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HOLOMORPHIC MAPPINGS ON l_1

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ABSTRACT. We describe the holomorphic mappings of bounded type, and the arbitrary holomorphic mappings from the complex Banach space l_1 into a complex Banach space X. It is shown that these mappings have monomial expansions and the growth of the norms of the coefficients is characterized in each case. This characterization is used to give new descriptions of the compact open topology and the Nachbin ported topology on the space $\mathcal{H}(l_1; X)$ of holomorphic mappings, and to prove a lifting property for holomorphic mappings on l_1. We also show that the monomials form an equicontinuous unconditional Schauder basis for the space $(\mathcal{H}(l_1), \tau_0)$ of holomorphic functions on l_1 with the topology of uniform convergence on compact sets.

1. Introduction. The purpose of this article is to give a complete description of the holomorphic mappings from the Banach space l_1 over the field of complex numbers into an arbitrary complex Banach space X. This is achieved by showing that every such mapping has a monomial expansion of the form $\sum_{m \in \mathbf{N}^{(\mathbf{N})}} a_m z^m$, where $\mathbf{N}^{(\mathbf{N})}$ is the set of all multi-indices, $a_m \in X$, and z^m is the monomial $\prod_{k=1}^{\infty} z_k^{m_k}$. We show that the coefficients a_m which appear can be characterized by a set of conditions on the growth of their norms $\|a_m\|$, and we examine some of the consequences of this characterization. For example, we show that the well-known lifting property for bounded linear mappings on l_1 extends to holomorphic mappings; thus if π is a bounded linear mapping of X onto Y then for every holomorphic mapping $f: l_1 \to Y$ there exists a holomorphic mapping $\tilde{f}: l_1 \to X$ such that $\pi \circ \tilde{f} = f$. We also exploit the monomial expansion to give some new generating families of continuous seminorms for the natural locally convex topologies on the spaces of holomorphic mappings on l_1.

The paper is organized as follows: in §2 we outline our notation and definitions and recall some of the properties of holomorphic mappings on Banach spaces. §3 is concerned with holomorphic mappings from l_1 into a complex Banach space which are bounded on the bounded subsets of l_1. It is shown that every such function has a monomial expansion which is uniformly absolutely convergent on the bounded subsets of l_1, and the growth rate of the norms of the coefficients is characterized. We show that these mappings on l_1 have a lifting property of the type already described, and we give a description of the natural topology of uniform convergence on bounded set on the space of all such mappings. We also give some estimates for the norm of a continuous homogeneous polynomial on l_1 in terms of the coefficients of the monomial expansion.

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In §4 we study the vector space $\mathcal{H}(l_1;X)$ of all holomorphic mappings from l_1 into X. Our basic method is as follows: if K is an absolutely convex compact subset of l_1, we may form the Banach space $(l_1)_K$ generated by K, which has K as its unit ball. Now if $f \in \mathcal{H}(l_1;X)$ then the restriction of f to $(l_1)_K$ is not only holomorphic, but is bounded on every bounded set. We show that there is a fundamental system of absolutely convex compact subsets K of l_1 for which the Banach space $(l_1)_K$ is isometrically isomorphic to l_1 in a natural way. Thus the results of §3 can be applied to the corresponding family of restrictions of f. In this way we can again construct a monomial expansion, and characterize the growth of the norms of the coefficients. This characterization enables us to prove the lifting property described above, and yields new descriptions of the compact-open topology τ_0, and the Nachbin topology τ_ω, on $\mathcal{H}(l_1;X)$. It is shown that the set of monomials forms an equicontinuous unconditional Schauder basis for the space $\mathcal{H}(l_1)$ of scalar-valued holomorphic mappings on l_1 with the topology τ_0.

The use of monomial expansions in modern infinite-dimensional holomorphy was initiated by Boland and Dineen in their study of holomorphic functions on fully nuclear spaces with a basis [**2**, **3**, **5**, **6**]. Their work has inspired many of the ideas which are developed here.

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2. Notation and definitions. All spaces considered will be Banach spaces over the field of complex numbers. For each $n \in \mathbf{N}$, $P(^nX;Y)$ denotes the Banach space of continuous n-homogeneous polynomials from X into Y, where the norm is given by $\|P\| = \sup\{\|P(x)\|: x \in X, \|x\| \leq 1\}$. A mapping $f: X \to Y$ is holomorphic if there exists a sequence $P_n \in P(^nX;Y)$ such that the series $\sum_{n=0}^\infty P_n(x)$ converges to $f(x)$ for every $x \in X$. This is equivalent to f having a complex Fréchet derivative at every point of X. $\mathcal{H}(X;Y)$ denotes the vector space of holomorphic mappings from X into Y. $\mathcal{H}_b(X;Y)$ denotes the subspaces of $\mathcal{H}(X;Y)$ consisting of holomorphic mappings of bounded type, that is, holomorphic mappings which are bounded on every bounded subset of X. If X is infinite dimensional and $Y \neq \{0\}$, then $\mathcal{H}_b(X;Y)$ is always a proper subspace of $\mathcal{H}(X;Y)$. When $Y = \mathbf{C}$, the spaces which have been introduced here are denoted by $P(^nX)$, $\mathcal{H}(X)$, and $\mathcal{H}_b(X)$. We refer to [**4** and **7**] for further details. All of the properties of the space l_1 which we use can be found in [**8**].

We shall make extensive use of multi-indices. The set of multi-indices is $\mathbf{N}^{(\mathbf{N})} = \{m = (m_k)_{k=1}^\infty: m_k \in \mathbf{N},\ m_k = 0 \text{ for } k \text{ sufficiently large}\}$. For $m \in \mathbf{N}^{(\mathbf{N})}$, the degree on m is $|m| = \sum_{k=1}^\infty m_k$, and for each natural number n, we denote by $\mathbf{N}_n^{(\mathbf{N})}$ the set of multi-indices of degree n. We let $m! = \prod_{k=1}^\infty m_k!$, where the usual convention $0! = 1$ is observed. If $a = (a_n)$ is a sequence of complex numbers, then $a^m = \prod_{k=1}^\infty a_k^{m_k}$, where 0^0 is defined to be 1. For each $m \in \mathbf{N}^{(\mathbf{N})}$ the monomial z^m is the mapping $z \in l_1 \to z^m \in \mathbf{C}$. z^m is a continuous homogeneous polynomial on l_1 of degree $|m|$. We follow the usual abuse of notation by using the symbol z^m both for this monomial and for its value at a point z in l_1. We shall also make use of the multinomial theorem, which states that if n is a natural number and

$z = (z_k) \in l_1$, then

$$\left(\sum_{k=1}^{\infty} z_k\right)^n = \sum_{m \in \mathbf{N}_n^{(\mathbf{N})}} \frac{n!}{m!} z^m.$$

3. Holomorphic mappings of bounded type on l_1. We begin with a computation of the norm of the monomial z^m. Recall that $z^m \in P(^n l_1)$ where $n = |m|$, and that the norm of $P(^n l_1)$ is given by $\|P\| = \sup\{|P(z)| \colon z \in l_1, \|z\| \le 1\}$.

LEMMA 3.1. $\|z^m\| = m^m/|m|^{|m|}$ *for every* $m \in \mathbf{N}^{(\mathbf{N})}$.

PROOF. Let $m \in \mathbf{N}^{(\mathbf{N})}$. Choose $k \in \mathbf{N}$ so that $m_j = 0$ when $j > k$. Now using the inequality between the geometric and arithmetic means, we obtain

$$\begin{aligned} \frac{|z^m|}{m^m} &= \left(\frac{|z_1|}{m_1}\right)^{m_1} \cdots \left(\frac{|z_k|}{m_k}\right)^{m_k} \le \left(\frac{|z_1| + \cdots + |z_k|}{|m|}\right)^{|m|} \\ &\le \frac{\|z\|^{|m|}}{|m|^{|m|}} \quad \text{for every } z \in l_1. \end{aligned}$$

Therefore $\|z^m\| \le m^m/|m|^{|m|}$. On the other hand, for $z_0 = (m_1/|m|, \ldots, m_k/|m|, 0, \ldots)$ we have $\|z_0\| = 1$ and $z_0^m = m^m/|m|^{|m|}$. Therefore $\|z^m\| = m^m/|m|^{|m|}$. Q.E.D.

We shall see that the space $\mathcal{H}(l_1; X)$ can be viewed as a vector space of sequences in X. The description of these sequences will entail the use of the numbers $m^m/|m|^{|m|}$ and $m!/|m|!$ as weights. The following lemma will enable us to pass from on to the other.

LEMMA 3.2. $m^m/|m|^{|m|} \le m!/|m|! \le e^{|m|} m^m/|m|^{|m|}$ *for every* $m \in \mathbf{N}^{(\mathbf{N})}$.

PROOF. Let $m \in \mathbf{N}^{(\mathbf{N})}$. Choose $k \in \mathbf{N}$ so that $m_j = 0$ for $j > k$. Let $P(\lambda) = (\lambda_1 + \cdots + \lambda_k)^{|m|}$ for $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbf{C}^k$. The coefficient of the term $\lambda_1^{m_1} \cdots \lambda_k^{m_k}$ in the Taylor series of $P(\lambda)$ at the origin is $|m|!/m!$. Applying the Cauchy integral formula to $P(\lambda)$ on the polydisc of polyradius $(m_1, \ldots, m_k)$, we have

$$\frac{|m|!}{m!} \le \frac{1}{m_1^{m_1} \cdots m_k^{m_k}} \sup\{|P(\lambda)| \colon |\lambda_j| = m_j,\ 1 \le j \le k\} = \frac{|m|^{|m|}}{m^m}.$$

Therefore $m^m/|m|^{|m|} \le m!/|m|!$.

To establish the second inequality, we note first that $|m|^{|m|}/|m|! \le e^{|m|}$. Since $(m_j)! \le m_j^{m_j}$ for every j, it follows that $m! \le m^m$, and hence $m!/|m|! \le e^{|m|} m^m/|m|^{|m|}$. Q.E.D.

THEOREM 3.3. *Let X be a complex Banach space.*

(a) *Let $a_m \in X$ for each $m \in \mathbf{N}^{(\mathbf{N})}$. The following are equivalent:*

(i) *for every $R > 0$ there exists $C > 0$ such that $\|a_m\|(m^m/|m|^{|m|})R^{|m|} \le C$ for every $m \in \mathbf{N}^{(\mathbf{N})}$.*

(ii) $\lim_{|m|\to\infty}(\|a_m\| m^m/|m|^{|m|})^{1/|m|} = 0$.

(iii) *For every $R > 0$ there exists $D > 0$ such that $\|a_m\|(m!/|m|!)R^{|m|} \le D$ for every $m \in \mathbf{N}^{(\mathbf{N})}$.*

(iv) $\lim_{|m|\to\infty}(\|a_m\| m!/|m|!)^{1/|m|} = 0$.

(b) *Let $f\colon l_1 \to X$ be a holomorphic mapping of bounded type. There exists a unique family $\{a_m\colon m \in \mathbf{N}^{(\mathbf{N})}\} \subset X$ satisfying the equivalent conditions given in* (a) *such that the series $\sum_{m\in\mathbf{N}^{(\mathbf{N})}} a_m z^m$ converges absolutely to $f(z)$ for every $z \in l_1$, and the convergence is uniform on bounded subsets of l_1.*

(c) *If $\{a_m\colon m \in \mathbf{N}^{(\mathbf{N})}\} \subset X$ satisfies the equivalent conditions given in* (a) *then the series $\sum_{m\in\mathbf{N}^{(\mathbf{N})}} a_m z^m$ converges absolutely and uniformly on every bounded subset of l_1 and its sum defines a holomorphic mapping of bounded type from l_1 into X.*

PROOF. (a) It is obvious that (i) is equivalent to (ii) and that (iii) is equivalent to (iv). Lemma 3.2 shows that (i) is equivalent to (iii).

(b) Fix $R > 0$. Let $m \in \mathbf{N}^{(\mathbf{N})}$ and choose $k \in \mathbf{N}$ so that $m_j = 0$ for $j > k$. Let $\rho = (\rho_k)$ be a sequence of positive real numbers such that $\sum_{k=1}^{\infty} \rho_k \le R$. We define

$$a_m = \frac{1}{(2\pi i)^k} \int_{|\lambda_1|=\rho_1} \cdots \int_{|\lambda_k|=\rho_k} \frac{f(\lambda_1,\ldots,\lambda_k,0,\ldots)}{\lambda_1^{m_1+1}\cdots\lambda_k^{m_k+1}} d\lambda_1 \cdots d\lambda_k. \tag{1}$$

The Cauchy integral formula in several variables implies that a_m does not depend on the choice of $R, \rho_1, \ldots, \rho_k$. Since f is of bounded type we may define

$$C = \sup\{\|f(z)\|\colon \|z\| \le R\}.$$

By (1), $\|a_m\|\rho^m \le C$ for every $m \in \mathbf{N}^{(\mathbf{N})}$.

We can write $\rho^m = (R^{-1}\rho)^m R^{|m|}$, where $R^{-1}\rho$ is the sequence $(R^{-1}\rho_k)$. Therefore

$$\|a_m\|\sigma^m R^{|m|} \le C \quad \text{for every } m \in \mathbf{N}^{(\mathbf{N})},$$

where (σ_k) is an arbitrary sequence of positive real numbers satisfying $\sum_{k=1}^{\infty}\sigma_k \le 1$, and hence by Lemma 3.1

$$\|a_m\| \frac{m^m}{|m|^{|m|}} R^{|m|} \le C \quad \text{for every } m \in \mathbf{N}^{(\mathbf{N})}.$$

Since $R > 0$ is arbitrary, the family $\{a_m\colon m \in \mathbf{M}^{(\mathbf{N})}\}$ satisfies the condition described in (a).

To see that the series $\sum_{m\in\mathbf{N}^{(\mathbf{N})}} a_m z^m$ converges absolutely and uniformly on bounded sets, let $R > 0$, $\varepsilon > 0$, and let $z \in l_1$, $\|z\| \le R$. Then using condition (iii) of (a) and the multinomial theorem, we obtain

$$\begin{aligned}
\sum_{m\in\mathbf{N}^{(\mathbf{N})}} \|a_m z^m\| &= \sum_{m\in\mathbf{N}^{(\mathbf{N})}} \|a_m\|\,|z|^m \\
&\le \sup\left\{\|a_m\|\frac{m!}{|m|!}(R+\varepsilon)^{|m|}\colon m \in \mathbf{N}^{(\mathbf{N})}\right\} \sum_{m\in\mathbf{N}^{(\mathbf{N})}} (R+\varepsilon)^{-|m|}\frac{|m|!}{m!}|z|^m \\
&= \sup\left\{\|a_m\|\frac{m!}{|m|!}(R+\varepsilon)^{|m|}\colon m \in \mathbf{N}^{(\mathbf{N})}\right\} \sum_{n=0}^{\infty}(R+\varepsilon)^{-n} \sum_{m\in\mathbf{N}_n^{(\mathbf{N})}} \frac{n!}{m!}|z|^m \\
&= \sup\left\{\|a_m\|\frac{m!}{|m|!}(R+\varepsilon)^{|m|}\colon m \in \mathbf{N}^{(\mathbf{N})}\right\} \sum_{n=0}^{\infty}(R+\varepsilon)^{-n}\|z\|^n \\
&\le \sup\left\{\|a_m\|\frac{m!}{|m|!}(R+\varepsilon)^{|m|}\colon m \in \mathbf{N}^{(\mathbf{N})}\right\} \left(\frac{R+\varepsilon}{\varepsilon}\right).
\end{aligned}$$

It follows that $\sum_{m\in\mathbf{N}^{(\mathbf{N})}} \|a_m z^m\|$ converges uniformly on $\{z \in l_1 : \|z\| \le R\}$ for every $R > 0$. Therefore $g(z) = \sum_{m\in\mathbf{N}^{(\mathbf{N})}} a_m z^m$ defines a holomorphic mapping of bounded type from l_1 into X. But, by the definition of a_m, $g(z) = f(z)$ for $z \in \bigcup_{k=1}^{\infty} \pi_k(l_1)$, where $\pi_k : l_1 \to l_1$ is the projection onto the first k coordinates: $\pi(z) = (z_1, \dots, z_k, 0, \dots)$. Since $\bigcup_{k=1}^{\infty} \pi_k(l_1)$ is dense in l_1, $g(z) = f(z)$ for every $z \in l_1$. Finally, it is obvious that the coefficients a_m, $m \in \mathbf{N}^{(\mathbf{N})}$, are uniquely determined by f, since these coefficients must satisfy (1).

(c) Suppose that $\{a_m : m \in \mathbf{N}^{(\mathbf{N})}\} \subset X$ satisfies the conditions given in (a). Then, using condition (iii) of (a) and proceeding as in the latter part of the proof of (b), it is clear that the series $\sum_{m\in\mathbf{N}^{(\mathbf{N})}} a_m z^m$ converges absolutely and uniformly on bounded subsets of l_1 and that its sum is a holomorphic mapping of bounded type. Q.E.D.

We shall refer to the series $\sum_{m\in\mathbf{N}^{(\mathbf{N})}} a_m z^m$ as the monomial expansion of f. The relation between the monomial expansion of f and the Taylor series of f at the origin, $\sum_{n=0}^{\infty} P_n$, is clear: $P_n(z) = \sum_{m\in\mathbf{N}_n^{(\mathbf{N})}} a_m z^m$ for every $z \in l_1$, $n \in \mathbf{N}$. We shall be discussing only entire functions, and it will not therefore be necessary to consider monomial expansions about points other than the origin. It is easy to see that the coefficients a_m can be expressed in the usual manner in terms of partial derivatives of f at the origin: we have

$$a_m = \frac{1}{m!}\frac{\partial^{|m|} f}{\partial z^m}(0) \quad \text{for every } m \in \mathbf{N}^{(\mathbf{N})}.$$

Our first application of the monomial expansion is to show that the linear lifting property of l_1 extends to holomorphic mappings of bounded type:

COROLLARY 3.4. *Let π be a bounded linear mapping from X onto Y, where X and Y are complex Banach spaces. Then for every $f \in \mathcal{H}_b(l_1; Y)$ there exists $\tilde{f} \in \mathcal{H}_b(l_1; X)$ such that $\pi \circ \tilde{f} = f$.*

PROOF. Let $\sum_{m\in\mathbf{N}^{(\mathbf{N})}} a_m z^m$ be the monomial expansion of $f \in \mathcal{H}_b(l_1; Y)$, so that $\{a_m : m \in \mathbf{N}^{(\mathbf{N})}\} \subset Y$ satisfies the conditions given in Theorem 3.3(a). Since π is onto, it follows from the open mapping theorem that there exists $A > 0$ such that we may choose $b_m \in X$ for each $m \in \mathbf{N}^{(\mathbf{N})}$ for which $\pi(b_m) = a_m$ and $\|b_m\| \le A\|a_m\|$. The second condition shows that $\{b_m : m \in \mathbf{N}^{(\mathbf{N})}\}$ satisfies the conditions of Theorem 3.3(a) and so we may define $\tilde{f} \in \mathcal{H}_b(l_1; X)$ by $\tilde{f}(z) = \sum_{m\in\mathbf{N}^{(\mathbf{N})}} b_m z^m$. Since $\pi(b_m) = a_m$ for every $m \in \mathbf{N}^{(\mathbf{N})}$, we have $\pi \circ \tilde{f} = f$. Q.E.D.

The natural topology on the vector space $\mathcal{H}_b(X; Y)$, which we denote by τ_b, is the locally convex topology of uniform convergence on bounded subsets of X. This topology is generated by the family of seminorms M_R, $R > 0$, where

$$M_R(f) = \sup\{\|f(z)\| : \|z\| \le R\}.$$

$(\mathcal{H}_b(X; Y), \tau_b)$ is a Fréchet space; $\{M_k\}_{k=1}^{\infty}$ is a fundamental sequence of continuous seminorms.

Theorem 3.3 enables us to introduce two further families of seminorms on $\mathcal{H}_b(l_1; X)$. For each $R > 0$, we define

$$p_R(f) = \sup\left\{\|a_m\| \frac{m^m}{|m|^{|m|}} R^{|m|} : m \in \mathbf{N}^{(\mathbf{N})}\right\}$$

and

$$q_R(f) = \sup\left\{ \|a_m\| \frac{m!}{|m|!} R^{|m|} : m \in \mathbf{N}^{(\mathbf{N})} \right\},$$

where $f(z) = \sum_{m \in \mathbf{N}^{(\mathbf{N})}} a_m z^m$ is the monomial expansion of $f \in \mathcal{H}_b(l_1; X)$. It follows easily from Theorem 3.3 that p_R and q_R are seminorms on $\mathcal{H}_b(l_1; X)$.

PROPOSITION 3.5. *Let X be a complex Banach space. For every $f \in \mathcal{H}_b(l_1; X)$, $R > 0$, and $\varepsilon > 0$,*

$$p_R(f) \le M_R(f) \le \frac{R+\varepsilon}{\varepsilon} q_{R+\varepsilon}(f) \le \frac{R+\varepsilon}{\varepsilon} p_{e(R+\varepsilon)}(f).$$

Therefore the families of seminorms $\{p_R : R > 0\}$, $\{q_R : R > 0\}$, and $\{M_R : R > 0\}$ each generate the topology τ_b on $\mathcal{H}_b(l_1; X)$.

PROOF. The proof of Theorem 3.3(b) shows that

$$p_R(f) \le M_R(f) \le \frac{R+\varepsilon}{\varepsilon} q_{R+\varepsilon}(f).$$

By Lemma 3.2,

$$\begin{aligned} q_R(f) &= \sup\left\{ \|a_m\| \frac{m!}{|m|!} R^{|m|} : m \in \mathbf{N}^{(\mathbf{N})} \right\} \\ &\le \sup\left\{ \|a_m\| \frac{m^m}{|m|^{|m|}} (eR)^{|m|} : m \in \mathbf{N}^{(\mathbf{N})} \right\} = p_{eR}(f), \end{aligned}$$

which proves the last inequality. Q.E.D.

We now restrict our attention to the Banach spaces $P(^n l_1; X)$, where the norm is given by $\|P\| = \sup\{\|P(z)\| : \|z\| \le 1\}$. Theorem 3.3 shows that every $P \in P(^n l_1; X)$ has a monomial expansion of the form $P(z) = \sum_{m \in \mathbf{N}_n^{(\mathbf{N})}} a_m z^m$, and we can use this expansion to define two other norms on $P(^n l_1; X)$:

$$\|P\|' = \sup\left\{ \|a_m\| \frac{m^m}{n^n} : m \in \mathbf{N}_n^{(\mathbf{N})} \right\} \quad \text{and} \quad \|P\|'' = \sup\left\{ \|a_m\| \frac{m!}{n!} : m \in \mathbf{N}_n^{(\mathbf{N})} \right\}.$$

Proposition 3.5 shows that the three norms $\|\cdot\|$, $\|\cdot\|'$, and $\|\cdot\|''$ on $P(^n l_1; X)$ are equivalent. However, we can improve on the estimates given there:

PROPOSITION 3.6. *$\|P\|' \le \|P\| \le \|P\|'' \le e^n \|P\|'$ for every $P \in P(^n l_1; X)$.*

PROOF. The first and last inequalities are given in Proposition 3.5. To establish the second inequality, let $P(z) = \sum_{m \in \mathbf{N}_n^{(\mathbf{N})}} a_m z^m$. Then

$$\begin{aligned} \|P(z)\| &\le \sum_{m \in \mathbf{N}_n^{(\mathbf{N})}} \|a_m\| \, |z|^m \le \sup\left\{ \|a_m\| \frac{m!}{n!} : m \in \mathbf{N}_n^{(\mathbf{N})} \right\} \sum_{m \in \mathbf{N}_n^{(\mathbf{N})}} \frac{n!}{m!} |z|^m \\ &= \|P\|'' \|z\|^n \quad \text{for every } z \in l_1. \end{aligned}$$

Therefore $\|P\| \le \|P\|''$. Q.E.D.

We give an example to illustrate the inequalities given in Proposition 3.6. For each $a > 0$ let P_a be the continuous 2-homogeneous polynomial

$$P_a(z) = z_1^2 + z_2^2 + a z_1 z_2.$$

Elementary computations yield the following table of values for the norms $\|P_a\|'$, $\|P_a\|$, and $\|P_a\|''$ for α in each of the ranges $[0,2)$, $[2,4)$, and $[4,\infty)$:

	$0 \le \alpha < 2$	$2 \le \alpha < 4$	$4 \le \alpha$
$\|P_a\|'$	1	1	$\frac{a}{4}$
$\|P_a\|$	1	$\frac{2+a}{4}$	$\frac{2+a}{4}$
$\|P_a\|''$	1	$\frac{a}{2}$	$\frac{a}{2}$

The norms $\|\cdot\|'$ and $\|\cdot\|''$ are convenient to work with since they can be computed directly from the norms of the coefficients on the monomial expansion. However these norms do not in general coincide with the natural norm and thus we have no representation of the natural norm in terms on the coefficients in the monomial expansion. Such a representation would enable us to derive a Cauchy-Hadamard formula for the radius of uniform convergence of an arbitrary monomial expansion.

4. Arbitrary holomorphic mappings on l_1. We begin by describing a general procedure for reducing the study of arbitrary holomorphic mappings to the study of holomorphic mappings of bounded type. If K is an absolutely convex compact subset of the complex Banach space X, then X_K will denote the Banach space generated by K, where the norm is given by the Minkowski functional of K. Thus $X_K = \bigcup_{n=1}^{\infty} nK$, and the norm of X_K is

$$\|x\|_K = \inf\{\lambda > 0 \colon x \in \lambda K\}.$$

The closed unit ball of X_K is K. We denote the canonical inclusion mapping from X_K into X by J_K. The following proposition summarizes the properties of the spaces X_K which are of interest to us. Although these properties are well known, we give a proof for the sake of completeness.

PROPOSITION 4.1. *Let X be a complex Banach space, and let $\mathcal{K}$ be a fundamental system of absolutely convex compact subsets of X.*

(a) *The norm topology of X coincides with the topological inductive limit of the spaces X_K, $K \in \mathcal{K}$.*

(b) *Let Y be a complex Banach space. A mapping $f \colon X \to Y$ is holomorphic if and only if $f \circ J_K \in \mathcal{H}_b(X_K; Y)$ for every $K \in \mathcal{K}$.*

PROOF. (a) Let τ be the inductive limit topology. since $J_K \colon X_K \to X$ is continuous for every $K \in \mathcal{K}$, the identity mapping $I \colon (X,\tau) \to X$ is continuous. To see that $I \colon X \to (X,\tau)$ is continuous, let (x_n) be a sequence in X such that $\|x_n\| \to 0$. Choose a sequence (λ_n) of positive real numbers such that $\lambda_n \to 0$ and $\lambda_n^{-1}\|x_n\| \to 0$. Let L be the absolutely convex hull of the sequence $(\lambda_n^{-1}x_n)$. Then L is compact, $x_n \in X_L$ for every n, and $\|x_n\|_L = \lambda_n$. There exists $K \in \mathcal{K}$ such that $L \subset K$. The sequence (x_n) lies in X_K and $\|x_n\|_K \to 0$. Therefore $x_n \to 0$ in (X,τ), and it follows that $I \colon X \to (X,\tau)$ is continuous.

(b) Suppose that $f \colon X \to Y$ is holomorphic. Let $K \in \mathcal{K}$. $f \circ J_K$ is a holomorphic mapping from X_K into Y and since f is bounded on each of the compact sets nK, it follows that $f \circ J_K$ is bounded on every bounded subset of X_K.

Conversely, suppose $f \colon X \to Y$ is such that $f \circ J_K \in \mathcal{H}_b(X_K; Y)$ for every $K \in \mathcal{K}$. In particular, $f \circ J_K$ is continuous for every $K \in \mathcal{K}$, and hence by (a) f

is continuous. We complete the proof by showing that f is Gâteaux holomorphic. Let $a, b \in X$. The affine line $Z = \{a + \lambda b \colon \lambda \in \mathbf{C}\}$ lies in the space X_K, where K is the absolutely convex hull of $\{a, b\}$. It follows that the restriction of f to Z is a holomorphic function of λ. Therefore f is Gâteaux holomorphic. Q.E.D.

In general, the usefulness of Proposition 4.1 is limited by a lack of knowledge of the Banach spaces X_K. We shall see that in the case $X = l_1$ this problem can be overcome.

The compact subsets of l_1 have a simple characterization; $K \subset l_1$ is relatively compact if and only if $\lim_{n\to\infty} \sum_{k=n}^{\infty} |z_k| = 0$ uniformly in $z \in K$ [8, p. 297]. Let c_0^+ denote the set of all sequences $\xi = (\xi_n)$ of positive real numbers for which $\lim_{n\to\infty} \xi_n = 0$. For each $\xi \in c_0^+$, we define the subset K_ξ of l_1 as follows:

$$K_\xi = \left\{ z \in l_1 \colon \sum_{n=1}^{\infty} \xi_n^{-1} |z_n| \le 1 \right\}.$$

It is easy to see that K_ξ is closed and absolutely convex. K_ξ is also compact, since for every $z \in K_\xi$,

$$\sum_{k=n}^{\infty} |z_k| = \sum_{k=n}^{\infty} \xi_k \xi_k^{-1} |z_k| \le \sup_{k \ge n} |\xi_k|$$

and hence $\lim_{n\to\infty} \sum_{k=n}^{\infty} |z_k| = 0$ uniformly in $z \in K_\xi$. Thus K_ξ is an absolutely convex compact subset of l_1 for every $\xi \in c_0^+$. K_ξ can also be described as the closed absolutely convex hull of the sequence $(\xi_n e_n)$.

PROPOSITION 4.2. *$\{K_\xi \colon \xi \in c_0^+\}$ is a fundamental system of absolutely convex compact subsets of l_1.*

PROOF. Let K be a compact subset of l_1. We construct a sequence $\xi \in c_0^+$ such that $K \subset K_\xi$. Since $\lim_{n\to\infty} \sum_{k=n}^{\infty} |z_k| = 0$ uniformly in $z \in K$, there exists a strictly increasing sequence of indices $(N_k)_{k=1}^{\infty}$ such that $\sum_{j > N_k} |z_j| \le (k2^{k+1})^{-1}$ for every $k \in \mathbf{N}$ and every $z \in K$. In particular, we have $\sum_{j=N_k+1}^{N_{k+1}} |z_j| \le (k2^{k+1})^{-1}$ for every $k \in \mathbf{N}$ and every $z \in K$. Let $c = \sup\{\sum_{j=1}^{N_1} |z_j| \colon z \in K\}$ and define the sequence $\xi = (\xi_n)$ by setting $\xi_n = 2(1+c)$ for $1 \le n \le N_1$, and $\xi_n = k^{-1}$ for $N_k < n \le N_{k+1}$. Now $\xi \in c_0^+$, and if $z \in K$, then

$$\begin{aligned}
\sum_{n=1}^{\infty} \xi_n^{-1} |z_n| &= \sum_{n=1}^{N_1} \xi_n^{-1} |z_n| + \sum_{k=1}^{\infty} \sum_{n=N_k+1}^{N_{k+1}} \xi_n^{-1} |z_n| \\
&= \frac{1}{2(1+c)} \sum_{n=1}^{N_1} |z_n| + \sum_{k=1}^{\infty} k \sum_{n=N_k+1}^{N_{k+1}} |z_n| \\
&\le \frac{1}{2(1+c)} c + \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} < 1.
\end{aligned}$$

Therefore $z \in K_\xi$. Q.E.D.

Let $c(l_\infty, l_1)$ be the locally convex topology on l_∞ of uniform convergence on the compact subsets of l_1. Since the weakly compact subsets of l_1 are compact in the norm topology, $c(l_\infty, l_1)$ coincides with the Mackey topology $\tau(l_\infty, l_1)$. Now if $w = (w_n) \in l_\infty$ and $z = (z_n) \in K_\xi$ for some $\xi \in c_0^+$, then

$$|\langle w, z\rangle| \leq \sum_{n=1}^{\infty} |w_n|\,|z_n| \leq \sup_n |w_n|\xi_n \sum_{n=1}^{\infty} \xi_n^{-1}|z_n| \leq \sup_n |w_n|\xi_n.$$

Also, we have $|\langle w, \xi_n e_n\rangle| = |w_n|\xi_n$ for every n, and so $\sup\{|\langle w, z\rangle|: z \in K_\xi\} = \sup_n |w_n|\xi_n$. Hence we have proved the following corollary.

COROLLARY 4.3. *The topology $c(l_\infty, l_1) = \tau(l_\infty, l_1)$ on l_∞ is generated by the seminorms $p_\xi(w) = \sup_n |w_n|\xi_n$, where $\xi \in c_0^+$.*

We now examine the Banach spaces $(l_1)_{K_\xi}$ generated by the compact sets K_ξ, $\xi \in c_0^+$. If $\xi = (\xi_n)$ and $z = (z_n)$ are two sequences of complex numbers, we shall denote by ξz the sequence $(\xi_n z_n)$.

PROPOSITION 4.4. *For each $\xi \in c_0^+$ the mapping $z \to \xi z$ is an isometric isomorphism from l_1 onto $(l_1)_{K_\xi}$.*

PROOF. It follows from the definition of K_ξ that a sequence (w_n) lies in $(l_1)_{K_\xi}$ if and only if $\sum_{n=1}^\infty \xi_n^{-1}|w_n| < \infty$, and so ξz is an element of $(l_1)_{K_\xi}$ for every $z \in l_1$. It is easy to see that the mapping $z \in l_1 \to \xi z \in (l_1)_{K_\xi}$ is linear and onto. Finally,

$$\|\xi z\|_{K_\xi} = \inf\{\lambda > 0: \xi z \in \lambda K_\xi\} = \inf\left\{\lambda > 0: \sum_{n=1}^{\infty} \xi_n^{-1}|\xi_n z_n| \leq \lambda\right\}$$

$$= \inf\left\{\lambda > 0: \sum_{n=1}^{\infty} |z_n| \leq \lambda\right\} = \sum_{n=1}^{\infty} |z_n| = \|z\|. \quad \text{Q.E.D.}$$

Now applying Proposition 4.1 to the fundamental system $\{K_\xi: \xi \in c_0^+\}$ of absolutely convex compact subsets of l_1, we have

PROPOSITION 4.5. *Let X be a complex Banach space. A mapping $f: l_1 \to X$ is holomorphic if and only if the mapping $f_\xi: l_1 \to X$, defined by $f_\xi(z) = f(\xi z)$, is a holomorphic mapping of bounded type for every $\xi \in c_0^+$.*

This proposition is also valid for mappings on l_p, where $1 < p < \infty$; methods similar to those we have used show that a fundamental system of absolutely convex compact subsets is given by $K_\xi^p = \{z \in l_p: \sum_{n=1}^\infty \xi_n^{-1}|z_n|^p \leq 1\}$, $\xi \in c_0^+$. However, the space $\mathcal{H}_b(l_p: X)$ does not have as simple a structure as $\mathcal{H}_b(l_1; X)$, and there is no analogue for the case $p > 1$ of our next result.

THEOREM 4.6. *Let X be a complex Banach space.*

(a) *Let $a_m \in X$ for each $m \in \mathbf{N}^{(\mathbf{N})}$. The following are equivalent:*

(i) *For every $\xi \in c_0^+$ there exists $C > 0$ such that $\|a_m\|(m^m/|m|^{|m|})\xi^m \leq C$ for every $m \in \mathbf{N}^{(\mathbf{N})}$.*

(ii) *For every $\xi \in c_0^+$ there exists $D > 0$ such that $\|a_m\|(m!/|m|!)\xi^m \leq D$ for every $m \in \mathbf{N}^{(\mathbf{N})}$.*

(b) *Let $f: l_1 \to X$ be a holomorphic mapping. There exists a unique family $\{a_m: m \in \mathbf{N}^{(\mathbf{N})}\} \subset X$ satisfying the equivalent conditions given in* (a) *such that the series $\sum_{m\in\mathbf{N}^{(\mathbf{N})}} a_m z^m$ converges absolutely to $f(z)$ for every $z \in l_1$, and the convergence is uniform on compact subsets of l_1.*

(c) *If* $\{a_m : m \in \mathbf{N}^{(\mathbf{N})}\} \subset X$ *satisfies the equivalent conditions given in* (a), *then the series* $\sum_{m\in\mathbf{N}^{(\mathbf{N})}} a_m z^m$ *converges absolutely and uniformly on every compact subset of* l_1 *and its sum defined a holomorphic mapping from* l_1 *into* X.

PROOF. (a) That (i) and (ii) are equivalent follows immediately from Lemma 3.2.

(b) If $f: l_1 \to X$ is holomorphic, then $f_\xi \in \mathcal{H}_b(l_1; X)$ for every $\xi \in c_0^+$ and so there exists a family $\{a_m(\xi): m \in \mathbf{N}^{(\mathbf{N})}\} \subset X$ for each $\xi \in c_0^+$ satisfying the conditions of Theorem 3.3(a) such that $f_\xi(w) = \sum_{m\in\mathbf{N}^{(\mathbf{N})}} a_m(\xi) w^m$ for every $w \in l_1$. Let $\{a_m : m \in \mathbf{N}^{(\mathbf{N})}\}$ be defined in exactly the same way as in the proof of Theorem 3.3(b). A change of variables in the integrals which are used to define $a_m(\xi)$ shows that $a_m(\xi) = a_m \xi^m$ for every $m \in \mathbf{N}^{(\mathbf{N})}$ and every $\xi \in c_0^+$. Applying Theorem 3.3, with $R = 1$, it follows that $\{a_m : m \in \mathbf{N}^{(\mathbf{N})}\}$ satisfies the conditions given in (a) above. Now for each $z \in l_1$ we may choose $\xi \in c_0^+$ and $w \in l_1$ such that $z = \xi w$. Since $a_m z^m = a_m \xi^m w^m = a_m(\xi) w^m$, it follows that the series $\sum_{m\in\mathbf{N}^{(\mathbf{N})}} a_m z^m$ converges absolutely to $f_\xi(w) = f(z)$. Furthermore, as z ranges over the compact set K_ξ, w ranges over the unit ball of l_1, and it follows that the convergence is uniform on compact subset of l_1. The uniqueness of the coefficients a_m follows as in the proof of Theorem 3.3.

(c) Let $\{a_m : m \in \mathbf{N}^{(\mathbf{N})}\} \subset X$ satisfy the conditions given in (a). Let $\xi \in c_0^+$ and let $R > 0$. Then $R\xi = (R\xi_n) \in c_0^+$, and $(R\xi)^m = R^{|m|}\xi^m$. Therefore $\{a_m \xi^m : m \in \mathbf{N}^{(\mathbf{N})}\}$ satisfies the conditions given in Theorem 3.3(a) for every $\xi \in c_0^+$. Hence we may define $f_\xi \in \mathcal{H}_b(l_1; X)$ for each $\xi \in c_0^+$ by $f_\xi(w) = \sum_{m\in\mathbf{N}^{(\mathbf{N})}} a_m \xi^m w^m$. Now let $f: l_1 \to X$ be defined as follows: for each $z \in l_1$ choose $\xi \in c_0^+$ and $w \in l_1$ such that $z = \xi w$, and let $f(z) = f_\xi(w)$. It is easy to see that this definition is independent of the choice of ξ and w, and that f is a holomorphic mapping with the required properties. Q.E.D.

We shall refer to the series $\sum_{m\in\mathbf{N}^{(\mathbf{N})}} a_m z^m$ as the monomial expansion of f. The monomial expansion and the Taylor series at the origin are related in the same way as for mappings of bounded type.

Our first application of this theorem is to a holomorphic lifting property of the space l_1. The proof is the same as that of Corollary 3.4.

COROLLARY 4.7. *Let* π *be a bounded linear mapping from* X *to* Y, *where* X *and* Y *are complex Banach spaces. Then for every* $f \in \mathcal{H}(l_1; Y)$ *there exists* $\tilde{f} \in \mathcal{H}(l_1; X)$ *such that* $\pi \circ \tilde{f} = f$.

We now consider the natural locally convex topologies which may be placed on $\mathcal{H}(l_1; X)$. We begin with the locally convex topology τ_0 of uniform convergence on compact subsets. This topology is generated by the seminorms

$$f \to \|f\|_K = \sup\{\|f(z)\| : z \in K\},$$

where K ranges over the compact subsets of l_1. By Proposition 4.2, it is sufficient to take compact sets of the form $K = K_\xi$, where $\xi \in c_0^+$.

Theorem 4.6 enables us to introduce two further families of seminorms on $\mathcal{H}(l_1; X)$. For each $\xi \in c_0^+$, we define

$$p_\xi(f) = \sup\left\{\|a_m\| \frac{m^m}{|m|^{|m|}} \xi^m : m \in \mathbf{N}^{(\mathbf{N})}\right\}$$

and

$$q_\xi(f) = \sup\left\{\|a_m\|\frac{m!}{|m|!}\xi^m : m \in \mathbf{N}^{(\mathbf{N})}\right\},$$

where $f(z) = \sum_{m\in\mathbf{N}^{(\mathbf{N})}} a_m z^m$. These seminorms are related to the seminorms p_R and q_R which we have defined on $\mathcal{H}_b(l_1; X)$ in the following way: $p_\xi(f) = p_1(f_\xi)$ and $q_\xi(f) = q_1(f_\xi)$ for every $\xi \in c_0^+$ and every $f \in \mathcal{H}(l_1; X)$. Applying Proposition 3.5 with $R = 1$ we obtain

PROPOSITION 4.8. *Let X be a complex Banach space. For every $f \in \mathcal{H}(l_1; X)$, $\xi \in c_0^+$, and $\varepsilon > 0$,*

$$p_\xi(f) \le \|f\|_{K_\xi} \le \frac{1+\varepsilon}{\varepsilon} q_{(1+\varepsilon)\xi}(f) \le \frac{1+\varepsilon}{\varepsilon} p_{e(1+\varepsilon)\xi}(f).$$

Therefore the families of seminorms $\{p_\xi : \xi \in c_0^+\}$, $\{q_\xi : \xi \in c_0^+\}$, and $\{\|\cdot\|_{K_\xi} : \xi \in c_0^+\}$ each generate the topology τ_0 on $\mathcal{H}(l_1; X)$.

In particular, the families of seminorms $\{p_\xi : \xi \in c_0^+\}$, $\{q_\xi : \xi \in c_0^+\}$, and $\{\|\cdot\|_{K_\xi} : \xi \in c_0^+\}$ each generate the topology τ_0 on $P({}^n l_1; X)$ for every $n \in \mathbf{N}$. From Proposition 3.6 we can improve on the estimates given in Proposition 4.8 when we restrict our attention to $P({}^n l_1; X)$:

PROPOSITION 4.9. *Let X be a complex Banach space, $n \in \mathbf{N}$, and $\xi \in c_0^+$. Then $p_\xi(P) \le \|P\|_{K_\xi} \le q_\xi(P) \le e^n p_\xi(P)$ for every $P \in P({}^n l_1; X)$.*

The topology τ_0 is in many ways an unsatisfactory topology. For example, if X is an infinite-dimensional complex Banach space, then $(\mathcal{H}(X), \tau_0)$ is neither barrelled nor bornological [**4**, p. 253]. Our next result will demonstrate that τ_0 does have some good properties. First we establish some terminology. A sequence (u_n) is a locally convex space E is a Schauder basis for E if every element x of E can be expressed as the sum of a convergent series of the form $\sum_{n=1}^\infty x_n u_n$ for some uniquely determined sequence of complex numbers (x_n). The Schauder basis (u_n) is unconditional if the series $\sum_{n=1}^\infty x_n u_n$ converges unconditionally for every $x \in E$, and an unconditional Schauder basis (u_n) is equicontinuous if the linear mappings $x \to \sum_{n\in F} x_n u_n$ are equicontinuous, where F ranges over the family of finite subset on $\mathbf{N}$.

THEOREM 4.10. *$\{z^m : m \in \mathbf{N}^{(\mathbf{N})}\}$ is an equicontinuous unconditional Schauder basis for $(\mathcal{H}(l_1), \tau_0)$.*

PROOF. Let $f \in \mathcal{H}(l_1)$, $f \ne 0$, and let $\sum_{m\in\mathbf{N}^{(\mathbf{N})}} a_m z^m$ be the monomial expansion of f. We claim that this series converges unconditionally to f in the topology τ_0. To establish this, let $\xi \in c_0^+$ and $\varepsilon > 0$. Choose $\nu, \rho \in c_0^+$, such that $\xi_n = \nu_n \rho_n$ for every n, and $\|\rho\|_\infty = \sup_n |\rho_n| < 1$. We may assume without loss of generality that $\varepsilon < p_\nu(f)$. Since $\rho \in c_0^+$, there exists $k_0 \in \mathbf{N}$ such that

$$\rho_k < \varepsilon / p_\nu(f) \quad \text{for } k > k_0.$$

Since $\|\rho\|_\infty < 1$, there exists $n_0 \in \mathbf{N}$ such that

$$\|\rho\|_\infty^n < \varepsilon / p_\nu(f) \quad \text{for } n > n_0.$$

Let $A = \{m \in \mathbf{N}^{(\mathbf{N})} \colon |m| \le r_0 \text{ and } m_j = 0 \text{ for } j > k_0\}$. A is a finite subset of $\mathbf{N}^{(\mathbf{N})}$. If $m \in \mathbf{N}^{(\mathbf{N})} - A$, then two possibilities arise. The first is that there exists $j_0 > k_0$ such that $m_{j_0} \ge 1$. In this case we have

$$\rho^m = \rho_{j_0}^{m_{j_0}} \prod_{j \ne j_0} \rho_j^{m_j} \le \rho_{j_0}^{m_{j_0}} < \left(\frac{\varepsilon}{p_\nu(f)}\right)^{m_{j_0}} \le \frac{\varepsilon}{p_\nu(f)}.$$

The second possibility is that $|m| > n_0$, and in this case,

$$\rho^m = \prod_{j=1}^{\infty} \rho_j^{m_j} \le \prod_{j=1}^{\infty} \|\rho\|_\infty^{m_j} = \|\rho\|_\infty^{|m|} < \frac{\varepsilon}{p_\nu(f)}.$$

Now let B be any finite subset of $\mathbf{N}^{(\mathbf{N})}$ for which $B \cap A = 0$. Then

$$\begin{aligned} p_\xi\left(\sum_{m \in B} a_m z^m\right) &= \sup_{m \in B} \|a_m\| \frac{m^m}{|m|^{|m|}} \xi^m = \sup_{m \in B} \|a_m\| \frac{m^m}{|m|^{|m|}} \nu^m \rho^m \\ &\le \left(\sup_{m \in \mathbf{N}^{(\mathbf{N})}} \|a_m\| \frac{m^m}{|m|^{|m|}} \nu^m\right)\left(\sup_{m \in B} \rho^m\right) = p_\nu(f) \sup_{m \in B} \rho^m. \end{aligned}$$

We have seen that $\rho^m < \varepsilon / p_\nu(f)$ for every $m \in \mathbf{N}^{(\mathbf{N})} - A$, and hence

$$p_\xi\left(\sum_{m \in B} a_m z^m\right) < \varepsilon.$$

This shows that $\sum_{m \in \mathbf{N}^{(\mathbf{N})}} a_m z^m$ converges unconditionally to f for the topology τ_0. We have already seen that the coefficients a_m are uniquely determined by f, and so $\{z^m \colon m \in \mathbf{N}^{(\mathbf{N})}\}$ is an unconditional Schauder basis for $(\mathcal{H}(l_1), \tau_0)$. Furthermore, for every finite subset F of $\mathbf{N}^{(\mathbf{N})}$, and every $\xi \in c_0^+$ we have

$$p_\xi\left(\sum_{m \in F} a_m z^m\right) \le p_\xi(f)$$

for $f = \sum_{m \in \mathbf{N}^{(\mathbf{N})}} a_m z^m \in \mathcal{H}(l_1)$. Therefore the basis is also equicontinuous. Q.E.D.

We now consider the Nachbin topology, τ_ω, on $\mathcal{H}(l_1; X)$. We begin with the following description: τ_ω is the locally convex topology generated by the seminorms $f \to \sum_{n=0}^{\infty} \|P_n\|_{K + \alpha_n B}$, where $\sum_{n=0}^{\infty} P_n$ is the Taylor series of f at the origin, K is an absolutely convex compact subset of l_1, (α_n) is a sequence of positive real numbers converging to zero, and B is the closed unit ball of l_1 [**7**, p. 196]. It is easy to see that this family of seminorms is equivalent to the family of seminorms of the form $f \to \sup_n \|P_n\|_{K + \alpha_n B}$ with K and (α_n) as above. Furthermore, we may restrict the compact sets K to be members of the fundamental system $\{K_\xi, \xi \in c_0^+\}$. Thus τ_ω is generated by the family of seminorms $f \to \sup_n \|P_n\|_{K_\xi + \alpha_n B}$, where ξ and $\alpha = (\alpha_n)$ range over c_0^+.

It is not easy to estimate the norm of a polynomial over a set of the form $K_\xi + \alpha_n B$, and so we propose to replace these sets by other neighborhoods of K_ξ which have a simpler geometric structure. For $\xi \in c_0^+$ and β a positive real number, let

$$D_{\xi, \beta} = \left\{ z \in l_1 \colon \sum_{n=1}^{\infty} (\beta + \xi_n)^{-1} |z_n| \le 1 \right\}.$$

Since $\sum_{n=1}^{\infty}(\beta+\xi_n)^{-1}|z_n|$ defines a norm on l_1 which is equivalent to the original norm, $D_{\xi,\beta}$ is a closed absolutely convex set whose interior is the set $\{z \in l_1\colon \sum_{n=1}^{\infty}(\beta+\xi_n)^{-1}|z_n| < 1\}$, and since $K_\xi = \{z \in l_1\colon \sum_{n=1}^{\infty}\xi_n^{-1}|z_n| \leq 1\}$ it follows that $D_{\xi,\beta}$ is a neighborhood of the compact set K_ξ for every $\beta > 0$.

LEMMA 4.11. (a) *Let $\xi \in c_0^+$. Then*

(i) *$D_{\xi,\beta} \subset K_\xi + \beta B$ for every $\beta > 0$;*

(ii) *for every $\alpha \in c_0^+$ there exists $\beta \in c_0^+$ such that $K_\xi + \alpha_n B \subset D_{\xi,\beta_n}$ for every n.*

(b) *Let X be a complex Banach space. The topology τ_ω on $\mathcal{H}(l_1;X)$ is generated by the seminorms $f \to \sup_n \|P_n\|_{D_{\xi,\beta_n}}$, where ξ and $\beta = (\beta_n)$ are arbitrary elements of c_0^+, and $\sum_{n=0}^{\infty} P_n$ is the Taylor series of $f \in \mathcal{H}(l_1;X)$ at the origin.*

PROOF. (a) If $z \in D_{\xi,\beta}$, then we may write $z = ((\beta+\xi_n)\lambda_n)$, where $\sum_{n=1}^{\infty}|\lambda_n| \leq 1$. Let $w = (\xi_n\lambda_n)$. Then $w \in K_\xi$ and $\|z-w\| \leq \beta$. Therefore $z \in K_\xi + \beta B$, and (i) is established.

To prove (ii), let $\alpha \in c_0^+$. Let (ε_n) be a sequence of real numbers such that $0 < \varepsilon_n < 1$ for every n, and $\lim_{n\to\infty}\alpha_n/\varepsilon_n = 0$. Let

$$\beta_n = \max\left\{\frac{\varepsilon_n}{1-\varepsilon_n}\|\xi\|_\infty, \frac{\alpha_n}{\varepsilon_n}\right\}.$$

Then $\beta_n > 0$ for every n and $\lim_{n\to\infty}\beta_n = 0$. We claim that $K_\xi + \alpha_n B \subset D_{\xi,\beta_n}$ for every n.

To prove this claim, fix n and let $z \in K_\xi + \alpha_n\beta$. Then there exist elements λ and μ of the closed unit ball of l_1 such that $z_k = \xi_k\lambda_k + \alpha_n\mu_k$ for every k. Now

$$\beta_n \geq \frac{\varepsilon_n}{1-\varepsilon_n}\|\xi\|_\infty \geq \frac{\varepsilon_n}{1-\varepsilon_n}\xi_k \quad \text{for every } k,$$

and it follows that $\xi_k/(\beta_n+\xi_k) \leq 1-\varepsilon_n$ for every k. Also, $\alpha_n/(\beta_n+\xi_k) \leq \alpha_n/\beta_n \leq \varepsilon_n$. Therefore

$$\begin{aligned}\sum_{k=1}^{\infty}(\beta_n+\xi_k)^{-1}|z_k| &\leq \sum_{k=1}^{\infty}\left(\frac{\xi_k}{\beta_n+\xi_k}\right)|\lambda_k| + \sum_{k=1}^{\infty}\left(\frac{\alpha_n}{\beta_n+\xi_k}\right)|\mu_k| \\ &\leq (1-\varepsilon_n)\sum_{k=1}^{\infty}|\lambda_k| + \varepsilon_n\sum_{k=1}^{\infty}|\mu_k| \leq (1-\varepsilon_n)+\varepsilon_n = 1,\end{aligned}$$

and hence $z \in D_{\xi,\beta_n}$.

(b) is an immediate consequence of (a). Q.E.D.

We are now in a position to use the monomial expansion to obtain two generating families of seminorms for $(\mathcal{H}(l_1;X),\tau_\omega)$. If $\xi = (\xi_n)$ is a sequence of complex numbers, and β is a complex number, then $\xi+\beta$ will denote the sequence $(\xi_n+\beta)$.

THEOREM 4.12. *Let X be a complex Banach space. For each $\xi, \beta \in c_0^+$ let*

$$p_{\xi,\beta}(f) = \sup\left\{\|a_m\|\frac{m^m}{|m|^{|m|}}(\xi+\beta_{|m|})^m \colon m \in \mathbf{N}^{(\mathbf{N})}\right\}$$

and

$$q_{\xi,\beta}(f) = \sup\left\{\|a_m\|\frac{m!}{|m|!}(\xi+\beta_{|m|})^m \colon m \in \mathbf{N}^{(\mathbf{N})}\right\},$$

where $f = \sum_{m\in\mathbf{N}^{(\mathbf{N})}} a_m z^m \in \mathcal{H}(l_1;X)$. *Then* $p_{\xi,\beta}$ *and* $q_{\xi,\beta}$ *are continuous seminorms on* $(\mathcal{H}(l_1;X),\tau_\omega)$ *and the families of seminorms* $\{p_{\xi,\beta}: \xi,\beta \in c_0^+\}$ *and* $\{q_{\xi,\beta}: \xi,\beta \in c_0^+\}$ *each generate the topology* τ_ω.

PROOF. Let $f = \sum_{n=0}^{\infty} P_n = \sum_{m\in\mathbf{N}^{(\mathbf{N})}} a_m z^m \in \mathcal{H}(l_1;X)$, where $P_n = \sum_{m\in\mathbf{N}_n^{(\mathbf{N})}} a_m z^m$ for every n. Let $\xi,\beta \in c_0^+$, and fix $n \in \mathbf{N}$. Now $z \in D_{\xi,\beta_n}$ if and only if there exists $\lambda \in l_1$, $\|\lambda\| \le 1$, such that $z_k = (\xi_k + \beta_n)\lambda_k$ for every k. Therefore

$$\begin{aligned}\|P_n\|_{D_{\xi,\beta_n}} &= \sup\{\|P_n((\xi+\beta_n)\lambda)\|: \lambda\in l_1, \|\lambda\|\le 1\}\\ &= \sup\left\{\left\|\sum_{m\in\mathbf{N}_n^{(\mathbf{N})}} a_m(\xi+\beta_n)^m\lambda^m\right\| : \lambda\in l_1, \|\lambda\|\le 1\right\}.\end{aligned}$$

Applying Proposition 3.6 to the n-homogeneous polynomial

$$\lambda \to \sum_{m\in\mathbf{N}_n^{(\mathbf{N})}} a_m(\xi+\beta_n)^m\lambda^m$$

we have

$$\begin{aligned}&\sup\left\{\|a_m\|\frac{m^m}{n^n}(\xi+\beta_n)^m : m\in\mathbf{N}_n^{(\mathbf{N})}\right\}\\ &\quad\le \|P_n\|_{D_{\xi,\beta_n}} \le \sup\left\{\|a_m\|\frac{m!}{n!}(\xi+\beta_n)^m : m\in\mathbf{N}_n^{(\mathbf{N})}\right\}\\ &\quad\le e^n\sup\left\{\|a_m\|\frac{m^m}{n^n}(\xi+\beta_n)^m : m\in\mathbf{N}_n^{(\mathbf{N})}\right\}\\ &\quad= \sup\left\{\|a_m\|\frac{m^m}{n^n}(e\xi+e\beta_n)^m : m\in\mathbf{N}_n^{(\mathbf{N})}\right\}\end{aligned}$$

Since these inequalities hold for every $n \in \mathbf{N}$, it follows that

$$p_{\xi,\beta}(f) \le \sup_n \|P_n\|_{D_{\xi,\beta_n}} \le q_{\xi,\beta}(f) \le p_{e\xi,e\beta}(f)$$

for every $f \in \mathcal{H}(l_1;X)$. The assertions of the theorem follow immediately from these inequalities. Q.E.D.

5. Further developments. For the sake of simplicity we have worked with the space l_1 rather than the more general space $l_1(I)$, where I is an arbitrary indexing set. The results we have presented for l_1 are also valid for $l_1(I)$ provided some obvious minor modifications are made.

Every Banach space X is a quotient of $l_1(I)$ for a suitable choice of I. Composition with the quotient mapping yields an injective linear mapping from $\mathcal{H}(X)$ into $\mathcal{H}(l_1(I))$. For the topologies τ_0 and τ_ω, this inclusion is even topological. Thus $\mathcal{H}(l_1(I))$ can be viewed as a universal space for spaces of holomorphic functions. We refer to [**1**] for further details.

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