Analytic Functions on $\mathbf{c_0}$

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Abstract. Let $\mathcal{F}$ be a space of continuous complex valued functions on a subset of c_0 which contains the standard unit vector basis $\{e_n\}$. Let $R : \mathcal{F} \to C^{\mathcal{N}}$ be the restriction map, given by $R(f) = (f(e_1), \ldots, f(e_n), \ldots)$. We characterize the ranges $R(\mathcal{F})$ for various "nice" spaces $\mathcal{F}$. For example, if $\mathcal{F} = \mathcal{P}(^n c_0)$, then $R(\mathcal{F}) = \ell_1$, and if $\mathcal{F} = A^\infty(B_{c_0})$, then $R(\mathcal{F}) = \ell_\infty$.

Let c_o be the Banach space of complex null sequences $\vec{x} = (x_n)$, with the normal *sup*-norm and usual basis vectors $\vec{e}_n = (0, \ldots, 0, 1, 0, \ldots)$, and let $\mathcal{F}$ be a space of continuous complex-valued functions on some subset of c_o which contains the standard basis of c_o. Let $R : \mathcal{F} \to C^{\mathcal{N}}$ be the mapping which assigns to each function $f \in \mathcal{F}$ the sequence $(f(e_1), \ldots, f(e_n), \ldots)$. Our attention in this article will be focussed on characterizing the range of R for various spaces $\mathcal{F}$ of interest. For example, if $\mathcal{F} = \mathcal{C}(c_o)$, the space of all continuous complex valued functions on c_o, then a trivial application of the Tietze extension theorem shows that $R(\mathcal{F}) = C^{\mathcal{N}}$. On the other hand, c_0 is weakly normal (Corson [6], see also Ferrera [9]). Since $\{0\} \cup \{e_n : \ n \in N\}$ is weakly compact, we see that $R(\mathcal{F}) = c$, the space of convergent sequences, if we take $\mathcal{F}$ to be the subspace of $\mathcal{C}(c_0)$ consisting of weakly continuous functions. Recently Jaramillo [11] has examined the relationship between reflexivity of the space $\mathcal{F}$ and the range of R, for certain spaces of *real* valued infinitely differentiable functions and polynomials on a Banach space E with unconditional basis $\{e_n; \ n \in N\}$.

We concentrate here on analogous spaces of *complex* valued functions on c_0. After a review of relevant notation and definitions, we show in Section 1 that $R(\mathcal{F}) = \ell_1$ when $\mathcal{F} = \mathcal{P}(^n c_0), n \in N$. As a consequence, we prove that if $\mathcal{F} = \{f \in \mathcal{H}_b(B_R(c_0)) : \ f(0) = 0\}$, then $R(\mathcal{F}) = \ell_1$. Taking $n = 2$ in the above result, we see that every $2-$homogeneous polynomial P on c_0 satisfies $\sum_{j=1}^{\infty} |P(e_j)| < \infty$. This result is reminiscent of classical work of Littlewood [13] , who proved that every continuous bilinear form A on $c_0 \times c_0$ satisfies $(A(e_j, e_k))_{j,k=1}^{\infty} \in \ell_{\frac{4}{3}}$. Littlewood's work was extended by Davie [7], who showed that every continuous n-linear form $A : c_0 \times \cdots \times c_0 \to C$ satisfies $(A(e_{\alpha_1}, \cdot, \cdot, \cdot, e_{\alpha_n})) \in \ell_{\frac{2n}{n+1}}$. In Section 2, we prove that $R(\mathcal{A}^{\infty}(B(c_0))) = \ell_{\infty}$, and as a corollary of the proof of this result we show that $R(\mathcal{A}_U(B(c_0))) = \ell_1$.

Our notation for analytic functions is standard and follows, for example, Dineen [8] and Mujica [14]. For a Banach space $E, B_R(E)$ denotes the open $R-$ball centered at 0 in E with $B_1(E)$ abbreviated to $B(E)$. $\mathcal{L}(^n E)$ denotes the Banach space of continuous n-linear forms $A : E \times \cdots \times E \to C$, equipped with the norm n$\|A\| = sup\ \{|A(x_1, ..., x_n)| : \ x_j \in E, \|x_j\| \leq 1,\ j = 1, ..., n\}$. $\mathcal{P}(^n E)$ denotes the Banach space of continuous $n-$homogeneous polynomials on E. Each such polynomial P is associated with a unique symmetric continuous $n-$linear form A, by $P(x) = A(x, ..., x)$, and $\|P\|$ is defined to be $sup_{\|x\| \leq 1} |P(x)|$. A function f from an open subset U of E to C is said to be *holomorphic* if f has a complex Fréchet derivative at each point of U. Equivalently, f is holomorphic if for all points $a \in U$, the Taylor series $f(x) = \sum_{n=o}^{\infty} P_n(x - a)$, converges uniformly for all x in some neighborhood of a, where each $P_n \in \mathcal{P}(^n E)$.

$\mathcal{H}_b(B_R(E))$ is the space of all holomorphic functions on $B_R(E)$ which are bounded on $B_r(E)$ for every $r < R$. A useful characterization of $\mathcal{H}_b(B_R(E))$ is that it consists of all holomorphic functions f on $B_R(E)$ such that $limsup_{n\to\infty} \|P_n\|^{\frac{1}{n}} \leq 1/R$, where $\{P_n : n \in N\}$ represents the Taylor polynomials of f at the origin. The spaces $\mathcal{A}^\infty(B(E))$ and $\mathcal{A}_U(B(E))$ have been studied by Cole and Gamelin [4, 5], Globevnik [10] and others [1]. $\mathcal{A}^\infty(B(E)) = \{f : \overline{B(E)} \to C : f$ is holomorphic on B(E) and continuous and bounded on $\overline{B(E)}\}$. Unless E is finite dimensional, this space is always strictly larger than $\mathcal{A}_U(B(E)) = \{f : B(E) \to C : f$ is holomorphic and uniformly continuous on $B(E)\}$. Both of these spaces are natural infinite dimensional analogues of the disc algebra.

Section 1

We show here that for all $P \in \mathcal{P}(^n c_0)$ and all $n \in N$, $\sum_{j=1}^\infty |P(e_j)| \leq \|P\|$. This has already been done by K. John [12], in the case $n = 2$. In [13], Littlewood showed that for every $A \in \mathcal{L}(^n c_0)$, $(A(e_j, e_k))_{j,k=1}^\infty \in \ell_{\frac{4}{3}}$, and that $\frac{4}{3}$ is best possible; thus, Littlewood's $\frac{4}{3}$ result notwithstanding, John's result is that every $A \in \mathcal{L}(^n c_0)$ has a trace. Our proof will make use of a generalization of the classical Rademacher functions, which seems to be well-known to probabilists (see, for example, Chatterji [3]).

Definition 1.1. Fix $n \in N$, $n \geq 2$, and let $\alpha_1 = 1, \alpha_2, ..., \alpha_n$ denote the n^{th} roots of unity. Let $s_1 : [0, 1] \to C$ be the step function taking the value α_j on $(\frac{j-1}{n}, \frac{j}{n})$, for $j = 1, ..., n$. Assuming that s_{k-1} has been defined, define s_k in the following natural way. Fix any of the n^{k-1} sub-intervals I of $[0, 1]$ used in the definition of s_{k-1}. Divide I into n equal intervals $I_1, ..., I_n$, and set $s_k(t) = \alpha_j$ if $t \in I_j$. (The endpoints of the intervals are irrelevant for this construction and we may, for example, define s_k to be 1 on each endpoint.) Of course, when n = 2, Definition 1.1 gives us the classical Rademacher functions. The following lemma lists the basic properties of the functions s_k. Its proof is similar to the usual, induction proof for the Rademacher functions, and is omitted.

Lemma 1.2. For each $n = 2, 3, ...$, the associated functions s_k satisfy the following properties:

(a). $|s_k(t)| = 1$, for all $k \in N$ and all $t \in [0, 1]$.

(b). For any choice of $k_1, ..., k_n$,

$$\int_0^1 s_{k_1}(t) \cdots s_{k_n}(t)dt = \begin{cases} 1 & \text{if } k_1 = \cdots = k_n \\ 0 & \text{otherwise} \end{cases}$$

We are grateful to Andrew Tonge for suggesting an improvement in the proof of the following result.

Theorem 1.3. Let $P \in \mathcal{P}({}^n c_0)$. Then $\|(P(e_j))\|_{\ell_1} \leq \|P\|$.

Proof. Let $A \in \mathcal{L}({}^n c_0)$ be the symmetric n-linear form associated to P. Fix any $m \in N$. For each $i = 1, ..., m$, let $\lambda_i = |A(e_i, ..., e_i)|/A(e_i, ..., e_i)$, if $A(e_i, ..., e_i) \neq 0$, and 1 otherwise. Furthermore, let β_i denote any n^{th} root of λ_i. Thus, $\lambda_i A(e_i, ..., e_i) = |P(e_i)|$ for each $i = 1, ..., m$. Adding and applying Lemma 1.2 for the integer n, we get $\sum_{i=1}^m |P(e_i)| \ = \sum_{i=1}^m \lambda_i A(e_i, ..., e_i)$

$$= \sum_{i,j_2,...,j_n=1}^m \int_0^1 \lambda_i s_i(t) s_{j_2}(t) \cdots s_{j_n}(t) A(e_i, e_{j_2}, ..., e_{j_n})\ dt$$
$$= \int_0^1 A(\sum_{i=1}^m \lambda_i s_i(t) e_i, ..., \sum_{j_n=1}^m s_{j_n}(t) e_{j_n})\ dt$$
$$= \int_0^1 A(\sum_{j_1=1}^m \beta_{j_1} s_{j_1}(t) e_{j_1}, ..., \sum_{j_n=1}^m \beta_{j_n} s_{j_n}(t) e_{j_n})\ dt.$$

Since $\| \sum_{j=1}^m \beta_j s_j(t) e_j \| \leq 1$ for all t, the last expression is clearly less than or equal to $\|P\|$. Since m was arbitrary, the proof is complete. Q.E.D.

Rephrasing the above result in terms of the mapping R mentioned in the introduction, Theorem 1.3 implies that for any n, $R(\mathcal{P}({}^n c_0)) \subset \ell_1$. In fact, R is onto ℓ_1, since any $\vec{\lambda} = (\lambda_1, ..., \lambda_j, ...) \in \ell_1$ equals $R(P)$, where $P \in \mathcal{P}({}^n c_0)$ is given by $P(x) = \sum_{j=1}^\infty \lambda_j x_j^n$.

We conclude this section by proving that, up to a normalizing factor, $R(\mathcal{H}_b(B_R(c_0))) = \ell_1$, for every $R > 1$. Since $\mathcal{H}_b(B_R(c_0))$ "approaches" $\mathcal{A}^\infty(B(c_0))$ as $R \downarrow 1$, it is tempting to guess that Corollary 1.4 below is also true for the latter space. We will see in the next section that this is completely false.

Corollary 1.4. Let $R > 1$ and let $f \in \mathcal{H}_b(B_R(c_0))$, with $f(0) = 0$. Then $(f(e_n))_{n=1}^{\infty} \in \ell_1$.

Proof. By the characterization given earlier of $\mathcal{H}_b(B_R(c_0))$, we see that if S is such that $1 < S < R$, then $\|P_m\|^{\frac{1}{m}} < 1/S$, for all large m. Therefore,

$$\begin{aligned} \textstyle\sum_{n=1}^{\infty} |f(e_n)| &= \textstyle\sum_{n=1}^{\infty} |\sum_{m=1}^{\infty} P_m(e_n)| \\ \le \textstyle\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |P_m(e_n)| &\le \textstyle\sum_{m=1}^{\infty} \|P_m\| < \infty. \end{aligned}$$

Q.E.D.

Section 2

The following fundamental lemma shows in effect that any sequence of 0's and 1's can be interpolated by a norm one function in $\mathcal{A}^{\infty}(B(c_0))$.

Lemma 2.1. (i). Let $S \subset N$ be an arbitrary set. There exists a function $F \in \mathcal{A}^{\infty}(B(c_0))$ with the following properties:

$$\begin{aligned} \|F\| &= \sup_{x \in B(c_0)} |F(x)| = 1, \\ F(e_n) &= \begin{cases} 1 & \text{if } n \in S \\ 0 & \text{if } n \notin S \end{cases} \end{aligned}$$

(ii). If S is finite, then a function $F \in \mathcal{A}_U(B(c_0))$ can be found which satisfies the above conditions.

Proof. Let $\alpha_j \uparrow \infty$ so quickly that the following three conditions are satisfied:

(i). The function $\Phi(x) \equiv \Pi_{j \in S}(1 - x_j)^{\frac{1}{\alpha_j}}$ converges for all $x \in \overline{B(c_0)}$,
(ii). Re $\Phi(x) \ \geq \ 0$, for all $x \in B(c_0)$,
(iii). $\Phi(x) = 0$ for some $x \in \overline{B(c_0)}$ if and only if Re $\Phi(x) = 0$.

Note that $\Phi \in \mathcal{A}^\infty(B(c_0))$ and, if S is finite then in fact $\Phi \in \mathcal{A}_U(B(c_0))$. Also,

$$\Phi(e_n) = \begin{cases} 0 & \text{for } n \in S \\ 1 & \text{for } n \notin S \end{cases}$$

Now, let $G(x) \equiv e^{-\Phi(x)}$. From the above, it is clear that $G \in \mathcal{A}^\infty(B(c_0))$ for arbitrary S and that $G \in \mathcal{A}_U(B(c_0))$ for finite S. In addition, $|G(x)| \leq 1$ for all x and

$$G(e_n) = \begin{cases} 1 & \text{for } n \in S \\ \frac{1}{e} & \text{for } n \notin S \end{cases}$$

Finally, let $T : \overline{\Delta} \to \overline{\Delta}$ be the Mobius transformation $T(z) = \frac{z-\frac{1}{e}}{1-\frac{z}{e}}$ (where Δ is the complex unit disc). It is clear that $F \equiv T \circ G$ satisfies all the conditions of the lemma. Q.E.D.

We come now to the analogue of Corollary 1.4, for the polydisc algebras $\mathcal{A}^\infty(B(c_0))$ and $\mathcal{A}_U(B(c_0))$. Note that here the situation is completely different from the situation in Section 1.

Theorem 2.2. (i). $R(\mathcal{A}^\infty(B(c_0))) = \ell_\infty$. In fact, given $(\alpha_n) \in \ell_\infty$, there is $F \in \mathcal{A}^\infty(B(c_0))$ such that $F(e_n) = \alpha_n$ for all $n \in N$ and such that $\|F\| \leq 4\|(\alpha_n)\|_{\ell_\infty}$.

(ii). $R(\mathcal{A}_U(B(c_0))) = c$. In fact, given $(\alpha_n) \in c$, there is $F \in \mathcal{A}_U(B(c_0))$ such that $F(e_n) = \alpha_n$ for all $n \in N$ and such that $\|F\| \leq 8\|(\alpha_n)\|_{\ell_\infty}$.

Proof. (i). Without loss of generality, $\|(\alpha_n)\| \leq 1$. Let us first suppose that $\alpha_n \geq 0$ for all n. Write $\alpha_n = \sum_{j=1}^\infty 2^{-j}\alpha_{n_j}$, where each $\alpha_{n_j} = 0$ or 1. Let $S_j = \{n \in N : \alpha_{n_j} = 1\}$, and let F_j be the associated function obtained using Lemma 2.1. It is easy to see that $F \equiv \sum_{n=1}^\infty 2^{-j}F_j$ is the required function in this case, and that $\|F\| \leq \|(\alpha_n)\|$. The case of general α_n's is treated by writing $\alpha_n = p_n - q_n + iu_n - iv_n$.

(ii). Suppose first that $(\alpha_n) \in c$ with $\|(\alpha_n)\| \leq 1$, and write each $\alpha_n = \ell + \beta_n$ where $\ell = lim_{n\to\infty}\alpha_n$. As above, if each β_n is expressed in binary series form, then each of the associated sets S_j is finite. As a result, each F_j is finite, by Lemma 2.1 (ii), so that $F \in \mathcal{A}_U(B(c_0))$. The required function is $G \equiv F + \ell$.

Finally, note that for any $F \in \mathcal{A}_U(B(c_0))$, $F(x)$ can be approximated uniformly for $x \in B(c_0)$ by $F_r(x) = F(rx)$ for r sufficiently close to 1. Next, $F(rx)$ can be uniformly approximated on the unit ball of c_0 by a finite Taylor series, say $\sum_{k=0}^{M} P_k(x)$ (where P_0 is a constant). Next, it is well known (see, for example, [15]) that any k-homogeneous polynomial P_k on c_0 can be uniformly approximated on $B(c_0)$ by an k-homogeneous polynomial Q_k which is a finite sum of products of k continuous linear functionals on c_0. Summarizing, we see that the original function F can be uniformly approximated on $B(c_0)$ by $\sum_{k=0}^{\infty} Q_k$. Now, since $(e_n) \to 0$ weakly it follows that for each $k = 1, ..., M$, $Q_k(e_n) \to 0$ as $n \to \infty$. Hence $R(F) \in c$, and the proof is complete. Q.E.D.

It would be interesting to determine the best possible estimates in Theorem 2.2. In [2], we note that in this situation, the best estimate must be strictly larger than 1. To see this, suppose that there is $F \in \mathcal{A}^\infty(B(c_0))$ such that $\|F\| = 1$ and such that $F(e_1) = 1$, $F(e_2) = -1$, and $F(e_j) = 0$ for all $j \geq 3$. Then the function $f_1(z) \equiv F(1, z, 0, ...)$ would be in the disc algebra $\mathcal{A}(\Delta)$, and f_1 would attain its maximum at 0. Hence, f_1 would be a constant and, in particular, $1 = f_1(1) = F(1, 1, 0, ...)$. Similarly, the function $f_2(z) \equiv F(z, 1, 0, ...)$ would be constant, and so $-1 = f_2(1) = F(1, 1, 0, ...)$, a contradiction. In [2], the authors find necessary and sufficient conditions on the sequence $(x_n) \subset c_0$ in order that the mapping $F \in \mathcal{A}^\infty(B(c_0)) \to (F(x_n)) \in \ell_\infty$ be surjective and satisfy the following condition: For each $(\alpha_n) \in \ell_\infty$, there is $F \in \mathcal{A}^\infty(B(c_0))$ such that $F(x_n) = \alpha_n$ for each $n \in N$ and $\|F\| = sup_n|\alpha_n|$.

References

1. R.M. Aron, B. Cole, and T. Gamelin, The spectra of algebras of analytic functions associated with a Banach space, to appear.
2. R.M. Aron and J. Globevnik, Interpolation by analytic functions on c_0, Math. Proc. Camb. Phil. Soc. **104** (1988) 295-302.
3. S.D. Chatterji, Continuous functions representable as sums of independent random variables, Z. Wahrsch., **13** (1969) 338-341.
4. B. Cole and T. Gamelin, Representing measures and Hardy spaces for the infinite polydisc algebra, Proc. London Math. Soc (3) **53** (1986) 1, 112-142.
5. B. Cole and T. Gamelin, Weak-star continuous homomorphisms and a decomposition of orthogonal measures, Ann. Inst. Fourier **35** (1985) 149-189.
6. H.H. Corson, The weak topology of a Banach space, Trans. Amer. Math. Soc. **101** (1961), 1-15.
7. A.M. Davie, Quotient algebras of uniform algebras, J. London Math. Soc. 7 (1973) 31-40.
8. S. Dineen, Complex analysis in locally convex spaces, Mathematics Studies **83**, North Holland, Amsterdam, 1981.
9. J. Ferrera, Spaces of weakly continuous functions, Pacific J. Math. **102** (1982) 285-291.
10. J. Globevnik, Boundaries for polydisc algebras in infinite dimensions, Math. Proc. Camb. Phil. Soc. **85** (1979) 291-303.
11. J.A. Jaramillo, Algebras de funciones continuas y diferenciables, PhD dissertation, Univ. Complutense de Madrid, 1988.
12. K. John, On tensor product characterization of nuclear spaces, Math. Ann. 257 (1981) 341-353.
13. J. Littlewood, On bounded bilinear forms in an infinite number of variables, Quart. Journ. of Math. (2) **2** (1930), 164-174.

14. J. Mujica, Complex analysis in Banach spaces, Math. Studies **120**, North Holland, Amsterdam, 1986.

15. A. Pelczynski, A theorem of Dunford-Pettis type for polynomial operators, Bull. Acad. Pol. Sc. **XI**, 6 (1963) 379-386.

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