Fitting Data with Shift Invariant Spaces

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Good Models for Signal Processing

Shift Invariant Spaces (SIS)

Def.: A SIS is a closed subspace $V \subset L^2(\mathbb{R}^d)$ such that for all $k \in \mathbb{Z}^d$

$$f \in V \text{ if and only if } f(\cdot - k) \in V$$

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$V(\phi_1, ..., \phi_n)$ will denote the L^2-clousure of

Span$\{\phi_i(x-k) : k \in \mathbb{Z}, i = 1, ..., n\}$

These SISs are called **finitely generated**.

Advantages of using SIS

- Suitable for sampling
- Easy to handle through its generators
- Provide good algorithms for processing
- Include wavelet subspaces

Examples

- Spline Spaces
- Finite Elements
- Spaces of Band Limited Functions
- Wavelet subspaces

Usually in signal processing applications the signals are assumed to be band-limited. That means that the signals belong to one of the Paley-Wiener spaces

$$PW_\Omega = \{f \in L^2 : supp(\hat{f}) \subset [-\Omega, \Omega]\}.$$

Frequencies outside of the Ω interval are assumed to correspond to noise. This assumption has many theoretical and practical advantages. For example a function $f \in PW_\Omega$ can be recovered from its samples $f(\frac{k}{2\Omega})$ as

$$f(x) = \sum_k f(\frac{k}{2\Omega}) s(x - \frac{k}{2\Omega}) \quad \text{with } s(x) = \frac{sin(2\pi\Omega x)}{\pi x}.$$

However, in many applications, this assumption is not very realistic.

We will then consider finitely generated SIS

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Let H be a separable Hilbert Space.

$\{v_j : j \in J\} \subset H$ is a **frame** of H if there exist constants $0 < A \leq B < +\infty$ such that for every $f \in H$

$$A||f||^2 \leq \sum_{j \in J} |< f, v_j >|^2 \leq B||f||^2$$

If $\{v_j\}_{j \in J}$ is a frame of H, then every $f \in H$ has a representation

$$f = \sum_j < f, v_j > \tilde{v}_j$$

where $\{\tilde{v}_j\}$ is a dual frame. If $A = B = 1$ the frame is called **tight** and

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Frames produce redundant decompositions that have shown to be good for many signal processing applications, in particular for denoising.

Theorem: Every finitely generated SIS V has a tight frame of translates of a finite number of functions.

That is there exist $\phi_1, ..., \phi_r \in V$ such that

$\{\phi_i(x-k) : k \in \mathbb{Z}^d, i = 1, ..., r\}$ is a tight frame of V. In particular each $f \in V$ can be decomposed as

$$f(x) = \sum_{i=1}^{r} \sum_{k} c_k^i \phi_i(x-k),$$

with $\{c_k^i\}_k \in l^2(\mathbb{Z}^d), i = 1, ..., r$

Problem I

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If yes, in which way?

Determining Sets

The unknown space model:

$V = V(\Phi)$ a shift invariant space of length n.

$\{\phi_i(x-k) : i = 1, \dots, n,\ k \in \mathbb{Z}^d\}$ Riesz Basis of V.

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Main goal:

Find necessary and sufficient conditions on finite subsets $\mathcal{F} \subset V(\Phi)$ such that any $g \in V$ can be recovered from $\mathcal{F}$.

$\mathcal{F}$ is a determining set for $V(\Phi)$ if and only if

$$V(\Phi) = V(f_1, f_2, \ldots, f_m).$$

Here $V(f_1, f_2, \ldots, f_m) = \text{closure}_{L_2} \left\{ \text{span}\{ f_i(x-k) : \; f_i \in \mathcal{F} \, , \; k \in \mathbb{Z} \} \right\}.$

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Proposition: If $\mathcal{F}$ is a determining set for V then card $(\mathcal{F}) \geq n$.

Solution to Problem I

For each subset $F_\ell \subset \mathcal{F}$ of size n, we define the set

$$A_\ell = \{\omega : \det G_{F_\ell}(\omega) \neq 0\}, \qquad 1 \leq \ell \leq L = \binom{m}{n},$$

where G_{F_ℓ} is the $n \times n$ Gramian matrix for the vector F_ℓ.
(i.e.. $G_{F_\ell}(\omega) = \sum_{k \in \mathbb{Z}^d} \widehat{F_\ell}(\omega + k) \widehat{F_\ell}^*(\omega + k)$)

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Theorem 1 *A set* $\mathcal{F} = \{f_1, \cdots, f_m\} \subset V(\Phi)$ *is a determining set for* $V(\Phi)$ *if and only if the set*

$$Z = \bigcap_{\ell=1}^{L} A_\ell^c, \qquad \textit{has Lebesgue measure zero}$$

Moreover, if $\mathcal{F}$ *is a determining set for* $V(\Phi)$*, then the vector function*

$$\widehat{\Psi}(\omega) := G_{F_1}^{-\frac{1}{2}}(\omega)\widehat{F_1}(\omega)\chi_{B_1}(\omega) + \cdots + G_{F_L}^{-\frac{1}{2}}(\omega)\widehat{F_L}(\omega)\chi_{B_L}(\omega) \qquad (1)$$

where $B_1 := A_1, B_\ell := A_\ell - \bigcup_{j=1}^{\ell-1} A_j,\ \ell = 2, \ldots, L$*,*

generates an orthonormal basis $\{\psi_i(x-k) : i = 1, \ldots, n,\ k \in \mathbb{Z}^d\}$ *of* $V(\Phi)$*.*

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- the class of functions from which the data is drawn may not be a shift-invariant space.
- the shift-invariant space hypothesis is correct but the assumptions about the number of generators is wrong.
- the a priori hypothesis is correct but the data is corrupted by noise.

For these three more realistic cases, we must consider the following problem:

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Given a large set of experimental data $\mathcal{F} = \{f_1, f_2, \ldots, f_m\}$, where f_i are observed functions (signals) ,

let $\mathcal{V} = \{V \subset L^2(R^d) : V = V(\Phi), \Phi$ a frame of length $n\}$

we want to find a space $V \in \mathcal{V}$ such that

$$\sum_{i=1}^{m} \|f_i - P_V f_i\|^2 \leq \sum_{i=1}^{m} \|f_i - P_W f_i\|^2$$

for all $W \in \mathcal{V}$.

That is, given $\mathcal{F}$, find a space $V \in \mathcal{V}$ such that V minimizes the least square error

$$E(\mathcal{F}, n) = \sum_{i=1}^{m} \|f_i - P_V f_i\|^2,$$

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- Can we construct the generators of the space?

Results

Theorem 2 *Let* $F = (f_i, \ldots, f_m)^T$ *be a vector of functions with components in* $L^2(\mathbb{R}^d)$*, and* $n \leq m$ *be given.*

1. *There exists* $V \in \mathcal{V}_n$ *such that*

$$\sum_{i=1}^{m} \|f_i - P_V f_i\|^2 \leq \sum_{i=1}^{m} \|f_i - P_W f_i\|^2, \quad \forall\ W \in \mathcal{V}_n \qquad (2)$$

2. *The optimal shift-invariant space* V *in (2) can be chosen such that* $V \subset \mathcal{V}(F)$.

Theorem 3 *Under the same assumptions as in the previous theorem, let* $\lambda_1(\omega) \geq \lambda_2(\omega) \geq \cdots \geq \lambda_m(\omega)$ *be the eigenvalues of* $G_F(\omega)$*, then*

1. *The eigenvalues* $\lambda_i(\omega)$*,* $1 \leq i \leq m$ *are measurable functions in* $L^2([0,1]^d)$ *and*

$$\sum_{i=1}^{m} \|f_i - P_V f_i\|^2 = \sum_{i=n+1}^{m} \int_{[0,1]^d} \lambda_i(\omega) d\omega \qquad (3)$$

2. Let $E_i := \{\omega \ : \ \lambda_i(\omega) \neq 0\}$.

Define $\tilde{\sigma}_i(\omega) = \lambda_i^{-1/2}(\omega)$ on E_i and $\tilde{\sigma}_i(\omega) = 0$ on E_i^c.

Then, there exists a choice of eigenvectors $v_1(\omega), \ldots, v_n(\omega)$ associated with the first n largest eigenvalues such that the functions defined by

$$\hat{\phi}_i(\omega) = \tilde{\sigma}_i(\omega) \sum_{j=1}^{m} v_{ij}(\omega) \hat{f}_j(\omega), \quad i = 1, \ldots, n, \ \omega \in \mathbb{R}^d$$

are measurable functions in $L^2(\mathbb{R}^d)$.

Furhermore, the corresponding vector function $\Phi = (\phi_1, \ldots, \phi_n)$ is a generator for V and the set $\{\phi_i(x-k), k \in \mathbb{Z}^d, i = 1, \ldots, n\}$ is a tight frame for V.