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Weakly Continuous Mappings on Banach Spaces

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It is shown that every n-homogeneous continuous polynomial on a Banach space E which is weakly continuous on the unit ball of E is weakly uniformly continuous on the unit ball of E. Applications of the result to spaces of polynomials and holomorphic mappings on E are given.

INTRODUCTION

Let E and F be Banach spaces and let $C(E;F)$ denote the space of continuous functions from E to F. For any locally convex topology τ on E and any subspace $\mathscr{F}(E;F)$ of $C(E;F)$, let $\mathscr{F}_\tau(E;F)$ (resp. $\mathscr{F}_{\tau u}(E;F)$) denote the subspace of $\mathscr{F}(E;F)$ consisting of those functions which, when restricted to any bounded subset B of E, are τ-continuous (resp. uniformly τ-continuous) on B. Many interesting, natural spaces of functions arise in this manner, as either an $\mathscr{F}_\tau(E;F)$ or an $\mathscr{F}_{\tau u}(E;F)$. For example, if $\mathscr{F}(E;F)=H(E;F)$ (the space of complex analytic functions between the complex Banach spaces E and F) and β denotes the norm topology on E, then $H_\beta(E;F)=H(E;F)$ and $H_{\beta u}(E;F)=H_b(E;F)$ (the space of entire functions which are bounded on bounded subsets of E), while if $w=\sigma(E,E')$, $H_{wu}(E;C)$ coincides with the space of entire functions $f:E\to C$ such that $df:E\to E'$ is a locally compact mapping (see, for example, [1]). Also, in recent years, there has been attention focused on

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characterizing and describing topological properties of $C_w(E;F)$, $C_w^n(E;F)$, and $C_{wu}^n(E;F)$, in the case of real Banach spaces (see, for example, [3, 6, 7]).

In this note, our primary interest will be in the subspace $\mathscr{P}_w(^nE;F)$ of the space $\mathscr{P}(^nE;F)$ of continuous n-homogeneous polynomials from E to F. Thus, for example, we show in Theorem 2.9 that $\mathscr{P}_w(^nE;F)=\mathscr{P}_{wu}(^nE;F)$. This result, which answers a question posed in [3], can be used to prove that if E' has the approximation property, then each polynomial in $\mathscr{P}_w(^nE;F)$ is a uniform limit (on the ball of E) of n-homogeneous polynomials of finite rank. In Proposition 2.12, it is shown that if E does not contain a copy of l_1, then every weakly sequentially continuous polynomial P is weakly continuous on bounded sets, that is $P\in\mathscr{P}_w(^nE;F)$. This yields, as an immediate consequence, a new proof of the fact that every scalar valued continuous n-homogeneous polynomial on c_0 is a uniform limit (on the ball of c_0) of finite rank n-homogeneous polynomials. Perhaps the most interesting special cases of these results occur when F is the scalar field and the homogeneity of the polynomials $n=2$, for then we are essentially studying various classes of linear mappings from E to E', and when $n=3$, in which case our situation can be viewed as a study of classes of linear mappings from E to $\mathscr{L}(E;E')$.

Much of the work in this paper was originally motivated by the study of the relationship between the two classes of holomorphic mappings on a complex Banach space E, $H_w(E;F)$ and $H_{wu}(E;F)$, and in particular by the question of whether $H_w(E;F)=H_{wu}(E;F)$ if and only if E is reflexive. This question was, in turn, motivated by the fact [11] that a Banach space E is reflexive if and only if every weakly continuous function on E is bounded on bounded subsets of E. We discuss the situation for holomorphic functions in Section 3, examining the cases $E=c_0$ and $E=l_1$ in some detail. In the case of c_0, for example, we show that $H_w(c_0;F)$ coincides with the space of analytic functions from c_0 to F which are bounded on weakly compact subsets of c_0, while $H_{wu}(c_0;F)$ is the space of analytic functions which are bounded on bounded subsets of c_0. In fact, Dineen [5] has recently shown that if an analytic function $f\colon c_0\to F$ is bounded on weakly compact subsets of c_0, then it is bounded on all bounded subsets of c_0; thus $H_w(c_0;F)=H_{wu}(c_0;F)$. In light of Dineen's result and the equality of $\mathscr{P}_w(^nE;F)$ and $\mathscr{P}_{wu}(^nE;F)$ for all n, E, and F (Theorem 2.9) it is worth noting that we know of no example of spaces E and F for which $H_w(E;F)\neq H_{wu}(E;F)$.

1. Notation and Terminology

Throughout, E and F will be Banach spaces, with underlying field R or C to be specified. In addition $B_r(E)$ denotes the closed ball of center 0 and radius r in E. For each nonnegative integer n, $\mathscr{L}(^nE;F)$ is the space of continuous n-linear mappings from $E\times\cdots\times E$ to F and $\mathscr{P}(^nE;F)$ is the space of continuous n-homogeneous polynomials from E to F; $\mathscr{L}(^0E;F)$ and $\mathscr{P}(^0E;F)$ are associated to F. To each such polynomial P corresponds a unique symmetric mapping $A\in\mathscr{L}(^nE;F)$, via the polarization identity

$$A(x_1,\dots,x_n)=\frac{1}{2^n n!}\sum_{\substack{i=1\\ \varepsilon_i=\pm 1}}^{n}\varepsilon_1\cdots\varepsilon_n P(\varepsilon_1x_1+\cdots+\varepsilon_nx_n).$$

When E and F are complex Banach spaces, $H(E;F)$ is the space of holomorphic mappings from E to F. Recall that a mapping $f\colon E\to F$ is holomorphic if and only if f has a Frechet derivative at every point: that is for each $x\in E$ there is an element $df(x)\in\mathscr{L}(E;F)=\mathscr{L}(^1E;F)$ such that

$$\lim_{h\to 0}\frac{\|f(x+h)-f(x)-df(x)(h)\|}{\|h\|}=0.$$

Equivalently, f is holomorphic if and only if for each $x\in E$, there is a sequence of continuous n-homogeneous polynomials $(d^nf(x))$ from E to F such that $f(y)=\sum_{n=0}^{\infty}(d^nf(x)/n!)(y-x)$ uniformly for y in some neighborhood of x. See [4] for a very thorough discussion of holomorphic mappings in infinite dimensions.

We will also be interested in the following subspaces of the space $C(E;F)$ of continuous mappings from E to F. Let Φ be an arbitrary subset of E'.

$C_{\Phi u}(E;F)=\{f\in(E;F)$: for all balls B in E and for all $\varepsilon>0$, there is a finite subset θ of Φ and $\delta>0$ such that if $x, y\in B$, $|\varphi(x-y)|<\delta$, $(\varphi\in\theta)$, then $\|f(x)-f(y)\|<\varepsilon\}$.

$C_{\Phi}(E;F)=\{f\in C(E;F)$: for all balls B in E, all points $x\in B$, and all $\varepsilon>0$, there is a finite subset θ of Φ and $\delta>0$ such that if $y\in B$, $|\varphi(x-y)|<\delta$, $(\varphi\in\theta)$, then $\|f(x)-f(y)\|<\varepsilon\}$.

$C_{\Phi C}(E;F)=\{f\in C(E;F)$: for all bounded sequences (x_n) in E for which $(\varphi(x_n))$ is Cauchy $(\varphi\in\Phi)$ $(f(x_n))$ is a Cauchy sequence in $F\}$.

$C_{\Phi\mathrm{sc}}(E;F)=\{f\in C(E;F)$: for all bounded sequences (x_n) in E for which $\varphi(x-x_n)\to 0$ for some $x\in E$ $(\varphi\in\Phi)$ $(f(x_n))$ converges to $f(x)$ in $F\}$.

It is not hard to see that the following inclusions hold, and that all inclusions are strict in general:

$$C_{\Phi u}(E;F)\ \begin{matrix}\subset & C_{\Phi}(E;F) & \subset\\ \subset & C_{\Phi C}(E;F) & \subset\end{matrix}\ C_{\Phi\mathrm{sc}}(E;F)\subset C(E;F). \qquad (*)$$

Moreover, since balls in a Banach space are weakly precompact, every function in $C_{\Phi u}(E;F)$ is also a member of $C_b(E;F)=\{f\in C(E;F)$: f is bounded on balls in $E\}$.

When $\Phi = E'$, we will replace Φ in our notation by w, for "weak;" thus, for example, $C_{E'u}(E;F)$ will be denoted by $C_{wu}(E;F)$. $\mathscr{P}_{\Phi u}(^nE;F)$ (resp. $H_{\Phi u}(E;F)$) denotes the space $\mathscr{P}(^nE;F) \cap C_{\Phi u}(E;F)$ (resp. $H(E;F) \cap C_{\Phi u}(E;F)$). The spaces $\mathscr{P}_{\Phi}(^nE;F)$, $H_{\Phi}(E;F)$, etc. are defined analogously. $\mathscr{L}_{\Phi u}(^nE;F)$ denotes the subspace of $\mathscr{L}(^nE;F)$ consisting of those n-linear mappings which correspond, via the polarization formula, to elements of $\mathscr{P}_{\Phi u}(^nE;F)$; $\mathscr{P}_{\Phi}(^nE;F)$, $\mathscr{P}_{\Phi C}(^nE;F)$, etc. are defined similarly. When the range space $F = R$ or C, we will omit the second term in the parentheses; thus $\mathscr{L}(^nE;R)$ is denoted by $\mathscr{L}(^nE)$, etc.

In Section 2, we will examine the question of equality in $(*)$, when we restrict it to the corresponding spaces of polynomials. In Section 3, we briefly discuss this question for spaces of analytic functions.

2. The Polynomial Case

We will show in this section that the following diagram holds:

$$\mathscr{P}_{\Phi u}(^nE;F) \begin{array}{c} \subsetneq \ \mathscr{P}_{\Phi}(^nE;F) \ \subset \\ \subset \ \mathscr{P}_{\Phi C}(^nE;F) \ \subsetneq \end{array} \mathscr{P}_{\Phi \mathrm{sc}}(^nE;F) \subset \mathscr{P}(^nE;F), \qquad (**)$$

where the inclusion signs mean that strict inclusion can occur, depending on E and F. In fact, the same scalar valued polynomial $P(x) = \sum_{n=1}^{\infty} x_n^2$, acting on l_1 and l_2, shows this. Indeed, $P \in \mathscr{P}(^2l_2) - \mathscr{P}_{\mathrm{wsc}}(^2l_2)$ since the canonical basis vectors (e_n) in l_2 tend to 0 weakly but $P(e_n) = 1$ for all n. Also, $P \in \mathscr{P}_{\mathrm{wsc}}(^2l_1)$ since weak and norm sequence convergence coincide in l_1. However, an application of Proposition 2.8 (or Proposition 4 of [2]) shows that $P \notin \mathscr{P}_{wu}(^2l_1)$.

The following useful proposition is helpful in giving a geometric idea of some of the above spaces of polynomials.

Proposition 2.1. *A polynomial $P \in \mathscr{P}(^nE;F)$ belongs to $\mathscr{P}_{\Phi}(^nE;F)$ for some subset Φ of E' if and only if the following condition is satisfied: for any $x \in B_1(E)$ and $\varepsilon > 0$, there is a finite subset $\theta \subset \Phi$ such that if $y \in B_1(E)$ satisfies $\varphi(x-y) = 0$ $(\varphi \in \theta)$, then $\|P(x) - P(y)\| < \varepsilon$.*

Proof. Only the sufficiency needs to be proved. Using the homogeneity of P it is clear that the condition holds for $B_r(E)$ for any $r > 0$. Let $B = B_r(E)$, $x \in B$, and $\varepsilon > 0$ be given, and choose $\theta \subset \Phi$ as in the above condition corresponding to $2B$ and $\varepsilon/2$. There is clearly no loss in generality in assuming that the elements $\varphi_1, \ldots, \varphi_m$ of θ are linearly independent, and so we may choose points $z_1, \ldots, z_m \in E$ such that $\varphi_i(z_j) = \delta_{ij}$, for $1 \leqslant i, j \leqslant m$. Since P is uniformly continuous on $2B$, there is some constant γ, $0 < \gamma < r$, such that if $y,\ z \in 2B$, $\|y-z\| < \gamma$, then $\|P(y)-P(z)\| < \varepsilon/2$. Now, let $\delta = \gamma/(m \cdot \max\{\|z_i\|: 1 \leqslant i \leqslant m\})$ and let $y \in B$ satisfy $|\varphi_i(x-y)| < \delta$ $(1 \leqslant i \leqslant m)$. If we set $w = y - \sum_{i=1}^{m} \varphi_i(y-x)\, z_i$, then $\|w-y\| < \gamma$ and so $\|w\| < \|y\| + r < 2r$. Thus by the uniform continuity of P, $\|P(w)-P(y)\| < \varepsilon/2$. Also for each $i = 1,..., m$, $\varphi_i(w-x) = 0$, so that $\|P(w)-P(x)\| < \varepsilon/2$, by hypothesis. Therefore $\|P(x)-P(y)\| < \varepsilon$ as required. Q.E.D.

We remark that the above proof can be adapted to any situation in which the function in question is uniformly continuous on bounded sets. In particular, we have

PROPOSITION 2.2. *A polynomial $P \in \mathscr{P}({}^nE; F)$ belongs to $\mathscr{P}_{\Phi u}(E; F)$ for some subset Φ of E' if and only if for any $\varepsilon > 0$ there is a finite subset $\theta \subset \Phi$ such that if $x,\ y \in B_1(E)$ satisfy $\varphi(x-y) = 0$ $(\varphi \in \theta)$, then $\|P(x)-P(y)\| < \varepsilon$.*

THEOREM 2.3. *For any Banach spaces E and F, any $\Phi \subset E'$, and any integer n, $\mathscr{P}_{\Phi C}({}^nE; F) = \mathscr{P}_{\Phi sc}({}^nE; F)$.*

For the proof, it will be convenient to call a sequence (y_k) in E Φ-convergent to y in E (resp. Φ-Cauchy) if for all $\varphi \in \Phi$, $\varphi(y-y_k) \to 0$ (resp. $(\varphi(y_k))$ is Cauchy). We first need

LEMMA 2.4. *Let $A \in \mathscr{L}_{\Phi sc}({}^nE; F)$ and let $(x_k^1),..., (x_k^n)$ be n bounded sequences in E. Suppose that at least one sequence is Φ-convergent to 0 and the others are Φ-Cauchy. Then the sequence $(A(x_k^1,..., x_k^n))$ converges to 0 in F.*

Proof. The proof is by induction. For $n = 1$, the result is immediate. Assuming the result for $j = 1,..., n-1$, let A and the sequences (x_k^i) $(i = 1,..., n)$ be as in the hypothesis; to fix the notation, assume that (x_k^1) is Φ-convergent to 0. If the result is false, then for some $\varepsilon > 0$ $\|A(x_k^1,..., x_k^n)\| > \varepsilon$ for all k in an infinite subset J of natural numbers. Now for each fixed $k \in J$, the mapping Ax_k^n defined by

$$Ax_k^n(z^1,..., z^{n-1}) = A(z^1,..., z^{n-1}, x_k^n)$$

is an element of $\mathscr{L}_{\Phi sc}({}^{n-1}E; F)$. Therefore by the induction hypothesis, for some index $m(k) \in J$, it follows that $\|Ax_k^n(x_j^1,..., x_j^{n-1})\| < \varepsilon/2$ whenever $j \geqslant m(k)$; there is clearly no loss in generality in supposing $m(k+1) > m(k)$ for all k. In particular, for each $k \in J$ we have

$$\begin{aligned} \varepsilon/2 &\leqslant \|A(x_{m(k)}^1,..., x_{m(k)}^n) - A(x_{m(k)}^1,..., x_{m(k)}^{n-1}, x_k^n)\| \\ &= \|A(x_{m(k)}^1,..., x_{m(k)}^{n-1}, x_{m(k)}^n - x_k^n)\|. \end{aligned}$$

Consider now the sequences (y_k^i) $(i=1,\dots,n)$, where $y_k^i = x_{m(k)}^i$ for $i=1,\dots,n-1$ and $y^n = x_{m(k)}^n - x_k^n$. These n sequences have the property that all are Φ-Cauchy, and at least two are Φ-convergent to 0. By repeating the above argument, we can thus obtain n bounded sequences (z_k^i) $(i=1,\dots,n)$ which are all Φ-convergent to 0, such that $\|A(z_k^1,\dots,z_k^n)\| \geqslant \varepsilon/2^{n-1}$. However, this contradicts our assumption that $A \in \mathscr{L}_{\Phi sc}({}^nE;F)$, which completes the proof. Q.E.D.

Proof of Theorem 2.3. Let $P \in \mathscr{P}_{\Phi sc}({}^nE;F)$ and let $A \in \mathscr{L}_{\Phi sc}({}^nE;F)$ be the associated symmetric multilinear mapping. By the polarization formula, it suffices to show that if (x_k^i) is a bounded Φ-Cauchy sequence in E $(i=1,\dots,n)$, then $\|A(x_j^1,\dots,x_j^n) - A(x_k^1,\dots,x_k^n)\| \to 0$ as $j,\ k \to \infty$. But

$$\begin{aligned}\|A(x_j^1,\dots,x_j^n) &- A(x_k^1,\dots,x_k^n)\| \\ &\leqslant \|A(x_j^1 - x_k^1, x_j^2,\dots,x_j^n)\| + \|A(x_k^1, x_j^2 - x_k^2, x_j^2,\dots,x_j^n)\| \\ &\quad + \cdots + \|A(x_k^1,\dots,x_k^{n-1}, x_j^n - x_k^n)\|.\end{aligned}$$

In each of the above terms, at least one of the sequences is Φ-convergent to 0 as $j,\ k \to \infty$, and the other sequences are Φ-Cauchy. An application of Lemma 2.4 completes the proof. Q.E.D.

For Banach spaces $E_1,\dots,E_n$, F, let $\mathscr{L}(E_1,\dots,E_n;F)$ be the space of continuous n-linear mappings from $E_1 \times \cdots \times E_n \to F$. Let $\Phi_i \subset E_i'$ be arbitrary, and let $\Phi = (\Phi_1,\dots,\Phi_n)$. The subspaces $\mathscr{L}_{\Phi C}(E_1,\dots,E_n;F)$ and $\mathscr{L}_{\Phi sc}(E_1,\dots,E_n;F)$ are defined in the obvious manner, by analogy with $\mathscr{L}_{\Phi C}({}^nE;F)$ and $\mathscr{L}_{\Phi sc}({}^nE;F)$. Only trivial modifications in Lemma 2.4 and Proposition 2.2 are needed to prove

COROLLARY 2.5. *For any Banach spaces* $E_1,\dots,E_n$, F *and arbitrary* $\Phi = (\Phi_1,\dots,\Phi_n) \subset E_1' \times \cdots \times E_n'$, $\mathscr{L}_{\Phi C}(E_1,\dots,E_n;F) = \mathscr{L}_{\Phi sc}(E_1,\dots,E_n;F)$.

We now turn our attention to the proof that a polynomial which is Φ-continuous, when restricted to any ball in E, is in fact uniformly Φ-continuous on each ball; in particular, weak continuity of polynomials on balls implies uniform weak continuity. In connection with this, it is worth noting that unlike the linear or 2-homogeneous case, it is not in general true that a continuous n-homogeneous polynomial P belongs to $\mathscr{P}_w({}^nE;F)$ if its restriction to each ball is weakly continuous at 0 (cf. [1]).

In order to prove the equality of $\mathscr{P}_\Phi({}^nE;F)$ and $\mathscr{P}_{\Phi u}({}^nE;F)$ in general, it will be convenient to first restrict to separable Banach spaces E. In particular, the next result holds for general E although the proof of this will come later (Theorem 2.9).

LEMMA 2.6. *Let E and F be Banach spaces, E being separable, and let*

$P \in \mathscr{P}_w(^nE;F)$. Then there is a countable set $\Phi \subset E'$ such that $P \in \mathscr{P}_\Phi(^nE;F)$.

Proof. Let (x_j) be a dense sequence in E, $\|x_j\| \leqslant j$. For each pair of natural numbers (j,m), there is a finite subset $\Phi_j^m \subset E'$ such that if a point $y \in E$, $\|y\| \leqslant 2j$, is such that $\varphi(y-x_j)=0$ $(\varphi \in \Phi_j^m)$, then $\|P(y)-P(x_j)\| < 1/m$. Letting $\Phi = \bigcup_{j,m} \Phi_j^m$ we will show that $P \in \mathscr{P}_\Phi(^nE;F)$. To do this, let $x_0 \in B_1(E)$ and $\varepsilon > 0$ be arbitrary (it clearly suffices to restrict our attention to the unit ball $B_1(E)$). Since P is uniformly continuous on bounded sets, there is δ, $0<\delta<1$, such that if $x, y \in B_2(E)$, $\|x-y\|<\delta$, then $\|P(x)-P(y)\|<\varepsilon/3$. Choose x_j such that $\|x_j - x_0\| < \delta$, let $m > 3/\varepsilon$, and suppose that $z \in B_1(E)$ is such that $\varphi(z-x_0)=0$, $(\varphi \in \Phi_j^m)$. Let $w = z - x_0 + x_j$, noting that $\|w\| \leqslant \|z\| + \delta < 2$. Then $\varphi(w-x_j)=0$ for $\varphi \in \Phi_j^m$, so that $\|P(x_j)-P(w)\| < 1/m < \varepsilon/3$. Also, $\|w-z\| = \|x_j - x_0\| < \delta$, so that $\|P(w)-P(z)\| < \varepsilon/3$ since both z and $w \in B_2(E)$. Therefore, $\|P(x_0)-P(z)\| \leqslant \|P(x_0)-P(x_j)\| + \|P(x_j)-P(w)\| + \|P(w)-P(z)\| < \varepsilon$, and an application of Proposition 2.2 completes the proof. Q.E.D.

For the sake of completeness, we include a proof of the following well-known result:

LEMMA 2.7. *Let $\Phi = \{\varphi_j\}$ be any countable set in E' and let (x_j) be any bounded sequence in E. Then (x_j) has a Φ-Cauchy subsequence.*

Proof. Let $N_0 = N$ and for each $j > 1$, let $N_j \subset N_{j-1}$ be an infinite set such that the smallest element n_j in N_j is not in N_{j+1} and such that $(\varphi_j(x_k))_{k \in N_j}$ converges. Then the sequence (x_{n_j}) is Φ-Cauchy. Q.E.D.

Let now $P \in \mathscr{P}(^nE;F)$ be an arbitrary polynomial, with associated symmetric n-linear $A \in \mathscr{L}(^nE;F)$. To this mapping A, there is a uniquely associated linear mapping $C: E \to \mathscr{L}_s(^{n-1}E;F)$, the space of symmetric $(n-1)$-linear mappings of $E \times \cdots \times E$ into F, given by $C(x)(y_1,\ldots,y_{n-1}) = A(x, y_1,\ldots,y_{n-1})$ $(x, y_1,\ldots,y_{n-1} \in E)$.

PROPOSITION 2.8. *If E is a separable Banach space and $P \in \mathscr{P}_w(^nE;F)$, then the associated mapping C is a compact linear mapping.*

Proof. By Lemma 2.6, $P \in \mathscr{P}_\Phi(^nE;F) \subset \mathscr{P}_{\Phi sc}(^nE;F)$ for some countable set $\Phi \subset E'$, so that the n-linear mapping A is an element of $\mathscr{L}_{\Phi sc}(^nE;F)$. In fact, we now show that the associated linear mapping C is an element of $\mathscr{L}_{\Phi sc}(E;\mathscr{L}(^{n-1}E;F))$, which is equal to $\mathscr{L}_{\Phi c}(E;\mathscr{L}(^{n-1}E;F))$ by Theorem 2.3. In fact, if $C \notin \mathscr{L}_{\Phi sc}(E;\mathscr{L}(^{n-1}E;F))$, then for some bounded sequence (x_j) which is Φ-convergent to 0 and some $\varepsilon > 0$, $\|C(x_j)\| > \varepsilon$. This means that for each j there is a point $y_j \in B_1(E)$ such that $\|C(x_j)(y_j,\ldots,y_j)\| > ((n-1)!/(n-1)^{n-1})(\varepsilon/2) = \varepsilon'$ say. By Lemma 2.7, we can extract a subsequence (y_{j_k}) which is Φ-Cauchy. Therefore for all k, $\|A(x_{j_k}, y_{j_k},..., y_{j_k})\| > \varepsilon'$, which contradicts Lemma 2.4. Thus, $C \in \mathscr{L}_{\Phi sc}(E; \mathscr{L}(^{n-1}E; F))$. Now to show that C is a compact mapping, let $(x_j) \subset B_1(E)$ be an arbitrary sequence. Using Lemma 2.7 again, there is a Φ-Cauchy subsequence (x_{j_k}) of (x_j). Finally, since $C \in \mathscr{L}_{\Phi C}(E; \mathscr{L}(^{n-1}E; F))$, $(C(x_{j_k}))$ is Cauchy in $\mathscr{L}(^{n-1}E; F)$. Q.E.D.

Finally, we are ready to prove that a polynomial which is weakly continuous on balls is in fact weakly uniformly continuous on balls.

THEOREM 2.9. *For any Banach spaces E and F, let* $P \in \mathscr{P}(^nE; F)$ *and the associated linear mapping* $C: E \to \mathscr{L}_s(^{n-1}E; F)$ *be given. Then* $P \in \mathscr{P}_w(^nE; F)$ *if and only if* C *is compact. Consequently,* $\mathscr{P}_w(^nE; F) = \mathscr{P}_{wu}(^nE; F)$.

Proof. Let $P \in \mathscr{P}_w(^nE; F)$ and suppose that the associated mapping C is not compact. Thus there is a sequence $(x_j) \subset B_1(E)$ such that $(C(x_j))$ has no convergent subsequence in $\mathscr{L}_s(^{n-1}E; F)$. That is, for some $\varepsilon > 0$, $\|C(x_j - x_k)\| > \varepsilon$ whenever $j \neq k$. As a result, for each pair (j, k), where $j \neq k$ there is a point $y_{jk} \in B_1(E)$ such that $\|C(x_j - x_k)(y_{jk},..., y_{jk})\| > (\varepsilon/2)((n-1)!/(n-1)^{n-1}) = \varepsilon'$. Thus if we define G to be the closed subspace of E generated by the vectors $\{x_j : j \in N\}$, $\{y_{jk} : j, k \in N\}$ then $C|_G : G \to \mathscr{L}(^{n-1}G; F)$ is a noncompact linear mapping. On the other hand, $C|_G$ is the linear mapping associated to $P|_G \in \mathscr{P}(^nG; F)$. Since it is immediate that $P|_G \in \mathscr{P}_w(^nG; F)$ we have obtained a contradiction to Proposition 2.8. Therefore, C is compact. Conversely, let $C: E \to \mathscr{L}_s(^{n-1}E; F)$ be a compact operator. By an easy argument (cf. [3]), C is weakly uniformly continuous on $B_1(E)$, and so for each $\varepsilon > 0$, there is a finite set $\Phi_\varepsilon \subset E'$ such that if $v, w \in B_1(E)$ with $\varphi(v - w) = 0$ $(\varphi \in \Phi_\varepsilon)$, then $\|C(v) - C(w)\| < \varepsilon/n$. Therefore, if $A \in \mathscr{L}_s(^nE; F)$ is the n-linear mapping associated to P, we conclude that

$$\begin{aligned}\|P(v) - P(w)\| &= \|A(v, v,..., v) - A(w, w,..., w)\| \\ &\leqslant \|A(v - w, v,..., v)\| + \|A(w, v - w, v,..., v)\| + \cdots + \|A(w,..., w, v - w)\|.\end{aligned}$$

By symmetry, each $A(v,..., v - w, w,..., w) = C(v - w)(v,..., v, w,..., w)$, and we conclude that $\|P(v) - P(w)\| < \varepsilon$, as required. Q.E.D.

One consequence of the above result is that for every polynomial $P \in \mathscr{P}_w(^nE; F)$ there is a countable subset $\Phi \subset E'$ such that $P \in \mathscr{P}_{\Phi sc}(^nE; F)$. Indeed, by Theorem 2.9, $P \in \mathscr{P}_{wu}(^nE; F)$, and so for each $k \in N$ there is a finite set $\Phi_k \subset E'$ and $\delta_k > 0$ such that if $x, y \in B_1(E)$ satisfy $|\varphi(x - y)| < \delta_k$ $(\varphi \in \Phi_k)$ then $\|P(x) - P(y)\| < 1/k$. Let $\Phi = \bigcup_k \Phi_k$. It is easy to conclude that if (x_j) is a sequence in $B_1(E)$ which is Φ-convergent to a point

$x \in B_1(E)$ then $P(x_j) \to P(x)$, and thus $P \in \mathscr{P}_{\Phi sc}(^nE; F)$. In fact, we have already proved the converse implication, namely that if $P \in \mathscr{P}_{\Phi sc}(^nE; F)$ for some countable subset Φ of E', then $P \in \mathscr{P}_{wu}(^nE; F)$. To see this, note that if $P \in \mathscr{P}_{\Phi sc}(^nE; F)$, then by the proof of Proposition 2.8 the associated linear mapping $C: E \to \mathscr{L}_s(^{n-1}E; F)$ is compact. Thus we have

COROLLARY 2.10. *For a polynomial $P \in \mathscr{P}(^nE; F)$, the following are equivalent*:

(a) $P \in \mathscr{P}_{wu}(^nE; F)$.

(b) *For some countable subset Φ of E', $P \in \mathscr{P}_{\Phi sc}(^nE; F)$.*

In particular, $\mathscr{P}_{wsc}(^nE; F) = \mathscr{P}_{wu}(^nE; F)$ whenever E' is separable, for all n and all Banach spaces F; this situation is described more fully in Proposition 2.12.

COROLLARY 2.11. *If E' has the approximation property, then $\mathscr{P}_w(^nE; F) = \overline{\mathscr{P}_f(^nE) \otimes F}$ for all n, where $\mathscr{P}_f(^nE)$ is the vector subspace of $\mathscr{P}(^nE)$ spanned by the monomials φ^n, $\varphi \in E'$.*

Proof. Suppose that E' has the approximation property. Then, to any element $P \in \mathscr{P}_w(^nE; F)$ corresponds a compact linear mapping $D: E \to \mathscr{P}_w(^{n-1}E; F)$, given by $D(x)(y) = C(x)(y, ..., y)$, where C is defined in Proposition 2.8. Thus D can be approximated by elements of $E' \otimes \mathscr{P}_w(^{n-1}E; F)$, and one proceeds by induction. Details are given in [3].

Q.E.D.

We do not know whether the converse to Corollary 2.11 holds. In particular, we do not know if (in the case $n = 1$) it can happen that every compact operator from E to F can be approximated by finite rank linear operators, with E' not having the approximation property.

We conclude this section with some additional remarks on when every weakly sequentially continuous polynomial is weakly continuous on bounded sets. First, we recall (cf. [9]) that an infinite dimensional Banach space E either contains an isomorphic copy of l_1 or every bounded sequence (x_j) in E has a weak Cauchy subsequence (x_{j_k}), that is $(\varphi(x_{j_k}))$ is a Cauchy sequence for each $\varphi \in E'$.

PROPOSITION 2.12. *If the Banach space E does not contain a copy of l_1, then for any Banach space F and $n \in N$, $\mathscr{P}_{wsc}(^nE; F) = \mathscr{P}_{wu}(^nE; F)$.*

Proof. Let $P \in \mathscr{P}_{wsc}(^nE; F)$ be given, with associated linear mapping $C: E \to \mathscr{L}(^{n-1}E; F)$. Since $l_1 \not\subset E$, the argument of Proposition 2.8 can be applied to prove that $C \in \mathscr{L}_{wsc}(E; \mathscr{L}(^{n-1}E; F))$. Thus, again using the fact that $l_1 \not\subset E$, it follows that C is compact. But then the conclusion follows from Theorem 2.9. Q.E.D.

The authors are grateful to Joe Diestel for pointing out that the converse to Proposition 2.12 is true. Indeed, let E be any Banach space containing a copy of l_1, and let T be a quotient mapping of l_1 onto l_2. Then T can be extended to a continuous linear, noncompact mapping $T: E \to l_2$, and an application of the Grothendieck–Pietsch theorem [8] yields that T is absolutely summing, hence weakly sequentially continuous. Note also that for fixed E and F, the equality $\mathscr{P}_{\text{wsc}}(^nE; F) = \mathscr{P}_{wu}(^nE; F)$, for some n, does not imply this equality for larger values of n. For example, every continuous linear operator between $E = l_\infty$ and $F = l_1$ is compact, so $\mathscr{P}(^1E; F) = \mathscr{P}_{\text{wsc}}(^1E; F) = \mathscr{P}_{wu}(^1E; F)$. On the other hand, there is a noncompact linear mapping $T = (T_n)$: $E \to l_2$, and thus the 2-homogeneous polynomial $Q(x) = ((T_n x)^2)$ cannot belong to $\mathscr{P}_{wu}(^2E; F)$. Also, using the fact that E has the polynomial Dunford–Pettis property [10], it is not difficult to see that $Q \in \mathscr{P}_{\text{wsc}}(^2E; F)$.

Proposition 2.12 implies, in particular, that if E does not contain a copy of l_1 then for any Banach space F, any n-homogeneous polynomial $P: E \to F$ which is weakly sequentially continuous is automatically weakly continuous on weakly compact subsets of E. In fact, this result holds for any Banach space E. To see this, we first show that a weakly sequentially continuous linear mapping $T: E \to G$ is weakly continuous on weakly compact subset of E, where G is a seminormed space. We begin by observing that if $K \subset E$ is weakly compact, then $T(K)$ is precompact in G. Indeed, if this were false, then there would be a sequence (x_n) in K such that $\|T(x_n) - T(x_m)\| > \varepsilon$ whenever $m \neq n$, where $\varepsilon > 0$ is fixed. However, since there is a subsequence (x_{n_j}) converging weakly to a point x in K, $\|T(x_{n_j}) - T(x)\| \to 0$, a contradiction. Applying this to the weakly compact set $K - K$, we may argue as in [3] to conclude that for each $\varepsilon > 0$ there is a finite set $\{\varphi_1, \ldots, \varphi_k\} \subset G'$ such that for all $x, y \in K$, $\|T(x-y)\| \leqslant \max_i |\varphi_i \circ T(x-y)| + \varepsilon$. Therefore, T is weakly continuous on K, for if $x, y \in K$ are such that $|\varphi_i \circ T(x-y)| < \varepsilon$ $(i = 1, \ldots, k)$, then $\|T(x-y)\| < 2\varepsilon$. Now suppose that we have proved that every polynomial $P \in \mathscr{P}_{\text{wsc}}(^nE; F)$ is weakly continuous on weakly compact subsets of E, and let $P \in \mathscr{P}_{\text{wsc}}(^{n+1}E; F)$. Fix a weakly compact subset K of E, and let G be the space $\mathscr{L}_{\text{wsc}}(^nE; F)$ seminormed by

$$\theta \in \mathscr{L}_{\text{wsc}}(^nE; F) \to \sup\{\|\theta(x^1, \ldots, x^n)\| : x^1, \ldots, x^n \in K\}.$$

Consider the multilinear mapping $A \in \mathscr{L}_{\text{wsc}}(^{n+1}E; F)$ and the linear mapping $C: E \to \mathscr{L}(^nE; F)$ associated to P. It is an easy exercise to show that in fact the range of C lies in G. Moreover, C is weakly sequentially continuous. To see this, suppose that (x_k) tends weakly to 0 in E but $\|C(x_k)\| > \varepsilon$ for all k, where $\varepsilon > 0$ is fixed. Thus, for each k, there are points $x_k^1, \ldots, x_k^n \in K$ such that $\|C(x_k)(x_k^1,\dots,x_k^n)\| > \varepsilon$. Using the weak compactness of K, we may suppose that for each $i = 1,\dots,n$, $x_k^i \to x^i$ weakly in K. But since $C(x_k)(x_k^1,\dots,x_k^n) = A(x_k, x_k^1,\dots,x_k^n)$ converges to $A(0, x^1,\dots,x^n) = 0$, we have a contradiction. Therefore $C: E \to G$ is weakly continuous on K. That is for all $\varepsilon > 0$, there is a finite set $\{\varphi_1,\dots,\varphi_k\} \subset E'$ and $\delta > 0$ such that if $x, y \in K$, $|\varphi_i(x-y)| < \delta$, then $\|C(x) - C(y)\| < \varepsilon$. Using the symmetry of A, it follows that

$$\begin{aligned}\|P(x) - P(y)\| &= \|C(x)(x,\dots,x) - C(y)(y,\dots,y)\| \\ &\leqslant \|C(x)(x,\dots,x) - C(y)(x,\dots,x)\| \\ &\quad + \|C(x)(y,x,\dots,x) - C(y)(y,x,\dots,x)\| \\ &\quad + \cdots + \|C(x)(y,\dots,y) - C(y)(y,\dots,y)\| \leqslant (n+1)\varepsilon.\end{aligned}$$

Therefore, we have proved

PROPOSITION 2.13. *For any Banach spaces E and F and any n, $\mathscr{P}_{\mathrm{wsc}}({}^nE;F)$ is the space of all continuous n-homogeneous polynomials $P: E \to F$ such that for any weakly compact subset $K \subset E$, $P|_K: K \to F$ is continuous when K is given the weak topology.*

In connection with this, the following is worth noting. Let E be any Banach space and for each $r > 0$, let X_r be the ball $B_r(E)$ with the induced weak topology. Let X be the (topological) direct limit of the X_r, $X = \lim_{r\to\infty} X_r$. It is not difficult to see that a set is compact in X if and only if it is weakly compact in E, and that $C(X)$ is exactly those scalar-valued functions f on E such that f is weakly continuous when restricted to all balls $B_r(E)$. The space $C(X)$ has been extensively studied by Ferrera [6], who proved the following result.

PROPOSITION 2.14 ([6, Theorem 3.6]). *If E is a separable Banach space which does not contain a copy of l_1, then X is a k-space.*

Therefore, if f is a function on E which is continuous on each weakly compact subset of E, then f is weakly continuous when restricted to balls in E. Of course, in the case of polynomials, this result is subsumed by Propositions 2.12 and 2.13.

Last, we mention that $\mathscr{P}_f({}^nE;F)$ is not in general dense in $\mathscr{P}_{\mathrm{wsc}}({}^nE;F)$ with respect to the topology of uniform convergence on weakly compact subsets of E. For instance, if E is a reflexive Banach space without the approximation property, then for some Banach space F there is a compact linear mapping $T: E \to F$ which is not approximable on $B_1(E)$ by finite rank operators. Thus $\overline{\mathscr{P}_f({}^1E;F)} \neq \mathscr{P}_{\mathrm{wsc}}({}^1E;F)$.

3. THE HOLOMORPHIC CASE

In this section, we shall restrict our attention to the case $\Phi = E'$, to complex Banach spaces E and F, and to subspaces of the space of holomorphic mappings $H(E;F)$ from E to F. It is obvious that the following diagram holds:

$$H_{ui}(E;F) \begin{array}{c} \subset \; H_w(E;F) \; \subset \\ \subset \; H_{wC}(E;F) \; \subset \end{array} H_{wsc}(E;F) \subset H(E;F).$$

However, in this situation, we are unable to prove that any of the above inclusions are equalities in general. Thus, we will confine ourselves to briefly outlining certain situations where equality does occur and to discussing two open problems of some interest. We being with the following simple lemma, which is proved by an application of Cauchy's inequality.

LEMMA 3.1. *Let $f \in H(E;F)$ and let $S \subset E$ be an absolutely convex set such that $\|f\|_{rS} = \sup\{\|f(x)\| : x \in rS\} < \infty$, for some $r > 1$. Then $f(x) = \sum_{n=0}^{\infty} (d^n f(0)/n!)(x)$ uniformly for $x \in S$.*

This lemma, together with Theorem 2.9, yields the fact that a function $f \in H_w(E, F)$ belongs to $H_{wu}(E;F)$ if and only if f is bounded on bounded subsets of E.

Using Lemma 3.1, we can extend Proposition 2.13 to holomorphic functions.

PROPOSITION 3.2. *$H_{wsc}(E;F)$ is the space of analytic functions $f: E \to F$ which are weakly continuous on each weakly compact subset of E.*

Proof. Let $f \in H_{wsc}(E;F)$ and let K be an absolutely convex weakly compact subset of E. Note that $\|f\|_{rK}$ is finite, where $r > 1$ is fixed, since otherwise f is unbounded on a sequence in rK and hence on a weakly convergent sequence in rK. Therefore, the result follows by first observing that $d^n f(0) \in \mathscr{P}_{wsc}(^n E;F)$ for all n [1] and by applying Proposition 2.13, Lemma 3.1, and the fact that a uniform limit of weakly continuous functions on K is weakly continuous on K. Q.E.D.

The following is a partial analogue of Proposition 2.12 in the holomorphic situation.

PROPOSITION 3.3. *Let E be a complex Banach space.*

(a) *If E is separable and does not contain a copy of l_1, then $H_w(E;F) = H_{wsc}(E;F)$ for any complex Banach space F. In particular, $H_w(E;F) = H_{wsc}(E;F)$ if E' is separable.*

(b) *If E does not contain a copy of l_1, then $H_{wC}(E;F) = H_{wu}(E;F)$ for any complex Banach space F.*

Proof. The proof of (a) follows by applying Proposition 3.2 and Ferrera's result (Proposition 2.14). If E' is separable, then it is classical that E is separable and E does not contain l_1. To prove (b), we first show that any function $f \in H_{wC}(E;F)$ is bounded on bounded subsets of E. Indeed, if not, then for some bounded sequence (x_n) in E, $(f(x_n))$ is unbounded. However, since the sequence has a weak Cauchy subsequence, this contradicts the fact that $f \in H_{wC}(E;F)$. Thus by Lemma 3.1, if $f \in H_{wC}(E;F)$ then $f(x) = \sum_{n=0}^{\infty} (d^n f(0)/n!)(x)$ uniformly on bounded subsets in E. Since $H_{wu}(E:F)$ is complete with the topology of uniform convergence on bounded sets, the result follows by noting that for each n, $d^n f(0) \in \mathscr{P}_{wsc}({}^nE;F)$ (cf. [1]) and then applying Proposition 2.12. Q.E.D.

A continuous version of Proposition 3.3 holds, at least for the case when E' is separable. Specifically, if E' is separable, then we have

(a) $C_w(E;F) = C_{wsc}(E;F)$,

(b) $C_{wC}(E;F) = C_{wu}(E;F)$.

The proof of (a) is trivial, since if E' is separable, then the weak topology induced on any bounded subset of E is metrizable. Hence, in this topology, continuity is just the same as sequential continuity. To prove (b), let $\{\varphi_i\} \subset E'$ be dense, let $f \in C_{wC}(E;F)$ and suppose that $f \notin C_{wu}(E;F)$. Thus for some ball $B_r(E)$ and some $\varepsilon > 0$, the following holds: for each n, there are points $x_n, y_n \in B_{r/2}(E)$ with $x_n - y_n \in U_n = \{z \in B_r(E): |\varphi_i(z)| < 1/n$ for $1 \leqslant i \leqslant n\}$ such that

$$\|f(x_n) - f(y_n)\| > \varepsilon. \tag{$*$}$$

Applying Lemma 2.7, we may choose subsequences (x_{n_k}), (y_{n_k}) which are weakly Cauchy. Using the fact that $(x_{n_k} - y_{n_k})$ tends weakly to 0 and $f \in C_{wC}(E;F)$, it follows that $(f(x_{n_k}))$ and $(f(y_{n_k}))$ tend to the same limit. contradicting $(*)$ above.

Dineen [5] has recently shown that $H_{wsc}(c_0) = H_{wu}(c_0)$. However, we do not know whether the separability of E' implies that $H_{wsc}(E;F) = H_{wu}(E;F)$, or that $H_{wsc}(E) = H_{wu}(E)$, in general. To show equality of these two spaces, it would be equivalent to show that a function in $H_{wsc}(E;F)$ is bounded on bounded subsets of E, by Lemma 3.1 and Proposition 2.12. Dineen's result shows that there is no holomorphic analogue of the following characterization of reflexity for a Banach space E [11]: E is reflexive if and only if every weakly continuous function $f: E \to R$ is bounded on balls in E, and hence if and only if every function $f: E \to R$

which is weakly continuous on balls in E is weakly uniformly continuous on balls in E. On the other hand, Gil [7] has shown that there is a differentiable analog of the above characterization.

In fact, we have *no* example of a Banach space E for which $H_w(E) \neq H_{wu}(E)$ (see also Example 3.5). Another way of posing this problem is as follows: As noted in [1], given a function $f = \sum_{n=0}^{\infty} P_n \in H(E)$, $f \in H_w(E)$ if and only if for each ball B in E, the sequence $\{P_n|_B\}$ is weakly equicontinuous at each point of B, while $f \in H_{wu}(E)$ if and only if for each ball B in E, the sequence $\{P_n|_B\}$ is weakly uniformly equicontinuous on B. Thus, the question of whether $H_w(E) = H_{wu}(E)$ becomes whether every such weakly equicontinuous sequence is automatically weakly uniformly equicontinuous.

Finally, we briefly discuss the above questions in the cases $E = l_1$ and $E = c_0$.

EXAMPLE 3.4. $H_{wC}(c_0) = H_{wu}(c_0) = \{f \in H(c_0)$: f is bounded on bounded subsets of $c_0\}$, and $H_{wsc}(c_0) = H_w(c_0) = \{f \in H(c_0)$: f is bounded on weakly compact subsets of $c_0\}$. (In fact, by [5], all these spaces coincide.)

Proof. The first equality follows from Proposition 3.3(b), while the second follows from the remarks after Proposition 2.11 and Lemma 3.1. By Proposition 3.3(a), $H_{wsc}(c_0) = H_w(c_0)$. If $f \in H(c_0)$ is bounded on weakly compact sets, then $f = \sum_{n=0}^{\infty} d^n f(0)/n!$ uniformly on weakly compact sets. Since each $d^n f(0) \in \mathscr{P}_{wsc}({}^n c_0)$ and is therefore weakly continuous on weakly compact sets, an application of Proposition 3.2 proves that $f \in H_{wsc}(c_0)$. The converse if trivial. Q.E.D.

The following example shows that in a sense, l_1 is universal with respect to weak continuity of holomorphic mappings on bounded sets.

EXAMPLE 3.5. If $H_w(l_1) = H_{wu}(l_1)$, then for all complex spaces E and F, $H_w(E; F) = H_{wu}(E; F)$. Thus, if $H_w(l_1) = H_{wu}(l_1)$, Theorem 2.9 follows as a consequence.

Proof. We first show that for any index set Γ, $H_w(l_1(\Gamma)) = H_{wu}(l_1(\Gamma))$. By Lemma 3.1, it suffices to show that any function $f \in H_w(l_1(\Gamma))$ is bounded on bounded sets. Let us suppose that $(f(x_n))$ is unbounded for some bounded sequence $(x_n) \subset l_1(\Gamma)$. Let S be the union of the supports of the points of the sequence, that is, $S = \{\gamma \in \Gamma: x_n(\gamma) \neq 0$ for some $n\}$. Then it is clear that $f|_{l_1(S)} \in H_w(l_1(S)) = H_{wu}(l_1(S))$ since S is countable, and so f is unbounded on some bounded subset of $l_1(S)$, a contradiction. Now, let E and F be arbitrary Banach spaces and let $f \in H_w(E; F)$. As before, $f \in H_{wu}(E; F)$ if and only if f is bounded on bounded subsets of E. So let us suppose that $f(B)$ is unbounded on some ball B in E; it follows that $\varphi \circ f$ is unbounded on B, for some $\varphi \in F'$. Let $\pi: l_1(\Gamma) \to E$ be the canonical quotient mapping for an appropriate choice of Γ. Since $\varphi \circ f \in H_w(E)$, for each ball $B_r(E)$, $x \in B_r(E)$ and $\varepsilon > 0$, there are $\psi_1, \dots, \psi_k \in E'$ and $\delta > 0$ such that if $y \in B_r(E)$ satisfies $|\psi_i(x-y)| < \delta$ $(i = 1, \dots, k)$, then $|\varphi \circ f(x) - \varphi \circ f(y)| < \varepsilon$. From this it is easy to see that if $x, y \in B_r(l_1(\Gamma))$ are such that $|\psi_i \circ \pi(x-y)| < \delta$ $(i = 1, \dots, k)$, then $|\varphi \circ f \circ \pi(x) - \varphi \circ f \circ \pi(y)| < \varepsilon$. Thus, $\varphi \circ f \circ \pi \in H_w(l_1(\Gamma)) = H_{wu}(l_1(\Gamma))$. Moreover, this implies that $\varphi \circ f \in H_{wu}(E)$. To see this, note that for any ball $B_r(E)$, $\|\varphi \circ f\|_{B_r(E)} = \|\varphi \circ f \circ \pi\|_{B_r(l_1(\Gamma))} < \infty$, and therefore $\varphi \circ f$ is bounded on bounded subsets of E. Thus, we have obtained the desired contradiction. Q.E.D.

We conclude with two open problems which are implicit in the above discussion.

PROBLEM 1. Does $H_w(E;F) = H_{wu}(E;F)$ for every pair of complex Banach spaces E and F?

PROBLEM 2. Is the condition that E not contain a copy of l_1 necessary and sufficient to ensure that $H_w(E;F) = H_{wsc}(E;F)$ for any Banach space F?

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Note added in proof. Recently, J. Ferrera and J. G. Gil have used the fact that E with its weak topology is angelic to show that Propositions 2.13 and 3.2 hold in general. Specifically (in "Funciones debilmente continuas y debilmente secuencialmente continuas," to appear), they have shown that $C_{wsc}(E;F)$ is the space of continuous functions which are weakly continuous when restricted to weakly compact subsets of E.

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