Error estimates for anisotropic finite elements

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- Introduction.
- Error estimates for the Lagrange interpolation.
- Regularity conditions.
- Differences between 2D and 3D cases.
- Necessity of other interpolations.
- An average interpolation.
- Application to problems with boundary layers.
- Numerical examples.

The Finite Element Method is based on the variational formulation:

$$B(u, v) = F(v) \qquad \forall v \in H$$

H Hilbert space

B continuous bilinear form, F continuous linear form.

Classic examples:

Scalar second order elliptic equations:

$$\begin{cases} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j}) = f & \text{in} \quad \Omega \subset \mathbb{R}^n \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

$$\gamma|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leq M|\xi|^2 \quad \forall x \in \Omega \quad \forall \xi \in \mathbb{R}^n$$

$$H = H_0^1(\Omega) = \{v \in L^2(\Omega) : \frac{\partial v}{\partial x_j} \in L^2(\Omega) \text{ for } j = 1, \ldots, n \ , v = 0 \text{ on } \partial\Omega\}$$

which is a Hilbert space with the norm

$$\|v\|_{H^1}^2 = \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \quad ,$$

$$B(u, v) = \sum_{i,j=1}^{n} \int_\Omega a_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx$$

and

$$F(v) = \int_\Omega f v \, dx$$

The linear elasticity equations:

$$\begin{cases} -\mu\mathbf{\Delta u} - (\lambda + \mu)\nabla div\mathbf{u} = \mathbf{f} & \text{in} \quad \Omega \subset \mathbb{R}^3 \\ \mathbf{u} = 0 & \text{on} \quad \partial\Omega \end{cases}$$

$$H = H_0^1(\Omega)^3$$

$$B(\mathbf{u}, \mathbf{v}) = \int_\Omega \{2\mu\varepsilon_{i,j}(\mathbf{u})\varepsilon_{i,j}(\mathbf{v}) + \lambda div\mathbf{u}div\mathbf{v}\}\, dx$$

where

$$\varepsilon_{i,j}(\mathbf{v}) = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$$

The Stokes equations:

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in} \quad \Omega \subset \mathbb{R}^3 \\ div\,\mathbf{u} = 0 & \text{in} \quad \Omega \subset \mathbb{R}^3 \\ \mathbf{u} = 0 & \text{on} \quad \partial\Omega \end{cases}$$

$$B((\mathbf{u}, p)), (\mathbf{v}, q) = F(v)$$

$$H = H_0^1(\Omega)^3 \times L_0^2(\Omega)$$

FINITE ELEMENT METHOD

GENERAL SETTING: H Hilbert space

$$B(u,v) = F(v) \qquad \forall v \in H$$

B continuous bilinear form, F continuous linear form.

APPROXIMATE SOLUTION: $U \in H_{FE}$, $H_{FE} \subset H$

$$B(U,V) = F(V) \qquad \forall V \in H_{FE}$$

BASIC THEORY OF FINITE ELEMENT METHODS

ERROR ESTIMATES

They can be divided in two classes

-<u>A PRIORI ESTIMATES</u>

-<u>A POSTERIORI ESTIMATES</u>

Goals of a priori estimates:

- To prove convergence and to know the order of the error

- To know the dependence of the error on different things (geometry of the mesh, regularity of the solution, degree of the approximation).

A typical error estimate is of the form

$$\|error\| \leq Ch^{\alpha}|||u|||$$

where h is a mesh size parameter.

Goals of a posteriori estimates (using the computed solution!)

- To obtain quantitative information

- To develop adaptive methos

Typical error estimator

$$\eta = \left(\sum_{T \in \mathcal{T}_h} \eta_T^2 \right)^{1/2} .$$

η_T local contributions

$$\eta_T = \left(h_T^2 \|U\|_{0,T}^2 + \frac{1}{2} \sum_{F \in \mathcal{F}_T} h_F \|J_F\|_{0,F}^2 \right)^{1/2} ,$$

F edge in 2D or face in 3D

J_F jump of normal derivative:

$$J_F = \nabla \left(U|_{T_{\text{out}}} \right) \cdot n_F - \nabla \left(U|_{T_{\text{in}}} \right) \cdot n_F.$$

Typical result:

$$\|e\| \sim \eta$$

A PRIORI ESTIMATES

If

$$B(v, v) \geq \alpha \|v\|^2 \quad \forall v \in H$$

then

$$\|u - U\| \leq C \inf_{V \in H_{FE}} \|u - V\|$$

The computed approximate solution is, up to a constant, like the best approximation.

More generally:

If the inf-sup condition holds

$$\inf_{U \in H_{FE}} \sup_{V \in H_{FE}} \frac{B(U,V)}{\|U\|\|V\|} \geq \alpha > 0$$

then,

$$\|u - U\| \leq C \inf_{V \in H_{FE}} \|u - V\|$$

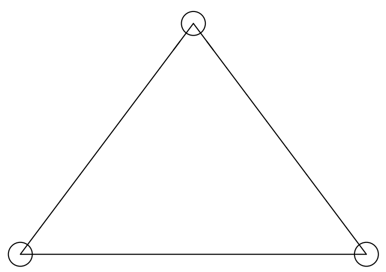
Example: Stokes equations, B is not coercive but satisfies inf-sup

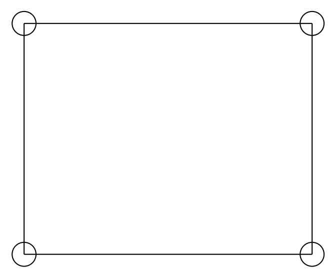
Therefore the problem reduces to obtain some approximation $V \in H_{FE}$ of u.

A natural approximation is

LAGRANGE INTERPOLATION

K triangle (3 nodes) or quadrilateral (4 nodes)





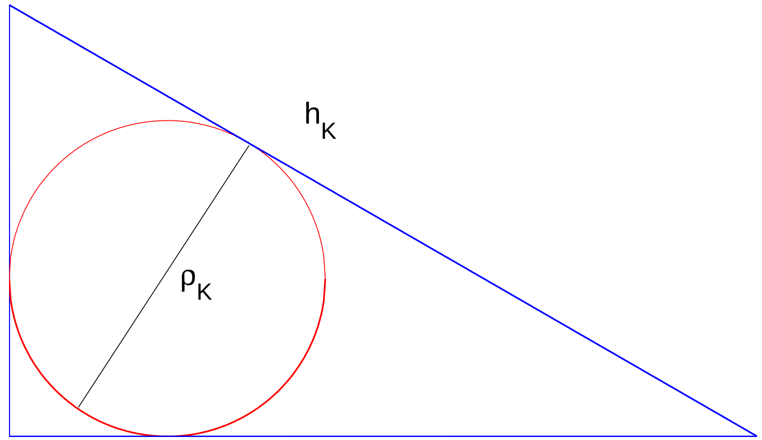
If P_i are the vertices, the Lagrange interpolation u_I of a function u is defined by:

$$\boxed{u_I(P_i) = u(P_i)}$$

ERROR ESTIMATES FOR LAGRANGE INTERPOLATION

They started to be developed around 1970

The classic theory is based on the "Regularity hypothesis"



$$\boxed{\frac{h_K}{\rho_K} \le \sigma}$$

h_K exterior diameter, ρ_K interior diameter.

In the estimate

$$\boxed{|u - u_I|_{1,K} \leq C h_K |u|_{2,K}}$$

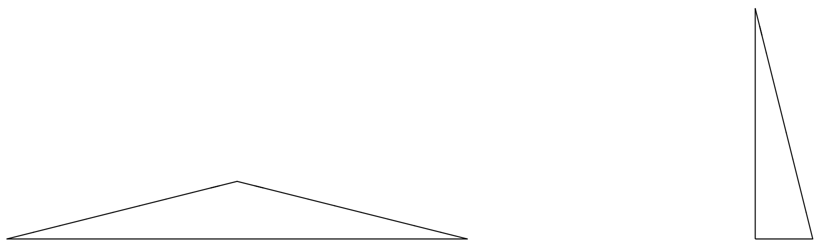
i. e.,

$$\boxed{\int_K |\nabla(u - u_I)|^2 \leq C h_K^2 \int_K |D^2 u|^2}$$

$C = C(\sigma) \to \infty$ when $\sigma \to \infty$.

BUT, IT IS KNOWN THAT THE REGULARITY HYPOTHESIS IS NOT NECESSARY

IN THE CASE OF TRIANGLES IT CAN BE REPLACED BY THE “MAXIMUM ANGLE CONDITION”



First results: Babuska-Aziz, Jamet (1976).

Other references: Krizek, Al Shenk, Dobrowolski, Apel, Nicaise, Formaggia, Perotto, Acosta, Lombardi, Durán, etc..

THE CASE OF QUADRILATERALS IS MORE COMPLICATED

SEVERAL CONDITIONS HAVE BEEN INTRODUCED

- Ciarlet-Raviart (1972): Regularity and non degeneracy of the angles.

- Jamet (1977): Regularity.

- Zenizek-Vanmaele (1995), Apel (1998): Allows anisotropic (flat) elements but far from triangles.

The most general condition seems to be

“THE REGULAR DECOMPOSITION PROPERTY” (G. Acosta, R.Durán, SIAM J. Numer. Anal. 2000)

RDP: K convex quadrilateral. Divide it in two triangles by the diagonal d_1. Then, the constant in the error estimate depends on the ratio $|d_2|/|d_1|$ and on the maximum angle of the two triangles.

In particular the maximum angle condition is a sufficient condition

CONSIDER THE CASE OF RECTANGULAR ELEMENTS

GOALS:

1- TO GIVE AN IDEA OF THE PROOF OF THE ERROR ESTIMATE

2- TO SHOW THE DIFFERENCES BETWEEN 2D AND 3D CASES

3- TO SHOW AN APPLICATION TO PROBLEMS WITH BOUNDARY LAYERS

K REFERENCE ELEMENT

IDEA: PROVE FIRST AN ESTIMATE IN K AND CHANGE VARIABLES

USUAL NOTATION:

$$\|v\|^2_{L^2(K)} = \int_K |v(x,y)|^2 dxdy$$

Given $u(x,y)$ let $p(x,y) = a + bx + cy$ be such that

$$\left\|\frac{\partial}{\partial x}(u-p)\right\|_{L^2(K)} \leq C \left\|\nabla \frac{\partial u}{\partial x}\right\|_{L^2(K)}$$

For example: p an average of Taylor polynomials of degree 1

$$p(x,y) = \frac{1}{|K|}\int_K \left\{ u(\overline{x},\overline{y}) + \frac{\partial u}{\partial x}(\overline{x},\overline{y})(x-\overline{x}) + \frac{\partial u}{\partial y}(\overline{x},\overline{y})(y-\overline{y}) \right\} d\overline{x}d\overline{y}$$

We want an analogous estimate for the interpolation error

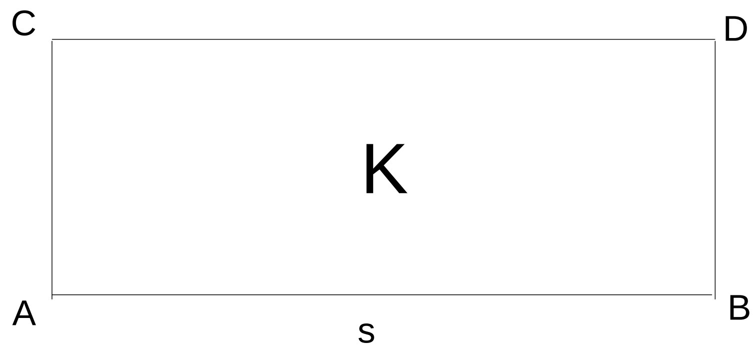
But,

$$\left\| \frac{\partial}{\partial x}(u-u_I) \right\|_{L^2(K)} \leq \left\| \frac{\partial}{\partial x}(u-p) \right\|_{L^2(K)} + \left\| \frac{\partial}{\partial x}(p-u_I) \right\|_{L^2(K)}$$

Therefore, it is enough to bound

$$\left\| \frac{\partial}{\partial x}(p - u_I) \right\|_{L^2(K)}$$

We use: for $\quad v = p - u_I = a + bx + cy + dxy$



$$\left\| \frac{\partial v}{\partial x} \right\|^2_{L^2(K)} \sim \quad |v(B) - v(A)|^2 + |v(D) - v(C)|^2$$

$$|v(B)-v(A)| = |(p(B)-u(B))-(p(A)-u(A))|$$

$$= \left|\int_s \frac{\partial}{\partial x}(p-u)\right| \le C\left\{\left\|\frac{\partial}{\partial x}(p-u)\right\|_{L^2(K)} + \left\|\nabla\frac{\partial u}{\partial x}\right\|_{L^2(K)}\right\}$$

where we have used a trace theorem

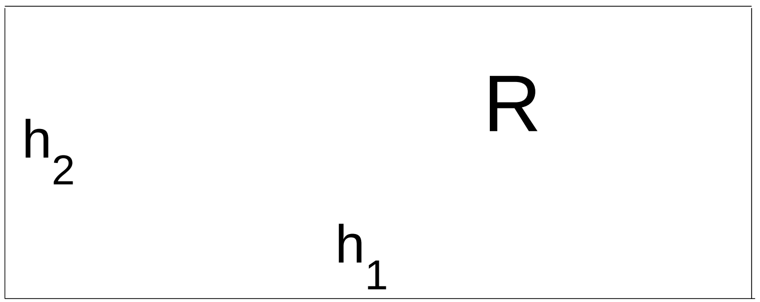
Analogously we estimate $|v(D)-v(C)|$ and so we obtain:

$$\left\|\frac{\partial}{\partial x}(u-u_I)\right\|_{L^2(K)} \le C\left\|\nabla\frac{\partial u}{\partial x}\right\|_{L^2(K)}$$

The important fact in this estimate is that

$$\frac{\partial^2 u}{\partial y^2} \qquad \text{does not appear!!}$$

and therefore, rescaling in each variable we obtain for a rectangle R



$$\left\| \frac{\partial}{\partial x}(u - u_I) \right\|_{L^2(R)} \leq C \left\{ h_1 \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L^2(R)} + h_2 \left\| \frac{\partial^2 u}{\partial x \partial y} \right\|_{L^2(R)} \right\}$$

THE CONSTANT C IS INDEPENDENT OF h_1 Y h_2 !

AN ANALOGOUS ESTIMATE IN 3D IS NOT TRUE!!

WHAT FAILS IN THE ARGUMENT?

$$\|u\|_{L^2(s)} \leq C\|u\|_{H^1(R)},$$

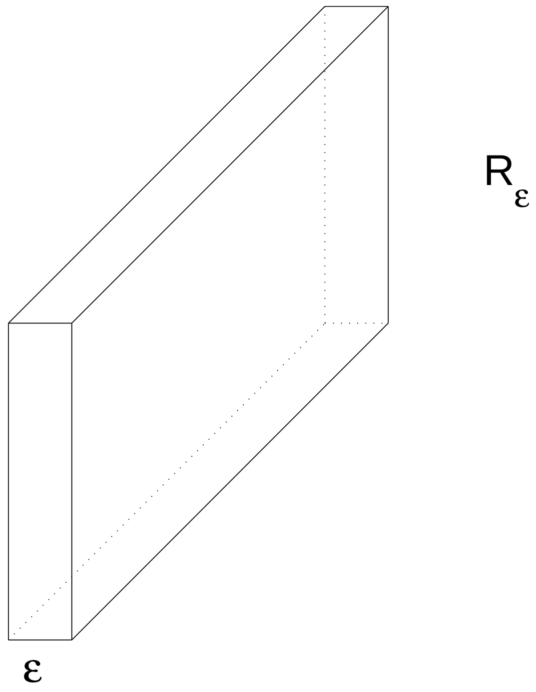
WHERE s IS AN EDGE OF R IS NOT TRUE

COUNTEREXAMPLES FOR THE INTERPOLATION ERROR ESTIMATE WERE GIVEN BY

Apel-Dobrowolski (Computing 1992), Al Shenk (Math. Comp. 1994).

$$\int_{R_\varepsilon} |\nabla(u - u_I)|^2 \sim C_\varepsilon h_{R_\varepsilon}^2 \int_{R_\varepsilon} |D^2 u|^2$$

C_ε goes to ∞ when $\varepsilon \to 0$



NATURAL QUESTION: IS THERE A BETTER APPROXIMATION?

YES !!

AVERAGE INTERPOLANTS

Originally they were introduced to approximate non smooth functions for which Lagrange interpolation is not even defined.

(P. Clement, 1976).

AN AVERAGE INTERPOLANT

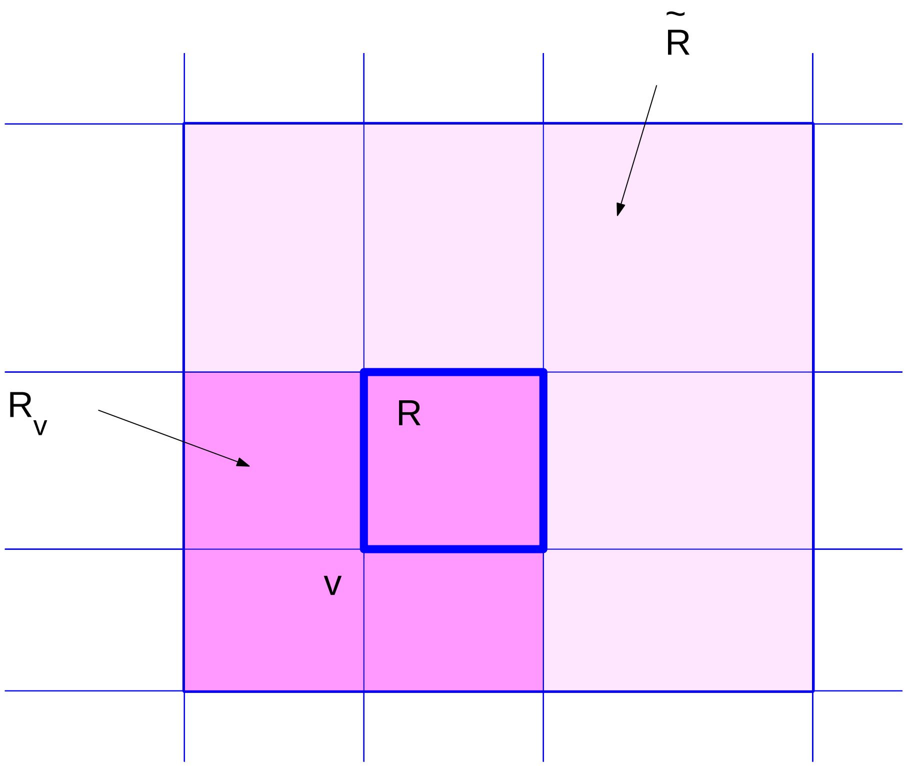
(A. Lombardi- R.Durán, Math. Comp. 2005)

HYPOTHESIS

R, S neighbor elements.

$$\boxed{\frac{h_{R,i}}{h_{S,i}} \leq \sigma \qquad 1 \leq i \leq n.}$$

THE CONSTANT IN THE ERROR ESTIMATE DEPENDS ONLY ON σ.



$\tilde{R}$
R_v
R
v

Consider the Taylor polynomial of degree 1 around $(\overline{x}, \overline{y})$

$$p_{\overline{x},\overline{y}}(x, y) = u(\overline{x}, \overline{y}) + \frac{\partial u}{\partial x}(\overline{x}, \overline{y})(x - \overline{x}) + \frac{\partial u}{\partial y}(\overline{x}, \overline{y})(y - \overline{y})$$

For each node V we take an average of $p_{\overline{x},\overline{y}}(x, y)$ around V obtaining the polynomial $q(x, y)$:

$$q(x, y) = \frac{1}{|R_V|} \int_{R_V} p_{\overline{x},\overline{y}}(x, y) d\overline{x}\, d\overline{y}$$

And define the approximation u_I of u by

$$\boxed{u_I(V) = q(V)}$$

ERROR ESTIMATES

Analogous to those for the Lagrange interpolation but:

1- The error on one element depends also on the values of u in neighbor elements

2- Valid also in 3D.

$$\left\| \frac{\partial}{\partial x_j}(u - u_I) \right\|_{L^2(R)} \leq C \sum_{i=1}^{n} h_{R,i} \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L^2(\tilde{R})}$$

APPLICATIONS

A. Lombardi, R. Durán, Appl. Num. Math., to appear.

Consider the convection-diffusion problem

$$\begin{aligned} -\varepsilon\Delta u + b\cdot\nabla u + cu &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{1}$$

$$b_i < -\gamma \qquad \text{with} \qquad \gamma > 0 \qquad \text{for} \qquad i = 1,2. \tag{2}$$

It is known that the solution obtained by standard FE with uniform meshes present oscillations unless the mesh is too fine.

SOLUTIONS?

Several special techniques have been introduced: up-wind, stream-line diffusion, Petrov-Galerkin, etc.

But, is it possible to obtain good results with the standard method by using appropriate meshes?

We prove error estimates valid uniformly in ε if graded meshes are used.

What is the difficulty in this problem?

Recall the FE theory:

Given the problem:

$$B(u,v) = L(v) \qquad \forall v \in V$$

If

$$B(u,v) \leq M\|u\|\|v\|$$

and

$$\alpha\|u\|^2 \leq B(u,u)$$

then,

$$\|u - U_{FE}\| \leq \frac{M}{\alpha}\|u - V\| \qquad \forall V \in H_{FE}$$

The bilinear form is:

$$B(v,w) = \int_\Omega (\varepsilon \nabla v \cdot \nabla w + b \cdot \nabla v\, w + c\, vw)\; dx.$$

Consider the norm:

$$\|v\|_\varepsilon^2 = \|v\|_{L^2(\Omega)}^2 + \varepsilon \|\nabla v\|_{L^2(\Omega)}^2.$$

Assuming

$$c - \frac{\operatorname{div} b}{2} \geq \mu > 0$$

the bilinear form is coercive with a constant α independent of ε.

But:

1- The constant M in the continuity of the form depends on ε.

2- The second derivatives arising in the standard error estimates depends on ε.

Using a graded mesh we have proved that

$$\boxed{\|u - U_{FE}\|_\varepsilon \leq C \frac{(\log(1/\varepsilon))^2}{\sqrt{N}},}$$

where N is the number of nodes in the mesh. The order with respect to the number of nodes is optimal in the sense that it is the same than the order obtained for a problem with a smooth solution with uniform meshes.

NUMERICAL EXAMPLES

$$
\begin{aligned}
-\varepsilon\Delta u + b\cdot\nabla u + cu &= f && \text{in } \Omega\\
u &= u_D && \text{in } \Gamma_D\\
\frac{\partial u}{\partial n} &= g && \text{in } \Gamma_N,
\end{aligned}
$$

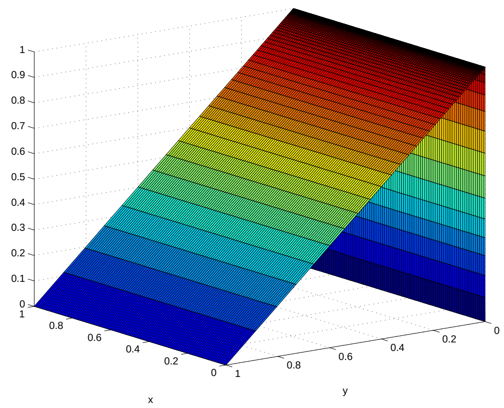
With different coefficients and data.

1. $b = (0, -1)$, $c = 0$, $\Gamma_D = [0,1] \times \{0,1\}$, $\Gamma_N = \{0,1\} \times [0,1]$, $u_D = 0$, $g = 0$ and $f = 1$,

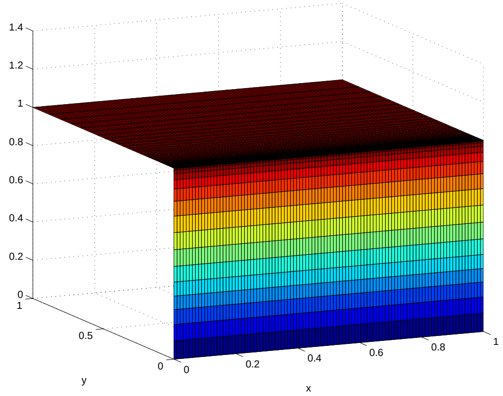
2. $b = (0, -1)$, $c = 0$, $\Gamma_D = [0,1] \times \{0,1\}$, $\Gamma_N = \{0,1\} \times [0,1]$, $u_D = 0$ on $\{0\} \times [0,1]$ and $u_D = 1$ on $\{1\} \times [0,1]$, $g = 0$ and $f = 0$,

3. $b = (-\frac{1}{2}, -1)$, $c = 2$, $\Gamma_D = \partial\Omega$, $u_D = 0$, and $f = 1$,

4. $b = (1-2\varepsilon)(-1,-1)$, $c = 2(1-\varepsilon)$, $\Gamma_D = \partial\Omega$, $u_D = 0$ and
$$f(x,y) = -\left[x - \left(\frac{1 - e^{-\frac{x}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}}\right) + y - \left(\frac{1 - e^{-\frac{y}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}}\right)\right] e^{x+y}.$$



(1)



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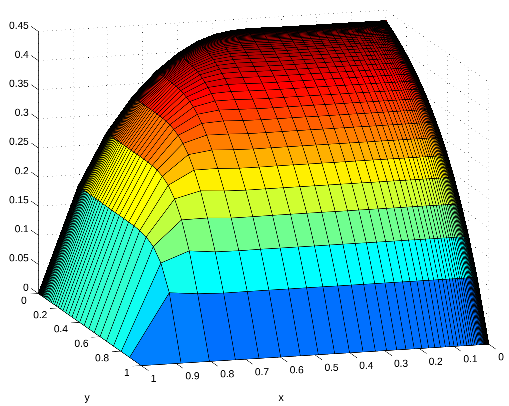

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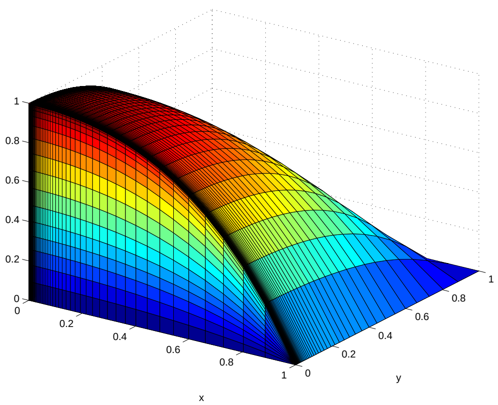

(4)

No oscillations are observed.

For the example 4 we know the exact solution

$$u(x,y) = \left[\left(x - \frac{1 - e^{-\frac{x}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}} \right) \left(y - \frac{1 - e^{-\frac{y}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}} \right) \right] e^{x+y},$$

and so we can compute the order of convergence.

N	Error
324	0.16855
961	0.097606
3249	0.052696
12100	0.025912
45796	0.013419

$\varepsilon = 10^{-4}$

N	Error
676	0.16494
2025	0.094645
6889	0.050256
25281	0.026023
96100	0.013427

$\varepsilon = 10^{-6}$

Different structure of our meshes and the well known Shishkin meshes.