Real elimination & application to the wavelet design

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TERA 2005—60th anniversary of Joos Heintz

Outline

1. Orthonormal systems
   - Sampling
   - Frequency bands and subbands
   - Wavelet Packets
2. Wavelets and their scaling functions
   - Refinement equation
   - Existence
   - Orthogonality
3. Real algebraic theory
   - Lagrange Theory

Outline

The Fourier Transform

Representation of functions as composition of waves

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\omega) e^{i(2\pi\omega)x} \, \mathsf{d}\omega \ ?$$

Everything is "legal" for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with

$$\widehat{f}(\omega) := \mathcal{F}(f)(\omega) := \int_{\mathbb{R}} f(x) e^{-i(2\pi\omega)x} \, \mathsf{d}x \ .$$

The *Fourier transform* is the unique extension $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$. $\mathcal{F}$ is unitary, $\mathcal{F}^{-1} = \mathcal{F}^*$,

$$\Longrightarrow \quad \text{Plancherels identity } \langle f, g \rangle = \langle \mathcal{F}(f), \mathcal{F}(g) \rangle \ .$$

Fourier series

$\{f_n : n \in \mathbb{Z}\}$ is *orthonormal system* (ONS)

$$\iff \langle f_k, f_n \rangle = \delta_{k,n} \text{ (Kronecker symbol).}$$

Plancherel identity implies: $\mathcal{F}$ transforms ONS into ONS.

Let $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $e_n(x) := e^{i(2\pi n)x}$ for $n \in \mathbb{Z}$,

$$\chi(x) := \chi_I(x) := \begin{cases} 1 & |x| \le \frac{1}{2} \\ 0 & \text{else} \end{cases}.$$

Fourier series:

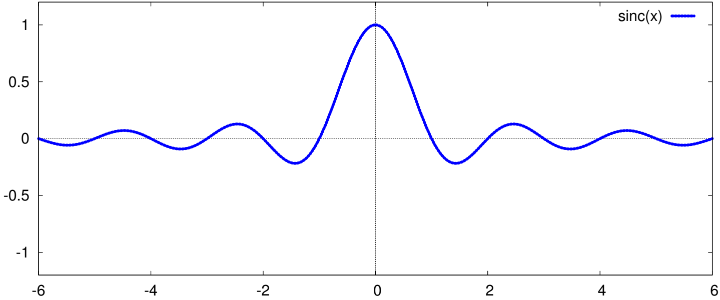
$\{\chi e_n : n \in \mathbb{Z}\}$ is an ONS in $L^2(\mathbb{R})$ and is a Hilbert basis of $L^2(I)$.

Fourier (ca. 1810)

Sinus cardinalis

With $\operatorname{sinc}(x) := \frac{\sin \pi x}{\pi x}$:

$$\mathcal{F}^{-1}(\chi e_n)(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i(2\pi\omega)(x+n)}\,\mathsf{d}\omega = \frac{\sin \pi(x+n)}{\pi(x+n)} = \operatorname{sinc}(x+n)$$



Basis band

Let $PW(I) := \{f \in L^2(\mathbb{R}) : \widehat{f} \in L^2(I)\}$, $s_n(x) := \operatorname{sinc}(x-n)$ for $n \in \mathbb{Z}$.

$\Longrightarrow \{s_n : n \in \mathbb{Z}\}$ is an ONS in $L^2(\mathbb{R})$ and is a Hilbert basis of $PW(I)$.

$f \in PW(I) \iff f = \sum_{n\in\mathbb{Z}} a_n s_n$ with $a_n = \langle f, s_n \rangle$.

Sampling Theorem

$$s_n(k) = \operatorname{sinc}(k-n) = \delta_{k,n} \text{ (Kronecker symbol).}$$

Thus, with $f = \sum_{n\in\mathbb{Z}} a_n s_n$, one gets $f(k) = a_k$ for each $k \in \mathbb{Z}$.

Sampling theorem:

$$f \in PW(I) \implies f = \sum_{n\in\mathbb{Z}} f(n) s_n \ .$$

More generally, $f \in PW\left(\left[-\frac{B}{2}, \frac{B}{2}\right]\right)$ for some $B > 0$, $d \in \mathbb{R}$

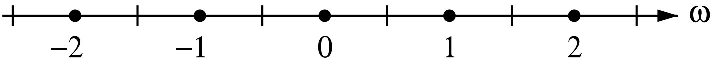
$$\implies f(x) = \sum_{n\in\mathbb{Z}} f\left(d + \tfrac{n}{B}\right) \operatorname{sinc}\Big(B(x-d) - n\Big) \ .$$

Whittaker (1915), Kotelnikov (1933), Shannon (1949)

Outline

Frequency Multiplex Method

$f \in L^2(\mathbb{R},\mathbb{R})$ a real function $\implies \widehat{f}(-\omega) = \overline{\widehat{f}(\omega)}$.



Band N: $\operatorname{supp} \widehat{f} \cap [0,\infty) \subset \left[N - \frac{1}{2}, N + \frac{1}{2}\right]$

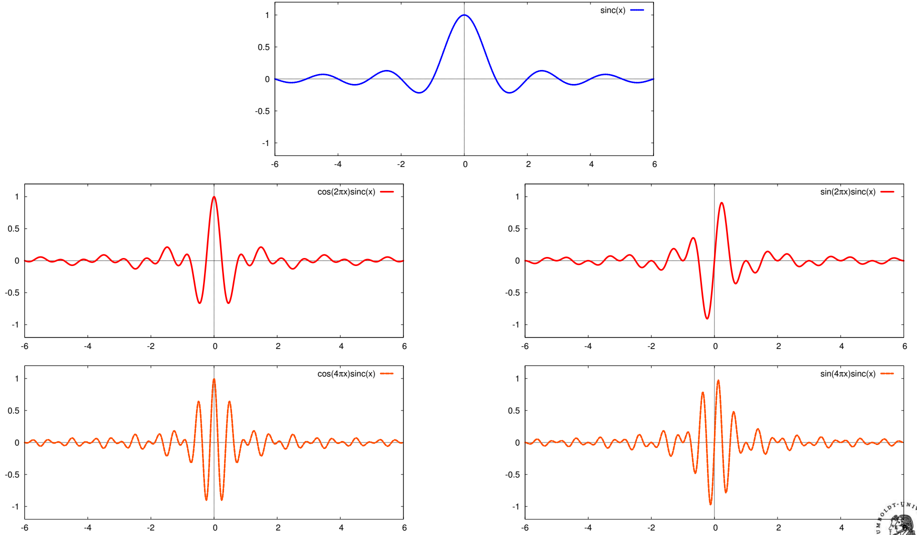
QAM pairs

Band 0: $\sum_{n\in\mathbb{Z}} a_{0,n} \operatorname{sinc}(x-n)$
Band N: $\sum_{n\in\mathbb{Z}} \big(a_{N,n} \cos(2\pi N)x + b_{N,n} \sin(2\pi N)x\big) \operatorname{sinc}(x-n)$.

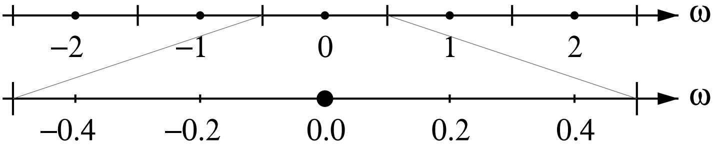
Broadband: $N = 1, \dots, 2^n$,
eg. *DAB*: $n = 2, 3, \dots, 6$, *DVB*: $n = 10, \dots, 12$.

"Cheat": replace sinc by χ and apply frequency filtering.
Downside: orthogonality is lost $\implies$ ISI: inter-symbol interference

Generating functions



Multi–scale properties



From the sampling theorem on $PW\left(\left[-\frac{5}{2},\frac{5}{2}\right]\right)$ one obtains e.g.

$$\cos(4\pi x)\,\mathrm{sinc}(x) = \sum_{n\in\mathbb{Z}} \cos\left(\frac{4\pi n}{5}\right)\mathrm{sinc}\left(\frac{n}{5}\right)\mathrm{sinc}(5x-n)$$

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Shift–orthonormal Functions and Sequences

$\varphi \in L^2(\mathbb{R})$ is said to be *shift–orthonormal*, if

$$\{\varphi_n := \mathcal{S}^n\varphi : n \in \mathbb{Z}\},\ \text{with the } \textit{shifts } (\mathcal{S}^n\varphi)(x) := \varphi(x-n),\ n \in \mathbb{Z},$$

is an ONS.

Fix $M, N \in \mathbb{N}$ with $0 < M \leq N.$ Given finite sequences $b_1 := \{b_{n,1}\}_{n\in\mathbb{Z}}, \ldots, b_M := \{b_{n,M}\}_{n\in\mathbb{Z}}$, they are said to be shift–N–orthonormal, if

$$\left\langle \mathcal{S}^{kN} b_m, \mathcal{S}^{lN} b_n \right\rangle := \sum_{j\in\mathbb{Z}} b_{j+kN,m} b_{j+lN,n} = \delta_{k,l}\delta_{m,n} \text{ (Kronecker symbol).}$$

Wavelet Packets

Let $\varphi \in L^2(\mathbb{R})$ be a shift–orthonormal function, $b_1, \ldots, b_M \in \mathbb{R}^{\mathbb{Z}}$ shift–N–orthonormal finite sequences.

Define functions $\psi_j(x) := \sqrt{N} \sum_{k\in\mathbb{Z}} b_{k,j} \varphi(Nx - k)$, $j = 1, \ldots, M$.

Wavelet Packet Theorem

Each of $\psi_1, \ldots, \psi_M$ is shift–orthonormal and

$$\{\psi_{j,n} := \mathcal{S}^n \psi_j : j = 1, \ldots, M,\ n \in \mathbb{Z}\}$$

is an ONS in $L^2(\mathbb{R})$.

Note that $\widehat{\psi}_j = \frac{1}{\sqrt{N}} \widehat{b}_j \left(\frac{\omega}{N}\right) \widehat{\varphi} \left(\frac{\omega}{N}\right)$ with $\widehat{b}_j = \sum_{k\in\mathbb{Z}} b_{k,j} e_{-k}$.

Daubechies (1992)

Interpretation I: Filling a noisy channel

Shannons channel capacity formula:

A channel with average power P, average noise power N and bandwidth of B cycles per second allows the transmission of up to

$$B \log_2 \left(1 + \frac{P}{N}\right) \frac{bits}{s}.$$

The Wavelet packet construction allows to split this channel orthonormally into M channels of bandwidth $\frac{B}{M}$ with power $\frac{P}{M}$ each and noise levels $N_1 + \cdots + N_M = N$.
$log_2 \left(1 + \frac{1}{x}\right)$ is concave, so for the bitrates one gets

$$\frac{B}{M} \sum_{j=1}^{M} \log_2 \left(1 + \frac{P}{MN_j}\right) \geq B \log_2 \left(1 + \frac{P}{N}\right).$$

Shannon (1949)

Interpretation II: Approximating a Function

Suppose $M = N$ and both φ and ψ_1 lie "almost" in $PW([-\frac{1}{2}, \frac{1}{2}])$.
Any $f \in L^2(\mathbb{R})$ with

$$f(x) = \sqrt{M} \sum_{k \in \mathbb{Z}} c_k \varphi(Mx - k)$$

lies almost in $PW([-\frac{M}{2}, \frac{M}{2}])$.
Define M coefficient sequences by

$$d_{j,k} = \sum_{n \in \mathbb{Z}} b_{j,n} c_{kM+n}.$$

Then also $f = \sum_{j=1}^{M} \sum_{k \in \mathbb{Z}} d_{j,k} \psi_{j,k}$ holds.

The sum for $j = 1$ represents the "trend", the sums for $j = 2, \ldots, M$ represent "details" of the function f

Haar (1910), Morlet/Grossmann (ca. 1985)

Outline

Wishlist

Make wavelet packets recursive via:

ψ_1 `becomes the next` φ,

while keeping interpretation II: "Approximation".

Preferably

- $\varphi = \psi_1$, i.e., with $a := \sqrt{M}\, b_1$ one gets the

$$\textit{Refinement equation: } \varphi(x) = \sum_{n\in\mathbb{Z}} a_n \varphi(Mx - n);$$

- φ should lie "almost" in $PW(I)$;
- φ should be "smooth" with compact support;
- φ should be symmetric.

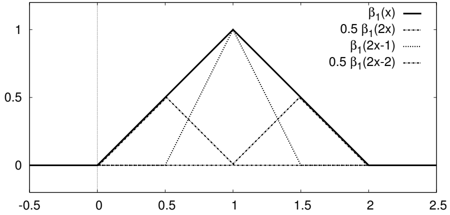
Notations

$a(Z) = \sum_{n\in\mathbb{Z}} a_n Z^n$ is a *Laurent–polynomial*,
$a(\mathcal{S}) := \sum_{n\in\mathbb{Z}} a_n \mathcal{S}^n$ is a bounded linear operator on $L^2(\mathbb{R})$.

With $\mathcal{D}_M : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $(\mathcal{D}_M f)(x) = f(Mx)$, the *refinement equation* reads

$$\varphi = \mathcal{D}_M a(\mathcal{S})\varphi$$

E.g., $\beta_1(x) := \max\big(0, 1 - |x-1|\big)$ satisfies $\beta_1 = \mathcal{D}_2 \frac{(1+\mathcal{S})^2}{2}\beta_1$.



Outline

Conditions for Existence and Smoothness

The *Haar–polynomial* is $H_M(Z) := \frac{1}{M}\left(1 + Z + \cdots + Z^{M-1}\right)$.

For a continuous solution with compact support of $\varphi = \mathcal{D}_M\, a(\mathcal{S})\varphi$ to exist,

- it is necessary that a has the structure

  $$a(Z) = M\, H_M(Z)^A\, p(Z) \quad \text{with } A \in \mathbb{N}\ \&\ A \geq 1,$$

  $p(Z)$ is a Laurent–polynomial with $p(1) = 1$.

- it is sufficient that additionally for some $r > 0$

  $$\|\widehat{p}\|_\infty := \sup_{\omega\in\mathbb{R}} \left| p(e^{i(2\pi\omega)}) \right| = M^{A-1-r} \text{ holds.}$$

  With $r = n + \alpha$ where $n \in \mathbb{N}$ & $\alpha \in (0, 1]$, one gets that φ is n times continuously differentiable.

Strang (ca. 1960), Daubechies (1992)

Algebraization of the Supremum

p as sequence is finite and real, i.e. there exists $\mathcal{J} \subset \mathbb{Z}$ finite with $p(Z) = \sum_{n\in\mathcal{J}} p_n Z^n$.
With the Cauchy–Schwarz inequality one gets

$$|p(e^{i\omega})| \le \sum_{n\in\mathcal{J}} |p_n| \le \sqrt{\#\mathcal{J}} \sqrt{\sum_{n\in\mathcal{J}} |p_n|^2}.$$

Task:

Minimize $\sum_{n\in\mathbb{Z}} p_n^2$ wrt. further conditions.

Advanced estimates of $r = n + \alpha$: Setting $p_{(j)} = \{p_{j+kM}\}_{k\in\mathbb{Z}}$,

$$M^{A-1-r} \text{ is the smaller of } \max_{j=1,\dots,M} \|p_{(j)}\|_1 \text{ and } \sqrt{\sum_{j=1,\dots,M} \|\widehat{p}_{(j)}\|_\infty^2}.$$

Heil (1992), Cabrelli-Heil-Molter (1996), Lehmann (2005)

Outline

Conditions for Shift–Orthonormality

For $\{\mathcal{S}^n \varphi : n \in \mathbb{Z}\}$ to be an ONS and $\varphi = \mathcal{D}_M a(\mathcal{S})\varphi$ to hold

- it is necessary, that $\frac{1}{\sqrt{M}} a$ is shift–M–orthonormal, i.e., for each $k \in \mathbb{Z}$
$$\sum_{n \in \mathbb{Z}} a_n a_{n+kM} = M \delta_{0,k} \text{ (Kronecker symbol).}$$
- it is sufficient, that additionally φ is continuous with compact support.

Cohen (ca. 1990)

Algebraic simplification

Using the structure $a(Z) = M\, H_M(Z)\, p(Z)$, an equivalent condition is

$$p(Z)p(Z^{-1}) = P_{M,A}\left(1 - \tfrac{Z+Z^{-1}}{2}\right) + \left(1 - \tfrac{Z+Z^{-1}}{2}\right)^A R(Z),$$

where $R(Z)$ is a Laurent–polynomial $R(Z) = \sum_{n\in\mathbb{Z}} R_n Z^n$, that satisfies $R_{kM} = 0$ for all $k \in \mathbb{Z}$.

$P_{M,A} \in \mathbb{Q}[X]$ is efficiently computable.

One obtains further simplifications if p is symmetric, i.e. $p(Z) = q\left(1 - \tfrac{Z+Z^{-1}}{2}\right)$ with $q(X) = Q_{M,A}(X) + X^A r(X)$.

$Q_{M,A} \in \mathbb{Q}[X]$ is again efficiently computable.
Heller (1995), Belogay/Wang (1999), Han (2002)

Equations

Parameters: M, A, $n \in \mathbb{N}$

Variables: $X_1, \ldots, X_n$, $\mathcal{R} := \mathbb{Q}[X_1, \ldots, X_n]$,

- Define
  $q(U) := Q_{M,A}(U) + U^A \left(X_1 + X_2 U + \cdots + U^{n-1} X_n\right) \in \mathcal{R}[U]$,
- $C(U) := 1$ unless both M and A are even, then $C(U) := 1 - \frac{1}{2}X$,
- Compute $s(U)$ from $C(U)q(U)^2 - P_{M,A}(U) = U^A s(U)$,
- Expand $r(Z) = s\left(1 - \frac{Z+Z^{-1}}{2}\right) \in \mathcal{R}[Z, Z^{-1}]$,
- Extract coefficients $f_{1+n} := R_{sn} \in \mathcal{R}$, $n = 0, \ldots, p-1$,
  $sp - s \leq \deg_U s(U) < sp$,
- Expand $p(Z) := q\left(1 - \frac{Z+Z^{-1}}{2}\right)$ and define $g := \sum_{k=1}^{n-1} p_k^2 \in \mathcal{R}$.

Minimize $g(x)$ under the conditions $f_1(x) = \cdots = f_p(x) = 0$.

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Varieties and Geometric Degree

Let $f_1, \dots, f_p \in \mathbb{Q}[X_1, \dots, X_n]$ be polynomials of degree bounded by d.
$V_{\mathbb{C}}(f_1, \dots, f_p) := \{x \in \mathbb{C}^n : f_1(x) = \cdots = f_p(x) = 0\}$ and
$V_{\mathbb{R}}(f_1, \dots, f_p) := V_{\mathbb{C}}(f_1, \dots, f_p) \cap \mathbb{R}^n$.

Suppose $V \subset \mathbb{C}^n$ is an irreducible and equidimensional algebraic variety, $\dim V = n - p$. Define $\deg(V, H) := \#(V_{\mathbb{C}} \cap H)$ for every hyperplane of dimension p.

$$\deg V := \max\left\{\deg(V, H) : H \text{ hyperplane with } \deg(V, H) < \infty\right\}.$$

If $V = C_1 \cup \cdots \cup C_N$, then $\deg V := \deg C_1 + \cdots + \deg C_N$.

Bezout–inequality: $\implies \deg V_{\mathbb{C}}(f_1, \dots, f_p) \leq d^n$.

Critical points

$x \in V_{\mathbb{C}}(f_1, \ldots, f_p)$ is *regular* $\iff \operatorname{rk} \frac{\partial(f_1,\ldots,f_p)}{\partial(x_1,\ldots,x_n)}(x) = p.$

Consider $g \in \mathbb{Q}[X_1, \ldots, X_n]$ as function on $V_{\mathbb{R}}(f_1, \ldots, f_p)$.

$x \in V_{\mathbb{C}}(f_1, \ldots, f_p)$ is *critical* $\iff \operatorname{rk} \frac{\partial(f_1,\ldots,f_p,g)}{\partial(x_1,\ldots,x_n)}(x) \leq p.$

$(f_1, \ldots, f_p)$ has *evaluation complexity* $L \iff$ it exists an arithmetic circuit of size L that evaluates $(f_1, \ldots, f_p)$.

Linear Functionals

From the theory of *classical polar varieties*:

Given $(f_1, \dots, f_p)$ with degree bound d and evaluation complexity L that is a *regular sequence* with geometric degree δ.

Then there is a dense subset $\mathcal{A} \subset \mathbb{Q}^n$ so that for any $a \in \mathcal{A}$ and the function $a^T : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto a^T x$

- all of the critical points of a^T on $V_{\mathbb{R}}(f_1, \dots, f_p)$ are regular and
- a "numerical easy" representation of them can be computed in time $\binom{n}{p} L^2 (nd\delta)^{O(1)}$.

TERA (since 1995), Mbakop (1999), Bank/Giusti/Heintz/Mbakop (2001),Lecerf (2001), B/G/H/Pardo (2003)

Nonlinear functionals

$f_1, \ldots, f_p, g \in \mathbb{Q}[X_1, \ldots, X_n] \subset \mathbb{Q}[X_1, \ldots, X_n, X_{n+1}]$ as above.

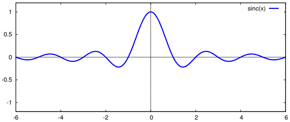
$V := V_{\mathbb{R}}(f_1, \ldots, f_p) \subset \mathbb{R}^n$ and $V_g := V_{\mathbb{R}}(f_1, \ldots, f_p, g - X_{n+1}) \subset \mathbb{R}^{n+1}$

$(x, x_{n+1}) \in V_g \iff x \in V$ and $a^T x + x_{n+1} = a^T x + g(x)$

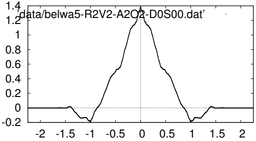
$(x, x_{n+1}) \in V_g$ regular $\iff x \in V$ regular

$(x, x_{n+1}) \in V_g$ critical for $(a, 1)^T \iff x \in V$ critical for $g + a^T$

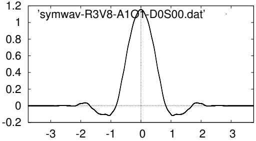
$\Longrightarrow$ we can—in a probabilistic way—decide if critical points of g on V (case $a = 0$) are regular and compute them.



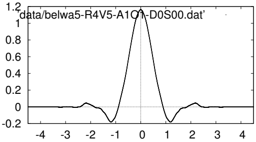
sinc as ideal



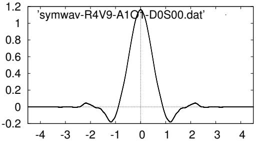
$A = 2,\ n = 2$



$A = 3,\ n = 8$



$A = 4,\ n = 5$



$A = 4,\ n = 9$