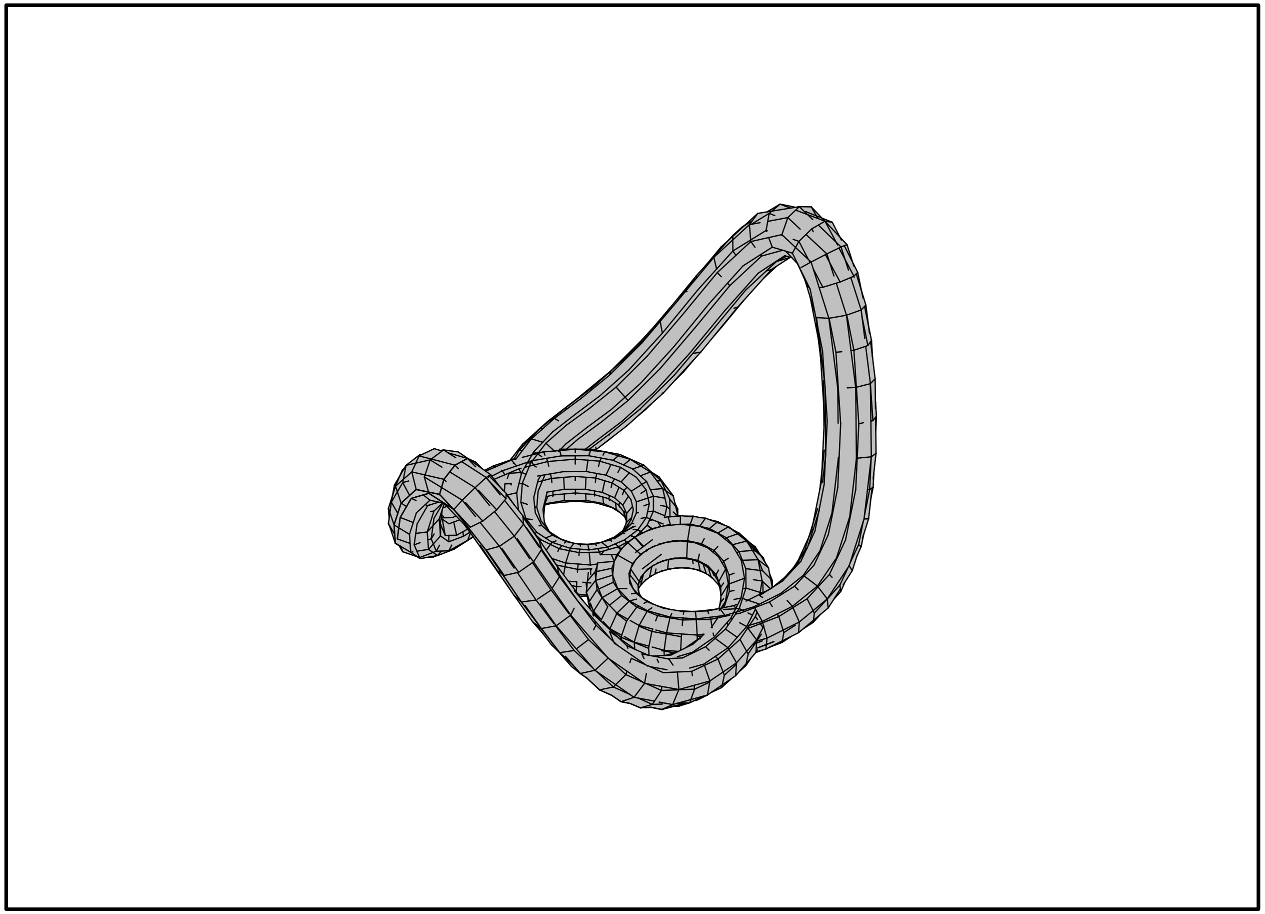
SHARP ESTIMATES ON THE VOLUME OF A TUBE ABOUT A PROJECTIVE ALGEBRAIC VARIETY.

⊙ EXTENSION OF THESE IDEAS TO GENERAL CLASSES OF MATRICES

Wrong Picture I:A curve



WRONG PICTURE II:A TUBE ABOUT THE SURFACE



⊙ A Tube about a curve.

* A topic of long tradition: Weyl, Wirtinger, Lelong, Wolf,.. (affine)

* Deepest (Main) contributions: A. Gray.

* Drawbacks: Gray's are only applicable to smooth, complete intersection varieties and for small values of ε (convergence radius of the exponential function).

IT DRAMATICALLY VANISHES IN PRESENCE OF SINGULARITIES.

Theorem 11 (Beltrán-P., 2005) *$V \subseteq \mathbb{P}_n(V)$ an equidimensional complex projective algebraic variety.*

$$V_\varepsilon := \{x \in \mathbb{P}_n(\mathbb{C}) \ : \ d_{\mathbb{P}}(x, V) < \varepsilon\}.$$

Then,

$$\frac{\nu_{\mathbb{P}}[V_\varepsilon]}{\nu_{\mathbb{P}}[\mathbb{P}_n(\mathbb{C})]} \leq 2deg(V) \left(\frac{en\varepsilon}{codim(V)}\right)^{2codim(V)}.$$

Corollary 12 *In the linear algebra case*

$$\frac{\nu_{\mathbb{P}}[\Sigma_\varepsilon]}{\nu_{\mathbb{P}}[\mathbb{P}(\mathcal{M}_n(\mathbb{C}))]} \leq 2n \left(\frac{en^2\varepsilon}{1}\right)^2 = 2e^2n^5\varepsilon^2.$$

$$\Sigma^{(r)} := \{A \in \mathbb{P}(\mathcal{M}_n(\mathbb{C})) \ : \ rank(A) \leq r\}.$$

$(\Sigma = \Sigma^{(n-1)})$.
$\Sigma^{(r)} \subseteq \mathbb{P}(\mathcal{M}_n(\mathbb{C}))$ is a projective variety of codimension $(n-r)^2$.

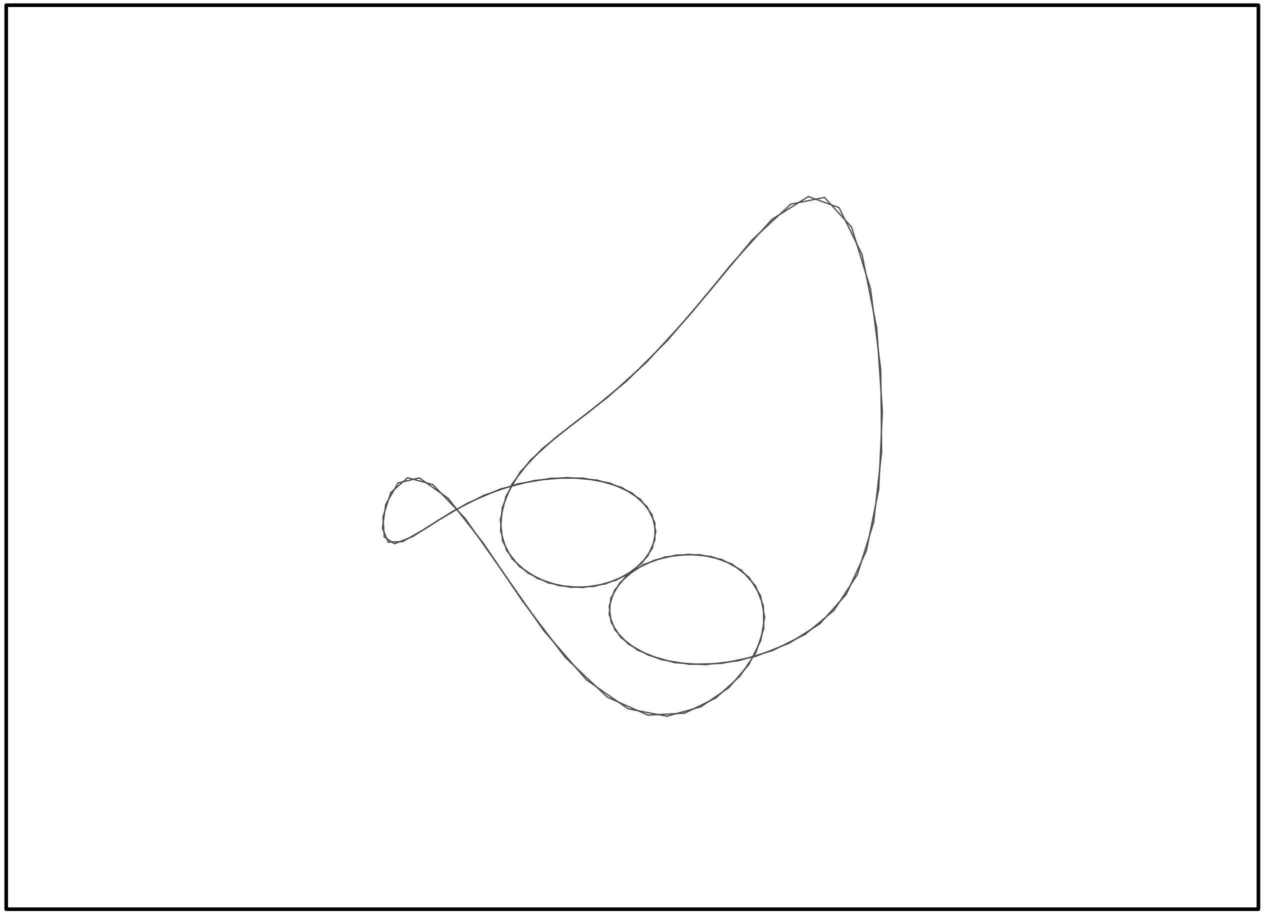
⊙ **Condition Number of a Singular Matrix:** $A \in \Sigma^{(r)}$

$$\kappa_D^{(r)}(A) := \kappa_D(A^{\dagger}) = \kappa_D(A \mid_{Ker(A)^{\perp}}).$$

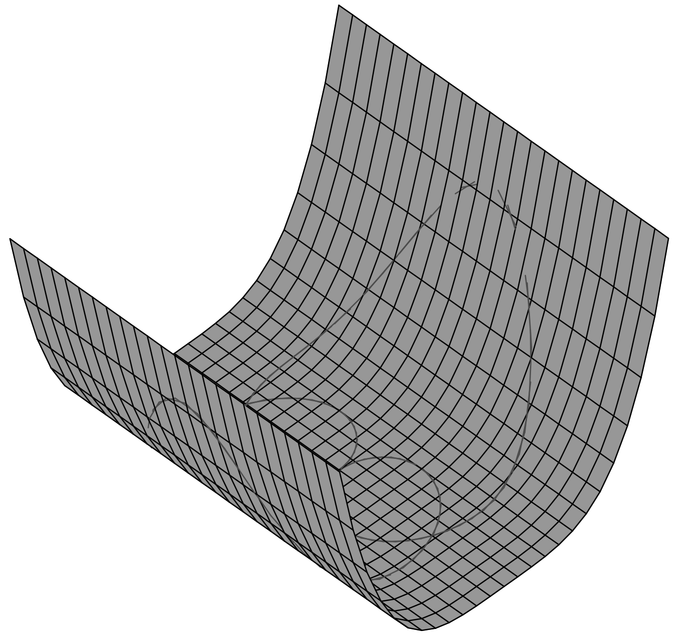
Proposition 13 (Several authors, easy from SVD) *The condition number $\kappa_D^{(r)}(A)$ measures the relative error if the computation of a basis of $Ker(A)$. It also verifies:*

$$\kappa_D^{(r)}(A) = \frac{1}{d_{\mathbb{P}}(A, \Sigma^{(r-1)})}.$$

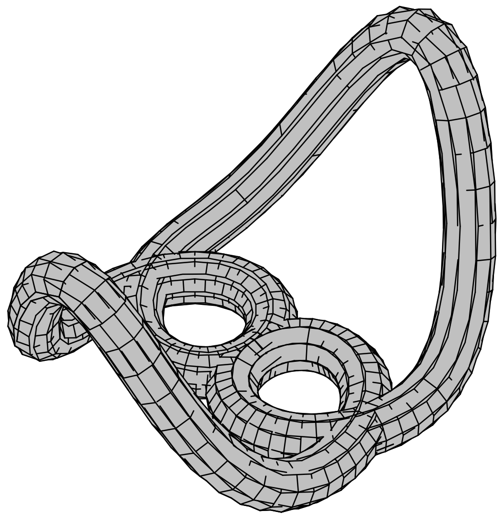
Wrong Picture I:A curve



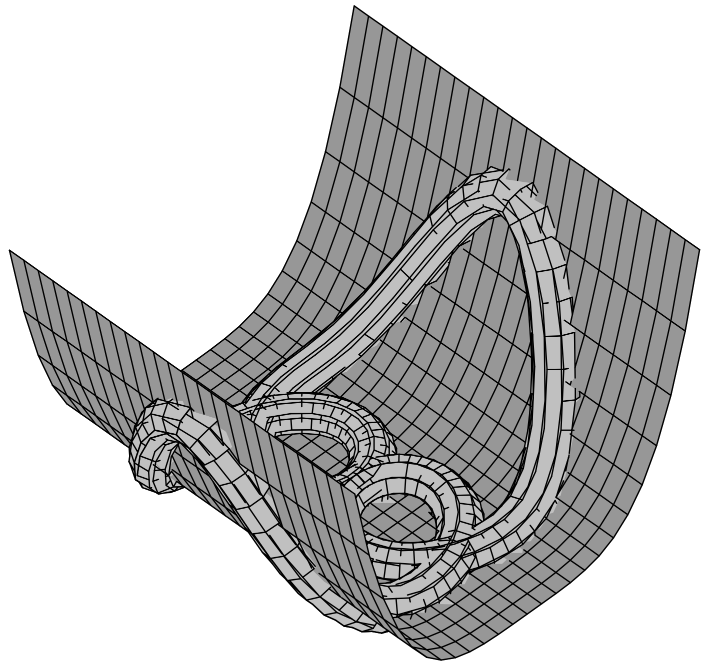
Wrong Picture II:The curve in a surface



Wrong Picture III:The Tube about the surface



Wrong Picture IV:The Intersection



⊙ $V \subseteq \mathbb{P}_n(\mathbb{C})$ an equi–dimensional algebraic variety of dimension m.

⊙ $V' \subseteq \mathbb{P}_n(\mathbb{C})$ equi–dimensional of dimension $1 < m' < m$.

⊙ $V'_\varepsilon := \{x \in \mathbb{P}_n(\mathbb{C}) \ : \ d_{\mathbb{P}}(x, V') < \varepsilon\}$.(The tube)

Theorem 14 (Beltrán-P., 2005) *The following holds:*

$$\frac{\nu_m[V'_\varepsilon \cap V]}{\nu_m[V]} \leq 2C(n, m, m')deg(V')\varepsilon^{2(m-m')}.$$

Corollary 15 The probability that a matrix of rank $n-1$ is at distance bigger than ε of $\Sigma^{(n-2)}$ is greater than

$$1 - Cn^{12}\varepsilon^{6},$$

for some constant C.

Otherwise said, the probability that a singular matrix satisfies

$$\kappa_D^{(n-1)}(A) \leq Cn^{3},$$

is greater than

$$1 - \frac{1}{n^{6}}.$$

Rk. Bounds are easily applicable to studies of some kinds of condition numbers of non–linear singular systems (based on fixed co–rank singularities...). But this approach is still ongoing...

Main 1: Universal Solving Procedures require exponential running time.

Main 2: There is an efficient non–universal (true) algorithm that solves most multivariate polynomial equations.

Ongoing Projects:

Find a more “aesthetic” questor class.

Find “simpler” questor class...until you get a deterministic efficient algorithm (if any).

Probability Distribution of Condition Numbers and other "special" classes of data: encodings (Straight–line Program Encoding,...)geometry (Singular systems), applications (systems for classes of real life problems)...

Approximate Zeros and Elimination Theory: What can be done with just a few approximate zeros?

Some results with Castro–Haegele and Morais...

Joos,

seguimos buscando nuevas fronteras.

Gracias por tu inspiración.