

# Analytic Disks in Fibers over the Unit Ball of a Banach Space

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## 1. Introduction

Fix an infinite-dimensional complex Banach space  $\mathfrak{X}$ , with open unit ball  $B$ . We are interested in studying the uniform algebra  $H^\infty(B)$  of bounded analytic functions on  $B$ , and its spectrum  $\mathfrak{M} = \mathfrak{M}(B)$  consisting of the non-zero complex-valued homomorphisms of  $H^\infty(B)$ . The spectrum  $\mathfrak{M}$  is fibered in a natural way over the closed unit ball  $\bar{B}^{**}$  of the bidual  $\mathfrak{X}^{**}$  of  $\mathfrak{X}$ . The projection of  $\mathfrak{M}$  onto  $\bar{B}^{**}$  is obtained by simply restricting  $\varphi \in \mathfrak{M}$  to  $\mathfrak{X}^*$ , regarded as a subspace of  $H^\infty(B)$ . As a straightforward application of the Josefson–Nissenzweig theorem [Jo; Ni], it is shown in [ACG] that each fiber consists of more than one point and is in fact quite large. Our aim here is to prove a sharpened form of the Josefson–Nissenzweig theorem, and to use this to embed analytic disks in the fiber over 0. This stands in contrast to the situation in finite-dimensional Banach spaces, where one expects (and can prove under certain hypotheses) that the natural projection is one-to-one over the open unit ball  $B = B^{**}$  and implements a homeomorphism of  $B$  and an open subset of  $\mathfrak{M}$ .

The Josefson–Nissenzweig theorem asserts that in any infinite-dimensional dual Banach space  $\mathfrak{Z}$ , there is a sequence  $\{z_j\}$  converging weak-star to 0 such that  $\|z_j\| = 1$ . The accessory condition we require is that the distance from  $z_j$  to the linear span of the preceding  $z_i$ 's tends to 1 as  $j \rightarrow \infty$ . Actually we prove that each  $z_j$  can be chosen to have unit distance from the linear span of the remaining  $z_i$ 's, and in fact the  $z_j$ 's can be taken as part of a unit biorthogonal system defined in Section 2.

In Sections 2 and 3 we establish the existence of unit biorthogonal systems having accessory properties involving weak or weak-star convergence to 0. In Section 4 we make some observations regarding infinite products in a uniform algebra. The embedding of analytic disks is accomplished in Section 5. In Section 6 we give conditions under which the unit ball of certain

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infinite-dimensional Banach spaces  $\mathfrak{Y}$  can be injected analytically into the fiber over 0. For instance, when  $\mathfrak{X}$  is  $c_0$ ,  $\mathfrak{Y}$  can be  $l^\infty$  and the injection isometric with respect to the Gleason metric, while if  $\mathfrak{X}$  is superreflexive and infinite-dimensional,  $\mathfrak{Y}$  can be a nonseparable Hilbert space and the injection uniformly bicontinuous. In Section 7 we indicate how certain of the embedding results extend to other algebras of analytic functions associated with  $\mathfrak{X}$ .

For background on analytic functions on Banach spaces see [Mu], and for uniform algebras see [Ga]. When convenient we will regard the functions in the algebra at hand to be continuous functions on the spectrum of the algebra, via the Gelfand transform. We denote the open unit disk in the complex plane  $\mathbf{C}$  by  $\Delta$ . The linear span of a set of vectors  $S$  will be denoted by  $\text{sp}(S)$ , and its closure by  $\overline{\text{sp}}(S)$ . The distance from a vector  $x$  to a subset  $Y$  of  $\mathfrak{X}$ , measured in the norm of  $\mathfrak{X}$ , will be denoted by  $\text{dist}(x, Y)$ .

## 2. Unit Biorthogonal Systems

Let  $\mathfrak{X}$  be an infinite-dimensional Banach space, with dual  $\mathfrak{X}^*$ . A *unit biorthogonal system* for  $\mathfrak{X}$  consists of sequences  $\{x_j\}$  in  $\mathfrak{X}$  and  $\{x_k^*\}$  in  $\mathfrak{X}^*$  satisfying

$$(2.1) \quad \|x_j\| = 1 = \|x_k^*\|,$$

$$(2.2) \quad x_k^*(x_j) = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

We will be interested in finding such systems with the additional property that  $x_j$  converges weakly to 0 in  $\mathfrak{X}$ . The standard basis and dual basis for  $l^p$ ,  $1 < p < \infty$ , form such a system. However,  $l^1$  cannot have such a system on account of the Schur property: weakly convergent sequences in  $l^1$  converge in norm. If a subspace  $\mathfrak{Y}$  of  $\mathfrak{X}$  has such a system, then by extending the  $x_j^*$ 's via the Hahn-Banach theorem we obtain a unit biorthogonal system for  $\mathfrak{X}$ .

To construct unit biorthogonal systems we use the following lemma, which is due to Krasnoselskii, Krein, and Milman [KKM].

**2.1. LEMMA.** *Let  $\mathfrak{X}_0$  and  $\mathfrak{X}_1$  be finite-dimensional subspaces of  $\mathfrak{X}$  such that the dimension of  $\mathfrak{X}_1$  exceeds that of  $\mathfrak{X}_0$ . Then there is a vector  $x \in \mathfrak{X}_1$  such that  $\|x\| = 1 = \text{dist}(x, \mathfrak{X}_0)$ .*

*Proof.* The proof is a simple application of Borsuk's theorem that there is no continuous antipodal map from one sphere to another of lower dimension. By perturbing the norm and taking limits, we can assume that every  $x \in \mathfrak{X}_1$  has a unique nearest point  $\phi(x) \in \mathfrak{X}_0$ . We must find  $x \in \mathfrak{X}_1$  such that  $\|x\| = 1$  and  $\phi(x) = 0$ . If there were no such point, then  $x \rightarrow \phi(x)/\|\phi(x)\|$  would be an antipodal map ( $\phi(-x) = -\phi(x)$ ) from the unit sphere of  $\mathfrak{X}_1$  to the unit sphere of  $\mathfrak{X}_0$ , contradicting Borsuk's theorem.  $\square$

Our basic method consists of repeated applications of the following.

2.2. LEMMA. *Let  $\{x_i\}_1^{n-1}, \{x_i^*\}_1^{n-1}$  be a unit biorthogonal system for  $\mathfrak{X}$  of length  $n-1$ , and let  $\mathcal{Z}$  be a subspace of  $\mathfrak{X}$  of dimension at least  $2n-1$ . Then there exist  $x_n \in \mathcal{Z}$  and  $x_n^* \in \mathfrak{X}^*$  so that  $\{x_i\}_1^n$  and  $\{x_i^*\}_1^n$  form a unit biorthogonal system of length  $n$ .*

*Proof.* The dimension of the space  $\mathcal{Z}_1$  of  $z \in \mathcal{Z}$  satisfying  $x_j^*(z) = 0$  for  $1 \leq j \leq n-1$  is at least  $n$ . By Lemma 2.1, there is  $x_n \in \mathcal{Z}_1$  such that

$$\|x_n\| = 1 = \text{dist}(x_n, \text{sp}\{x_1, \dots, x_{n-1}\}).$$

By the Hahn-Banach theorem, there is  $x_n^* \in \mathfrak{X}^*$  such that  $\|x_n^*\| = 1 = x_n^*(x_n)$ , and  $x_n^*(x_j) = 0$  for  $1 \leq j < n-1$ . □

2.3. LEMMA. *Let  $\{y_i\}$  be a basic sequence in  $\mathfrak{X}$ . Then there are a block basis  $\{x_j\}$  of  $\{y_i\}$  and functionals  $\{x_k^*\}$  in  $\mathfrak{X}^*$  which form a unit biorthogonal system for  $\mathfrak{X}$ .*

*Proof.* Suppose that  $x_1, \dots, x_{n-1} \in \mathfrak{X}$ ,  $x_1^*, \dots, x_{n-1}^* \in \mathfrak{X}^*$ , and integers  $1 \leq m_1 < m_2 < \dots < m_{n-1}$  have been chosen to satisfy (2.1) and (2.2), so that each  $x_j$  is a linear combination of the  $y_i$ 's for  $m_{j-1} < i \leq m_j$ . Choose  $m_n = m_{n-1} + 2n - 1$ , and apply Lemma 2.2 to the linear span of the  $y_i$ 's for  $m_{n-1} < i \leq m_n$ . The sequences constructed inductively in this way form a unit biorthogonal system. □

2.4. THEOREM. *For any infinite-dimensional Banach space  $\mathfrak{X}$ , there is a unit biorthogonal system  $\{x_j\}, \{x_k^*\}$  such that 0 is in the weak closure of the  $x_j$ 's.*

*Proof.* The  $x_j$ 's and  $x_k^*$ 's are constructed by an induction process similar to the procedure in the proof of Lemma 2.3, but with the following twist. By Dvoretzky's spherical sections theorem (cf. [MS]), there is for any  $\epsilon > 0$  a subspace of  $\mathfrak{X}$  of arbitrarily large dimension  $N$  which is  $(1 + \epsilon)$ -isomorphic to  $l_N^2$ . Now we repeatedly invoke Lemma 2.2 while arranging that successively larger blocks of the  $x_j$ 's lie in almost-Euclidean subspaces  $\mathfrak{M}_N$  of  $\mathfrak{X}$ . The  $x_j$ 's need not be orthogonal with respect to the Euclidean structure of  $\mathfrak{M}_N$ , but they are "almost" orthonormal, and the "almost" can be expressed quantitatively in terms of  $\epsilon$ , so that for  $\epsilon = \epsilon_N$  sufficiently small we obtain an estimate of the form

$$\sum_{x_j \in \mathfrak{M}_N} |x^*(x_j)|^2 \leq C \|x^*\|^2, \quad x^* \in \mathfrak{X}^*,$$

where  $C$  is independent of  $N$ . Since the number of summands grows larger with  $N$ , an increasingly large proportion of the values  $x^*(x_j)$  are small. It follows that 0 is a weak cluster point of the sequence  $\{x_j\}$ . □

2.5. THEOREM. *Let  $\mathfrak{X}$  be an infinite-dimensional Banach space such that  $l^1$  does not embed into  $\mathfrak{X}$ . Then there are sequences  $\{x_j\}$  in  $\mathfrak{X}$  and  $\{x_k^*\}$  in  $\mathfrak{X}^*$  forming a unit biorthogonal system such that  $x_j$  converges weakly to 0.*

*Proof.* Choose sequences  $\{x_j\}, \{x_k^*\}$  as in Theorem 2.4. Replacing  $\mathfrak{X}$  by the closed linear span of the  $x_j$ 's, we can assume that  $\mathfrak{X}$  is separable, so that the weak-star topology on the closed unit ball  $\bar{B}^*$  of  $\mathfrak{X}^*$  is a compact metric topology. By a theorem of Bourgain, Fremlin, and Talagrand ([BFT]; see also [Di2, p. 216]), the functions on  $\bar{B}^*$  in the first Baire class, endowed with the topology of pointwise convergence, form an angelic space  $\Omega$ . Since  $\mathfrak{X}$  contains no copies of  $l^1$ , Rosenthal's theorem (cf. [Di2, Ch. XI]) shows that the sequence  $\{x_j\}$ , viewed as a sequence of functions on  $\bar{B}^*$ , is sequentially precompact in  $\Omega$ . Since 0 lies in the pointwise closure of the sequence (by Theorem 2.4), the angelic property provides us with a subsequence which converges weakly to 0. Passing to subsequences then, we obtain the desired unit biorthogonal system.  $\square$

Odell, Rosenthal, and Schlumprecht [ORS] have obtained results which show in particular that in every infinite-dimensional Banach space which does not contain a copy of  $l^1$  there is a basic sequence  $\{y_i\}$  such that if  $x_j$  is in the unit ball of the linear span of the  $y_i$ 's for  $2^j < i \leq 2^{j+1}$ , then  $\{x_j\}$  converges weakly to 0. This result can be used instead of the theorem from [BFT] to prove Theorem 2.5.

The following result will be used for the embedding of analytic disks in Section 5.

**2.6. THEOREM.** *If  $\mathfrak{X}$  is any infinite-dimensional Banach space, then there is a unit biorthogonal system  $\{x_j^{**}\}, \{x_k^{***}\}$  for the bidual  $\mathfrak{X}^{**}$  of  $\mathfrak{X}$  such that  $\{x_j^{**}\}$  converges weakly to 0 in  $\mathfrak{X}^{**}$ .*

*Proof.* In view of Theorem 2.5, it suffices to find an infinite-dimensional subspace  $\mathfrak{Y}$  of  $\mathfrak{X}^{**}$  into which  $l^1$  does not embed. If  $l^1$  does not embed in  $\mathfrak{X}$ , then take  $\mathfrak{Y} = \mathfrak{X}$ . Suppose on the other hand that  $l^1$  embeds in  $\mathfrak{X}$ . Then  $(l^\infty)^*$  embeds in  $\mathfrak{X}^{**}$ . Now  $L^1[0, 1]$  embeds in  $(l^\infty)^*$ , for instance in  $L^1(\nu)$  for an appropriate measure  $\nu$  on the Stone-Ćech compactification of the integers, and  $l^2$  embeds in  $L^1[0, 1]$ , for instance as lacunary Fourier series. In this case, take  $\mathfrak{Y}$  to be the embedded image of  $l^2$  in  $\mathfrak{X}^{**}$ .  $\square$

### 3. Unit Biorthogonal Systems in a Dual Banach Space

In this section we aim to establish the existence of biorthogonal systems for a dual Banach space which converge weak-star to 0. The proof will be broken into cases, according to the following lemma.

**3.1. LEMMA.** *Let  $\mathfrak{Z}$  be a dual Banach space. Then at least one of the following conditions holds:*

- (i) *There is an embedding of  $l^1$  into  $\mathfrak{Z}$  such that the image of the standard basis converges weak-star to 0.*
- (ii) *There is an infinite dimensional subspace  $\mathfrak{Y}$  of  $\mathfrak{Z}$  into which  $l^1$  does not embed.*

*Proof.* Suppose that  $l^1$  embeds in  $\mathcal{Z}$  and (i) fails, that is, there is no embedding for which the image of the standard basis tends weak-star to 0. By [HJ] (or see [Di2, p. 219]),  $l^1$  embeds in the predual  $\mathfrak{X}$  of  $\mathcal{Z}$ . From a theorem of Pełczyński [Pe, Thm. 3.4; Di2, p. 213] it follows that  $L^1[0, 1]$  embeds in  $\mathfrak{X}^* = \mathcal{Z}$ . Since  $l^2$  embeds in  $L^1[0, 1]$ , we can take  $\mathfrak{Y}$  to be the embedded image of  $l^2$  in  $\mathcal{Z}$ . □

**3.2. THEOREM.** *Let  $\mathcal{Z}$  be an infinite-dimensional dual Banach space. Then there is a unit biorthogonal system  $\{z_j\}, \{z_k^*\}$  for  $\mathcal{Z}$  such that  $z_j$  converges weak-star to 0. Either the  $z_j$ 's can be chosen to converge weakly to 0; or else there is an embedding of  $l^1$  into  $\mathcal{Z}$  such that the image  $\{y_i\}$  of the standard basis for  $l^1$  converges weak-star to 0, in which case the  $z_j$ 's can be chosen to be a block basis of the  $y_i$ 's.*

*Proof.* Consider the two cases of Lemma 3.1. In case (ii) the unit biorthogonal system is provided by Theorem 2.5.

Suppose that we are in case (i) of Lemma 3.1. Applying Lemma 2.3 to the image  $\{y_i\}$  of the standard basis for  $l^1$ , we obtain  $z_j$ 's and  $z_k^*$ 's which form a unit biorthogonal system, such that the  $z_j$ 's are a block basis of the  $y_q$ 's. Then each  $z_j$  has the form  $\sum a_{jq} y_q$ , and since the  $z_j$ 's have unit norm we obtain a uniform bound

$$\sum_q |a_{jq}| \leq c, \quad 1 \leq j < \infty.$$

Hence if  $x \in \mathfrak{X}$ , the predual of  $\mathcal{Z}$ , we obtain

$$|z_j(x)| = |\sum a_{jq} y_q(x)| \leq c \max_{n_{j-1} < q \leq n_j} |y_q(x)|.$$

Since  $y_q(x)$  tends to 0, so does  $z_j(x)$ , and  $z_j$  converges weak-star to 0. □

While we are focusing on complex Banach spaces, the results of Sections 2 and 3 are all valid also for real Banach spaces.

### 4. Infinite Products in a Uniform Algebra

In this section we record some observations on the convergence of infinite products in a uniform algebra. For Blaschke products in this context, see [Gl].

Let  $A$  be a uniform algebra, with spectrum  $M_A$ . We regard  $A$  as a uniform algebra on  $M_A$ . For  $\psi \in M_A$  and  $0 < r < 1$ , let  $S_r(\psi)$  denote the hyperbolic ball of radius  $r$  centered at  $\psi$ , defined as the collection of  $\varphi \in M_A$  such that  $|f(\varphi)| \leq r$  for all  $f \in A$  satisfying  $\|f\| \leq 1$  and  $f(\psi) = 0$ . Thus the Gleason part of  $\psi$  is the union of the  $S_r$ 's for  $0 < r < 1$ . Harnack's estimate for uniform algebras [Ga, §VI.1] shows that if  $|f_j| \leq 1$  and  $f_j(\psi) \rightarrow 1$ , then  $f_j$  tends uniformly to 1 on each hyperbolic ball  $S_r(\psi)$ . Applying this remark to partial products, we are led to the following.

**4.1. LEMMA.** *Fix  $g_1, g_2, \dots \in A$  satisfying  $|g_j| \leq 1$ , and consider the infinite product  $G$  defined wherever it converges by*

$$G(\varphi) = \prod_{j=1}^{\infty} g_j(\varphi).$$

If the product converges at  $\psi \in M_A$  to a nonzero value, then it converges uniformly on each hyperbolic ball  $S_r(\psi)$ ,  $0 < r < 1$ .

We are interested in infinite products formed from elements of  $H^\infty(B)$ . Before discussing convergence, we provide some background. According to [DG, Thm. 5], the functions in  $H^\infty(B)$  have natural extensions to the open unit ball  $B^{**}$  of  $\mathfrak{X}^{**}$ , so that  $H^\infty(B)$  is embedded isometrically as a closed subalgebra of  $H^\infty(B^{**})$ . We can thus regard  $B^{**}$  as a subset of  $\mathfrak{M}$ , by identifying  $z \in B^{**}$  with the homomorphism which evaluates the natural extension of  $f \in H^\infty(B)$  at  $z$ . From the Schwarz lemma it is easy to see that the hyperbolic balls centered at 0 meet  $B^{**}$  in closed norm balls:

$$S_r(0) \cap B^{**} = r\bar{B}^{**}, \quad 0 < r < 1.$$

Following [ACG], we define a function  $g \in H^\infty(B^{**})$  to be *canonical* if  $g$  is the natural extension to  $B^{**}$  of its restriction to  $B$ . If  $g \in H^\infty(B^{**})$  is a pointwise limit on  $B^{**}$  of a bounded sequence of canonical functions which converges uniformly on each  $rB^{**}$ , then  $g$  is canonical [ACG, Lemma 10.3]. Combining these observations with Lemma 4.1, we obtain the following.

**4.2. LEMMA.** *Suppose  $\{g_j\}$  is a sequence of functions in  $H^\infty(B)$  satisfying  $|g_j| \leq 1$ , and suppose the infinite product  $\prod g_j$  converges at 0 to a nonzero value. Then the infinite product converges uniformly on  $rB$  for each  $r < 1$  to  $G \in H^\infty(B)$ . Furthermore, the canonical extensions to  $B^{**}$  of the partial products converge uniformly on each  $rB^{**}$  to the canonical extension of  $G$ .*

Note that any subproduct of  $G$  divides  $G$  in  $H^\infty(B)$ . Consequently, if  $\varphi \in \mathfrak{M}$  satisfies  $\varphi(g_m) = 0$  for some  $m$  then  $\varphi(G) = 0$ , because  $g_m$  divides  $G$ .

## 5. Embedding Analytic Disks in $\mathfrak{M}_0$

A sequence  $S$  in  $B$  is an *interpolating sequence for  $H^\infty(B)$*  if the restriction of  $H^\infty(B)$  to  $S$  coincides with  $l^\infty$ . We extend this definition to sequences  $S$  in  $B^{**}$  by defining such a sequence to be *interpolating for  $H^\infty(B)$*  if the restriction to  $S$  of the canonical functions in  $H^\infty(B^{**})$  coincides with  $l^\infty$ .

The Blaschke products of interpolating sequences in the open unit disk  $\Delta$  in the complex plane can be used to produce analytic disks in the fibers of  $\mathfrak{M}(\Delta)$  over points of the boundary of  $\Delta$ . For details, see [Ho]. We will use Blaschke products of interpolating sequences in  $B^{**}$  to embed analytic disks in fibers over  $B^{**}$ .

**5.1. THEOREM.** *Let  $\mathfrak{X}$  be an infinite-dimensional Banach space. Suppose  $\{z_k\}$  is a sequence in  $B^{**}$  which converges weak-star to 0, such that the*

distance from  $z_k$  to the linear span of  $z_1, \dots, z_{k-1}$  tends to 1 as  $k \rightarrow \infty$ . Then passing to a subsequence we can find a sequence of analytic disks  $\lambda \rightarrow z_k(\lambda)$  ( $\lambda \in \Delta, k \geq 1$ ) in  $B^{**}$  with  $z_k(0) = z_k$ , such that for each  $\lambda \in \Delta$ ,  $\{z_k(\lambda)\}$  is an interpolating sequence for  $H^\infty(B)$ . Furthermore, the correspondence  $(k, \lambda) \rightarrow z_k(\lambda)$  extends to an embedding

$$\Psi: \beta(N) \times \Delta \rightarrow \mathfrak{M}$$

such that

$$\Psi((\beta(N) \setminus N) \times \Delta) \subset \mathfrak{M}_0$$

and  $f \circ \Psi$  is analytic on each slice  $\{p\} \times \Delta$  for all  $f \in H^\infty(B)$  and  $p \in \beta(N)$ .

*Proof.* We can pass to a subsequence of the  $z_k$ 's whenever appropriate. Thus we can assume that  $\|z_k\|$  converges very rapidly to 1, and that in fact  $\text{dist}(z_k, \text{sp}\{z_1, \dots, z_{k-1}\}) \rightarrow 1$  very rapidly.

Fix  $\delta > 0$  small. For each  $z_k$ , choose  $r_k > 0$  so that

$$(5.1) \quad r_k < \|z_k\|,$$

$$(5.2) \quad \frac{r_k(1 - \|z_k\|)}{\|z_k\| - r_k^2} > \delta.$$

The condition (5.2) is satisfied for  $r_k = \|z_k\|$ , so both (5.1) and (5.2) are satisfied for  $r_k$  slightly less than  $\|z_k\|$ . Fix  $\epsilon > 0$  small. Choose  $0 < s_k < 1$  such that  $\sum(1 - s_k) < \infty$ , and such that (passing to a subsequence if necessary)

$$(5.3) \quad \left| \frac{r_j - \zeta}{1 - r_j \zeta} \right| > s_j, \quad |\zeta| < \epsilon.$$

The condition (5.3) means simply that the pseudohyperbolic disk centered at  $r_j$  with radius  $s_j$  is disjoint from the disk  $\{|\zeta| < \epsilon\}$ .

Fix  $\beta > 0$  small. We claim that, passing to a subsequence if necessary, we can find  $L_k \in \mathfrak{X}^*$ ,  $k \geq 1$ , such that

$$(5.4) \quad \|L_k\| < 1,$$

$$(5.5) \quad L_k(z_k) = r_k,$$

$$(5.6) \quad L_k(z_j) = 0, \quad 1 \leq j < k,$$

$$(5.7) \quad |L_k(z_j)| < \beta/2^k, \quad j > k.$$

Indeed, suppose we have chosen, after discarding some  $z_j$ 's and relabeling, functionals  $L_1, \dots, L_{m-1}$  satisfying (5.4) to (5.7) for  $1 \leq j, k \leq m-1$ . Since  $z_j$  tends weak-star to 0, we can also arrange (after more discarding and relabeling) that

$$|L_j(z_m)| < \beta/2^m, \quad 1 \leq j \leq m-1.$$

Using the distance hypothesis, we can then apply the Hahn-Banach theorem to find  $\Lambda \in (\text{sp}\{z_i\}_{i=1}^m)^*$  such that  $\|\Lambda\| < 1$ ,  $\Lambda(z_m) > r_m$ , and  $\Lambda(z_k) = 0$  for  $1 \leq k < m$ . Since the restriction mapping from  $\mathfrak{X}^*$  to  $(\text{sp}\{z_i\}_{i=1}^m)^*$  is a quotient mapping, we obtain  $L_m \in \mathfrak{X}^*$  with the asserted properties.

Now define

$$w_k(\lambda) = \frac{r_k - \lambda}{1 - \lambda r_k} \frac{z_k}{r_k}, \quad |\lambda| \leq \delta.$$

Then  $w_k(0) = z_k$ . Since the maximum of  $|r_k - \lambda|/|1 - \lambda r_k|$  over the disk  $|\lambda| \leq \delta$  is attained at  $\lambda = -\delta$ , we obtain the estimate

$$\|w_k(\lambda)\| \leq \frac{r_k + \delta}{1 + \delta r_k} \frac{\|z_k\|}{r_k}, \quad |\lambda| \leq \delta.$$

The condition (5.2) on  $r_k$  is equivalent to the right-hand side of this estimate being less than 1, so we obtain

$$\|w_k(\lambda)\| < 1, \quad |\lambda| < \delta.$$

We define a Blaschke product  $G$  by

$$G(z) = \prod_{j=1}^{\infty} \frac{r_j - L_j(z)}{1 - r_j L_j(z)}, \quad z \in B^{**}.$$

By Lemma 4.2 the product converges uniformly on  $rB^{**}$  for each  $0 < r < 1$ , and  $G \in H^\infty(B^{**})$  satisfies  $|G| < 1$ . Furthermore, the function  $G$  is canonical; that is,  $G$  represents the canonical extension to  $B^{**}$  of its restriction  $G|_B$  in  $H^\infty(B)$ .

The factors of the Blaschke product  $G$  are chosen so that

$$\frac{r_k - L_k(w_k(\lambda))}{1 - r_k L_k(w_k(\lambda))} = \lambda.$$

Hence we can express  $G(w_k(\lambda)) = \lambda g_k(\lambda)$ , where  $g_k(\lambda)$  is the product over  $j \neq k$  of

$$\frac{r_j - L_j(w_k(\lambda))}{1 - r_j L_j(w_k(\lambda))}.$$

From (5.6) we see that this reduces to  $r_j$  for  $j > k$ . For  $1 \leq j < k$  one computes that the factor has the form  $r_j + O(\Lambda_j(z_k))$ , where the error estimate is uniform for  $|\lambda| < \delta$  and  $j \neq k$ . The estimate

$$\sum_{j=1}^{k-1} |L_j(z_k)| \leq \beta \left( 1 + \frac{1}{2} + \cdots + \frac{1}{2^{k-1}} \right) < 2\beta$$

shows that by choosing  $\beta$  sufficiently small and the  $r_j$ 's sufficiently close to 1, we can arrange that  $g_k(\lambda)$  is close to 1 for all  $|\lambda| \leq \delta$ , uniformly in  $k$ . Hence if  $|\lambda| < \delta/2$ , there is a unique  $\zeta_k(\lambda)$  satisfying  $|\zeta_k(\lambda)| < \delta$  and

$$G(w_k(\zeta_k(\lambda))) = \lambda, \quad |\lambda| < \delta/2.$$

Now we define  $z_k(\lambda)$  to be the reparameterization of the  $w_k$ 's given by

$$z_k(\lambda) = w_k(\zeta_k(\lambda)), \quad |\lambda| < \delta/2.$$

Then  $z_k(\lambda)$  is a multiple of  $z_k$  which depends analytically on  $\lambda$  and satisfies



$$\begin{aligned} \|z_k(\lambda)\| < 1, \quad |\lambda| < \delta/2, \\ z_k(0) = z_k, \\ G(z_k(\lambda)) = \lambda, \quad |\lambda| < \delta/2. \end{aligned}$$

We claim that for each fixed  $\lambda$ ,  $|\lambda| < \delta/2$ , the sequence  $\{z_k(\lambda)\}$  is an interpolating sequence for  $H^\infty(B)$ . For this, it suffices to find for each subset  $J \subset N$  a canonical function  $f_J \in H^\infty(B)$  such that  $|f_J| \leq 1$  on  $B$ ,  $|f_J(z_k(\lambda))| < 1/3$  for  $k \in J$ , and  $|f_J(z_k(\lambda)) - 1| < 1/3$  for  $k \in N \setminus J$ . A function  $f_J$  having these properties is the Blaschke subproduct

$$f_J(z) = \prod_{k \in J} \frac{r_k - L_k(z)}{1 - r_k L_k(z)}.$$

From

$$f_J(w_k(\lambda)) = \lambda \prod_{j \in J, j \neq k} \frac{r_j - L_j(w_k(\lambda))}{1 - r_j L_j(w_k(\lambda))}$$

we obtain

$$|f_J(z_k(\lambda))| \leq \delta, \quad k \in J, \quad |\lambda| \leq \delta/2.$$

On the other hand, the earlier estimates for  $g_k(\lambda)$  are easily modified to show that  $f_J(z_k(\lambda))$  is near 1 when  $k \notin J$ . The claim is established.

Now let  $D$  be the open disk  $\{|\lambda| < \delta/2\}$  in the complex plane. It suffices to establish the theorem, with  $\Delta$  replaced by  $D$ . We define  $\Psi$  on  $N \times D$  by

$$\Psi(k, \lambda) = z_k(\lambda), \quad |\lambda| < \delta/2, \quad 1 \leq k < \infty,$$

where we regard  $z_k(\lambda)$  as a point in  $\mathfrak{M}$ ; that is, we identify  $z_k(\lambda)$  with the evaluation homomorphism. Since  $\mathfrak{M}$  is compact,  $\Psi$  extends to a map

$$\Psi: \beta(N) \times D \rightarrow \mathfrak{M},$$

which is continuous on  $\beta(N)$  for each fixed  $\lambda \in D$ . For all  $f \in H^\infty(B)$ ,  $f \circ \Psi$  is bounded and analytic on each slice  $\{p\} \times D$  of  $\beta(N) \times D$ , hence equicontinuous on the slices  $\{p\} \times \{|\lambda| < r\delta/2\}$ ,  $r < 1$ . It follows that  $\Psi$  is jointly continuous. Since  $\Psi(N \times \{\lambda\})$  is an interpolating sequence, for each fixed  $\lambda$ ,  $\Psi$  is one-to-one on  $\beta(N) \times \{\lambda\}$ . On account of the identity

$$G(\Psi(p, \lambda)) = \lambda, \quad (p, \lambda) \in \beta(N) \times D,$$

$\Psi$  is one-to-one on  $\beta(N) \times D$ . By shrinking  $\delta$  slightly we obtain then that  $\Psi$  is an embedding. Finally, note that  $z_k(\lambda)$  converges weak-star to 0 in  $\mathfrak{X}^{**}$ , so that  $\Psi$  maps  $(\beta(N) \setminus N) \times D$  into  $\mathfrak{M}_0$ . □

By Theorem 2.6 or Theorem 3.2, there is always a sequence  $\{z_k\}$  in  $\mathfrak{X}^{**}$  which converges weak-star to 0 and which satisfies

$$\|z_k\| = 1 = \text{dist}(z_k, \text{sp}\{z_1, \dots, z_{k-1}\}).$$

Thus we obtain the following.

5.2. COROLLARY. *If  $\mathfrak{X}$  is an infinite-dimensional Banach space, then there are analytic disks in the fiber  $\mathfrak{M}_0$  over 0 of the spectrum  $\mathfrak{M}$  of  $H^\infty(B)$ . In fact, there is an analytic embedding of  $(\beta(N) \setminus N) \times \Delta$  into  $\mathfrak{M}_0$ .*

## 6. Embedding Infinite-Dimensional Analytic Structure in $\mathfrak{M}_0$

The question now arises as to what sort of analytic objects are to be found in the fibers of the spectrum of  $H^\infty(B)$ . When is it possible to inject analytically the unit ball  $B$  of  $\mathfrak{X}$  into  $M_0$ ? What are the natural analytic objects appearing in  $\mathfrak{M}_0$ ? Can one learn something about  $\mathfrak{X}$  by peering into  $\mathfrak{M}_0$ ?

If the unit ball of some infinite-dimensional Banach space injects analytically in  $\mathfrak{M}_0$ , then so does the (countable-dimensional) infinite polydisk  $\Delta^\infty$ , the unit ball of  $l^\infty$ . This is because  $l^\infty$  can be mapped injectively into any infinite-dimensional Banach space. In the other direction, any separable Banach space maps injectively into  $l^\infty$ , so if one can inject  $\Delta^\infty$  analytically into  $\mathfrak{M}_0$  then one can inject the unit ball of any separable Banach space analytically into  $\mathfrak{M}_0$ .

Conceptually the simplest way to inject  $\Delta^\infty$  is by shifting along a basic sequence and passing to a limit. In Theorem 6.1 we give one result with this method of proof. The reader should note that all the results of this section through Theorem 6.5 apply to the spaces  $l^p$  and  $L^p[0, 1]$  for  $1 < p < \infty$ .

6.1. THEOREM. *Suppose  $\mathfrak{X}$  has a normalized basis  $\{x_j\}$  which is shrinking, that is, whose associated coefficient functionals  $\{L_j\}$  have linear span dense in  $\mathfrak{X}^*$ . Suppose furthermore that there is an integer  $N \geq 1$  such that*

$$\sum |L_j(x)|^N < \infty$$

*for all  $x = \sum L_j(x)x_j$  in  $\mathfrak{X}$ . Then there is an analytic injection of the (countable-dimensional) infinite polydisk  $\Delta^\infty$  into the fiber  $\mathfrak{M}_0$ .*

*Proof.* Define a map  $T_k$  of  $l^\infty$  into  $\mathfrak{X}$  by

$$T_k(y) = \sum_{n=1}^{\infty} 2^{-n} y_n x_{n+k}, \quad y \in l^\infty.$$

Then  $T_k$  injects  $\Delta^\infty$  into  $B$ . Since  $\mathfrak{M}$  is compact, the maps from  $B$  to  $\mathfrak{M}$  form a compact set in the topology of pointwise convergence. Let  $T$  be any map which is adherent to the sequence  $T_k$  as  $k \rightarrow \infty$ . Evidently  $f \circ T$  is analytic on  $B$  for all  $f \in H^\infty(B)$ , as it is a cluster point of the sequence  $f \circ T_k$  of bounded analytic functions on  $B$ . Thus  $T$  is an analytic mapping of  $\Delta^\infty$  into  $\mathfrak{M}$ .

If  $L \in \mathfrak{X}^*$  is a finite linear combination of the coordinate functionals  $L_j$ , then  $L \circ T_k$  converges pointwise to 0 on  $B$ , so that  $L \circ T = 0$ . Since these finite linear combinations are dense in  $\mathfrak{X}^*$ , every  $L \in \mathfrak{X}^*$  vanishes on the image of  $T$ , and the image of  $T$  is contained in the fiber  $\mathfrak{M}_0$ .

Suppose  $u$  and  $v$  are distinct points of  $\Delta^\infty$  with  $Tu = Tv$ . By the principle of uniform boundedness, there is  $C > 0$  such that  $\sum |L_j(x)|^N \leq C$  for all

$x \in B$ . Choose a net  $k_\alpha \rightarrow \infty$  such that  $T_{k_\alpha}$  converges pointwise on  $\Delta^\infty$  to  $T$ . For  $\lambda \in \mathbb{C}$  with  $|\lambda|=1$ , define an  $N$ -homogeneous analytic function  $f_\lambda$  on  $\mathfrak{X}$  by

$$f_\lambda(x) = \sum_{j=1}^{\infty} \lambda^j L_j(x)^N, \quad x \in \mathfrak{X}.$$

Evidently  $|f_\lambda| \leq C$  on  $B$ , and

$$f_\lambda(T_k(y)) = \lambda^k \sum_{j=1}^{\infty} \lambda^j (2^{-j} y_j)^N = \lambda^k f_\lambda(T_0(y)), \quad y \in \Delta^\infty, \quad k \geq 0.$$

We may assume  $\chi(\lambda) = \lim \lambda^{k_\alpha}$  exists for all  $|\lambda|=1$ . From the formula above we obtain  $f_\lambda(T(y)) = \chi(\lambda) f_\lambda(T_0(y))$  for all  $y \in \Delta^\infty$ . Since  $|\chi(\lambda)|=1$  and  $Tu = Tv$ , we have  $f_\lambda(T_0(u)) = f_\lambda(T_0(v))$  for all  $\lambda$ . Equating coefficients in the power series expansion, we obtain  $u_j^N = v_j^N$  for all  $j \geq 1$ . Replacing  $N$  by  $N+1$  in this argument, we also obtain  $u_j^{N+1} = v_j^{N+1}$  for all  $n \geq 1$ . It follows that  $u = v$ , and  $T$  is one-to-one. □

For subsequent embedding theorems, we will use the *Gleason metric*  $\rho$  on  $\mathfrak{M}$ , defined so that  $\rho(\varphi, \psi)$  is the supremum of  $|f(\varphi) - f(\psi)|$  over all  $f \in H^\infty(B)$  satisfying  $\|f\| \leq 1$  (see [Ga, Ch. VI]). The Gleason metric on  $B$ , regarded as a subset of  $\mathfrak{M}$ , is equivalent to the norm metric. In fact, it is a simple consequence of the Schwarz lemma that for any fixed  $r < 1$ , the Gleason metric is uniformly equivalent to the norm metric on the ball  $B_r = \{\|x\| \leq r\}$  via a bi-Lipschitz map.

A more natural problem, and a more difficult one, than the problem of simply embedding analytic structure is to embed analytic structure in  $\mathfrak{M}_0$  via an analytic injection which is uniformly bicontinuous with respect to the appropriate Gleason metrics. To obtain such an injection, we will look for limits along a sequence of subspaces.

Fix  $p > 1$ . A sequence  $\{\mathfrak{W}_n\}$  of subspaces of  $\mathfrak{X}^*$  satisfies an *upper  $p$ -estimate* if, for some constant  $C$ ,

$$\|\sum L_n\| \leq C \{\sum \|L_n\|^p\}^{1/p}$$

whenever  $L_n \in \mathfrak{W}_n$ ,  $n \geq 1$ . Let  $q = p/(p-1)$  be the conjugate index to  $p$ . Then, for any integer  $N \geq q$  and any functionals  $L_n \in \mathfrak{W}_n$  satisfying  $\|L_n\| \leq 1$ , the series

$$f = \sum_{n=1}^{\infty} L_n^N$$

converges to an analytic  $N$ -homogeneous function on  $\mathfrak{X}$  which satisfies

$$|f(x)| \leq C^q, \quad \|x\| \leq 1.$$

Indeed,  $|f(x)|$  is bounded by

$$\sum |L_n(x)|^N \leq \sum |L_n(x)|^q \leq \sup_{\sum |a_n|^p \leq 1} |\sum a_n L_n(x)|^q \leq C^q.$$

Let  $\mathfrak{U}$  be any free ultrafilter on the positive integers. Recall that the  *$\mathfrak{U}$ -ultraproduct* of a sequence of Banach spaces is the quotient space obtained

from their  $l^\infty$ -direct sum by dividing out by the subspace of sequences whose norms have limit 0 along the ultrafilter. The norm of an element  $\tilde{x}$  represented by a sequence  $(x_1, x_2, \dots)$  is given by  $\|\tilde{x}\| = \lim_{\mathcal{U}} \|x_n\|$ . As an example, the limit of the scalar products in  $l_n^2$  determines a scalar product which makes  $\lim_{\mathcal{U}} l_n^2$  into a Hilbert space of uncountable dimension. For background on ultraproducts see [He].

Recall that a subspace  $\mathfrak{W}$  of  $\mathfrak{X}^*$  is said to *c-norm* the subspace  $\mathfrak{E}$  of  $X$  if the supremum of  $|L(x)|$  over  $L \in \mathfrak{W}$ ,  $\|L\| \leq c$ , is at least  $\|x\|$  for every  $x \in \mathfrak{E}$ . One defines similarly what it means for  $\mathfrak{E}$  to *c-norm*  $\mathfrak{W}$ .

6.2. LEMMA. *Fix  $p > 1$  and  $C, c \geq 1$ . Let  $\{\mathfrak{E}_n\}$  be a sequence of subspaces of  $\mathfrak{X}$  and  $\{\mathfrak{W}_n\}$  a sequence of subspaces of  $\mathfrak{X}^*$  such that  $\mathfrak{W}_n$  *c-norms*  $\mathfrak{E}_n$  for each  $n$ , while  $\mathfrak{W}_n$  is orthogonal to  $\mathfrak{E}_k$  for all  $k \neq n$ . Assume that the sequence  $\{\mathfrak{W}_n\}$  satisfies an upper  $p$ -estimate, while the sequence  $\{\mathfrak{E}_n\}$  tends weakly to 0 (i.e., if  $x_n \in \mathfrak{E}_n$  is uniformly bounded then  $x_n \rightarrow 0$  weakly). Then, for any free ultrafilter  $\mathcal{U}$  on the positive integers, there is a natural analytic injection of the unit ball  $\tilde{B}$  of the  $\mathcal{U}$ -ultraproduct  $\tilde{\mathfrak{E}}$  of  $\{\mathfrak{E}_n\}$  into the fiber  $\mathfrak{M}_0$  of the spectrum of  $H^\infty(B)$  which is uniformly bicontinuous from the norm metric of any ball  $r\tilde{B}$  ( $r < 1$ ) to the Gleason metric of  $\mathfrak{M}$ .*

*Proof.* Suppose  $\tilde{x} \in \tilde{\mathfrak{E}}$  satisfies  $\|\tilde{x}\| < 1$ . Then  $\tilde{x}$  is represented by  $(x_1, x_2, \dots)$  with  $x_n \in \mathfrak{E}_n$  satisfying  $\|x_n\| < 1$ . Since  $\mathfrak{M}$  is compact,  $\lim_{\mathcal{U}} x_n$  exists in  $\mathfrak{M}$ . If  $(y_1, y_2, \dots)$  also represents  $\tilde{x}$ , then  $\lim_{\mathcal{U}} \|x_n - y_n\| = 0$ , so that  $\lim_{\mathcal{U}} \rho(x_n, y_n) = 0$  and consequently  $\lim_{\mathcal{U}} x_n = \lim_{\mathcal{U}} y_n$ . We denote this common limit by  $S(\tilde{x})$ . As before, one sees that  $S$  is an analytic map from  $\tilde{B}$  into  $\mathfrak{M}$ . Any such analytic map is nonexpansive from the Gleason metric of  $\tilde{B}$  determined by  $H^\infty(\tilde{B})$  to the Gleason metric of  $\mathfrak{M}$ . Since  $\{\mathfrak{E}_n\}$  tends weakly to 0,  $S$  maps into  $\mathfrak{M}_0$ .

It remains to check that  $S$  is one-to-one with uniformly continuous inverse. Take any  $\tilde{x}, \tilde{y}$  in  $\tilde{B}$  with  $\|\tilde{x} - \tilde{y}\| > \delta > 0$  and choose  $U \in \mathcal{U}$  so that  $\|x_n - y_n\| > \delta$  for all  $n \in U$ . For  $n \in U$  pick  $L_n$  in the unit ball of  $\mathfrak{W}_n$  so that

$$|L_n(x_n) - L_n(y_n)| > \delta/c.$$

Either for  $N = [q] + 1$  or  $N = [q] + 2$  the set

$$V = \{n \in U : |[L_n(x_n)]^N - [L_n(y_n)]^N| > \tau\}$$

(where  $\tau > 0$  depends on  $c$  and  $p$  but not on  $n$ ) is in the ultrafilter  $\mathcal{U}$ . Set

$$f = \sum_{n \in V} L_n^N.$$

Then by the earlier remark,  $f$  is in  $H^\infty(B)$  and  $\|f\| \leq C^q$ . But for each  $n \in V$ ,  $|[f(x_n)] - [f(y_n)]| = |[L_n(x_n)]^N - [L_n(y_n)]^N| > \tau$ , so  $|f(\tilde{x}) - f(\tilde{y})| \geq \tau$  and  $C^q \rho(S(\tilde{x}), S(\tilde{y})) \geq \tau$ . □

A Banach space  $\mathfrak{Y}$  is *finitely representable* in  $\mathfrak{X}$  if, for any  $c > 1$ , any finite-dimensional subspace of  $\mathfrak{Y}$  is isomorphic to a subspace of  $\mathfrak{X}$  via an iso-

morphism  $T$  which satisfies  $\|T\| < c$  and  $\|T^{-1}\| < c$ . The Banach space  $\mathfrak{X}$  is *superreflexive* if any Banach space finitely representable in  $\mathfrak{X}$  is reflexive. According to a theorem of Enflo, a Banach space is superreflexive if and only if it is uniformly convexifiable, and this occurs if and only if its dual space is so. For a discussion of this circle of ideas, see [Di1, pp. 86–87].

**6.3. THEOREM.** *If  $\mathfrak{X}$  is a superreflexive Banach space, then the unit ball of a nonseparable Hilbert space injects into the fiber  $\mathfrak{M}_0$  via an analytic map which is uniformly bicontinuous from the norm metric of the unit ball of the Hilbert space to the Gleason metric of its image in  $\mathfrak{M}$ .*

*Proof.* With Dvoretzky's theorem and a standard gliding hump argument, it is possible to construct for any infinite-dimensional Banach space  $\mathfrak{X}$  sequences of finite-dimensional subspaces  $\mathcal{E}_n$  of  $\mathfrak{X}$  and  $\mathfrak{W}_n$  of  $\mathfrak{X}^*$  such that the dimension of  $\mathcal{E}_n$  tends to  $\infty$ , each  $\mathfrak{W}_n$   $c$ -norms  $\mathcal{E}_n$ ,  $\mathfrak{W}_n$  is orthogonal to  $\mathcal{E}_k$  for all  $k \neq n$ , and the Banach–Mazur distance  $d(\mathcal{E}_n, l_n^2)$  tends to 1. The inductive step for constructing  $\{\mathcal{E}_n\}$  and  $\{\mathfrak{W}_n\}$  goes as follows. Suppose that the first  $n$  subspaces in each sequence have been constructed. Choose a finite-dimensional subspace  $\mathcal{E} \supset \bigcup_{k=1}^n \mathcal{E}_k$  of  $\mathfrak{X}$  which almost precisely norms the span of  $\bigcup_{k=1}^n \mathfrak{W}_k$  and choose a finite-dimensional subspace  $\mathfrak{W} \supset \bigcup_{k=1}^n \mathfrak{W}_k$  of  $\mathfrak{X}^*$  which almost precisely norms  $\mathcal{E}$ . A simple argument based on the Hahn–Banach theorem shows that  $\mathcal{E}^\perp$  almost 2-norms  $\mathfrak{W}^\perp$ . By Dvoretzky's theorem we may choose a subspace  $\mathcal{E}_{n+1}$  of  $\mathfrak{W}^\perp$  of arbitrarily large dimension which is almost isometric to  $l_{n+1}^2$ . Then choose  $\mathfrak{W}_{n+1} \subset \mathcal{E}^\perp$  finite-dimensional to almost 2-norm  $\mathcal{E}_{n+1}$ .

If  $\mathfrak{W}$  norms  $\mathcal{E}$  above with enough precision, a standard argument [Di2, pp. 38–39] shows that any sequence  $\{x_n\}$  with  $x_n$  a nonzero vector in  $\mathcal{E}_n$  forms a basic sequence. Since any normalized basic sequence in a reflexive Banach space tends weakly to 0 (because the coefficient functionals vanish on any weak adherent point), the sequence of subspaces  $\mathcal{E}_n$  tends weakly to 0.

If  $\mathcal{E}$  norms  $\mathfrak{W}$  above with enough precision, the same standard argument shows that any sequence  $\{L_n\}$  with  $L_n$  a unit vector in  $\mathfrak{W}_n$  forms a basic sequence, and furthermore the basis constants are uniformly bounded, independent of the normalized basic sequence. Thus the Gurarii–Gurarii–James theorem shows that any such sequence satisfies an upper  $p$ -estimate for some  $p > 1$ , and moreover by Theorem 2 of [Ja] (or by the discussion in [Di2]), the index  $p$  can be chosen independent of the normalized basic sequence  $\{L_n\}$ .

The hypotheses of Lemma 6.2 are now satisfied. Since  $d(\mathcal{E}_n, l_n^2) \rightarrow 1$ , any nontrivial ultraproduct of  $\{\mathcal{E}_n\}$  is a Hilbert space of uncountable dimension. If we restrict the map of Lemma 6.2 to a smaller ball, we obtain a uniformly bicontinuous injection.  $\square$

The same method of proof establishes the following.

6.4. THEOREM. *If  $\mathfrak{X}$  is superreflexive and  $l^p$  is finitely representable in  $\mathfrak{X}$  for some  $1 \leq p < \infty$ , then the unit balls of  $L^p[0, 1]$  and of  $l_{\aleph}^p$  inject into  $\mathfrak{M}_0$  via analytic maps which are uniformly bicontinuous from the norm of the unit ball to the Gleason metric of  $\mathfrak{M}$ .*

*Proof.* If  $l^p$  is finitely representable in  $\mathfrak{X}$  then it is also finitely representable in every finite codimensional subspace of  $\mathfrak{X}$ . The proof of Theorem 6.3 then shows that every ultraproduct of  $\{l_n^p\}$  injects into  $\mathfrak{M}_0$  via an analytic uniformly bicontinuous map, while  $L^p[0, 1]$  and  $l_{\aleph}^p$  inject isometrically into every nontrivial ultraproduct of  $\{l_n^p\}$  (see [Hn] or [He]). (To see that  $L^p[0, 1]$  injects, consider maps to  $l_n^p$  obtained by averaging functions over intervals of length  $1/n$ . To see that  $l_{\aleph}^p$  injects, consider for each  $0 < t \leq 1$  the  $\mathcal{U}$ -limit  $u_t$  of the sequence  $v_n \in l_n^p$  with only one nonzero entry, a 1 in the coordinate position  $[tn]$ , the integral part of  $tn$ .)  $\square$

In the case that  $\mathfrak{X}$  is  $L^p[0, 1]$  or  $l^p$  for  $1 < p < \infty$ , we can embed all of  $\mathfrak{M}$  into  $\mathfrak{M}_0$ . In fact, we have the following theorem.

6.5. THEOREM. *Suppose  $1 < p < \infty$ , and suppose  $\mathfrak{X}$  is isometric to an infinite  $l^p$ -direct sum of itself. Then there is a homeomorphism  $\Phi$  of  $\mathfrak{M}$  onto a compact subset of  $\mathfrak{M}_0$  satisfying  $f \circ \Phi \in H^\infty(B)$  for all  $f \in H^\infty(B)$ , which is uniformly bicontinuous with respect to the Gleason metric.*

*Proof.* For this proof, denote by  $\mathfrak{X}_j$  a copy of  $\mathfrak{X}$ , let  $\tilde{\mathfrak{X}}$  denote the  $l^p$ -direct sum of the  $\mathfrak{X}_j$ 's, and let  $\tilde{B}$  denote the unit ball of  $\tilde{\mathfrak{X}}$  which is by hypothesis isometric to  $B$ . Fix a free ultrafilter  $\mathcal{U}$  on the integers, and define an operator  $\Lambda$  from  $H^\infty(\tilde{B})$  to  $H^\infty(B)$  so that  $(\Lambda f)(x)$  is the pointwise limit along  $\mathcal{U}$  of the sequence of functions  $f(0, \dots, 0, x, 0, \dots)$ , where the  $x$  appears in the  $n$ th coordinate. Evidently  $\Lambda$  is linear and continuous and multiplicative, and  $\|\Lambda\| = 1 = \Lambda(1)$ . The dual map  $\Phi$  is a continuous map from  $\mathfrak{M}(B)$  into  $\mathfrak{M}(\tilde{B})$  satisfying  $f \circ \Phi \in H^\infty(B)$  for all  $f \in H^\infty(\tilde{B})$ . Since  $\tilde{\mathfrak{X}}^*$  is the  $l^q$ -direct sum of the duals of the  $\mathfrak{X}_j$ 's, it is clear that  $\Lambda(f) = 0$  for all  $f \in \tilde{\mathfrak{X}}^*$ . Hence the image of  $\mathfrak{M}(B)$  under  $\Phi$  is contained in  $\mathfrak{M}_0(\tilde{B})$ .

Let  $p_0$  be the least integer satisfying  $p_0 \geq p$ . Suppose  $g \in H^\infty(B)$  has order at least  $p_0$  at the origin, that is, the terms of the Taylor series of  $g$  at 0 of order less than  $p$  are all zero. Define an analytic function  $G$  on  $\tilde{\mathfrak{X}}$  by

$$G(x_1, x_2, \dots) = \sum_{j=1}^{\infty} g(x_j), \quad (x_1, x_2, \dots) \in \tilde{\mathfrak{X}}.$$

The estimate  $|g(x_j)| \leq \|g\|_B \|x_j\|^p$  leads to  $\|G\|_{\tilde{B}} \leq \|g\|_B$ . (In fact, equality holds here.) Evidently  $\Lambda(G) = g$ . Since such  $g$ 's separate the points of  $\mathfrak{M}(B)$ , the map  $\Phi$  is one-to-one, hence a homeomorphism onto its image.

The map  $\Phi$  is nonexpanding with respect to the Gleason metric. Let  $\varphi, \psi \in \mathfrak{M}(B)$ . Suppose  $h \in H^\infty(B)$  satisfies  $\|h\| \leq 1$  and  $|h(\varphi) - h(\psi)| = d$ . Then  $2^{-p_0}(h - h(0))^{p_0}$  and  $2^{-p_0-1}(h - h(0))^{p_0+1}$  vanish to order at least  $p$  at 0 and

have norms at most 1, and either one or the other of these functions satisfies  $|g(\varphi) - g(\psi)| \geq cd^{p+2}$ , where  $c$  depends only on  $p$ . Hence the corresponding  $G$  satisfies  $|G(\Phi(\varphi)) - G(\Phi(\psi))| \geq cd^{p+2}$ . This yields the uniform bicontinuity of  $\Phi$  with respect to the Gleason metric. □

Theorem 6.1 does not apply to the Banach space  $c_0$ . By making use of the automorphisms of the unit ball of  $c_0$ , we can still inject infinite-dimensional analytic objects in  $\mathfrak{M}_0$ .

**6.6. THEOREM.** *Let  $\mathfrak{X}$  be the Banach space  $c_0$  of null sequences. There is an analytic injection of the unit ball  $B^{**}$  of  $l^\infty$  into the fiber  $\mathfrak{M}_0$  which is an isometry from the Gleason metric of  $B^{**}$  to the Gleason metric of  $\mathfrak{M}$ .*

*Proof.* Choose  $0 < \alpha_k < 1$  such that  $\alpha_k$  increases to 1 and  $\sum(1 - \alpha_k) < \infty$ , and choose integers  $m_k$  such that  $m_{k+1} > m_k + k$ . Define  $\Phi_k: B^{**} \rightarrow B$  by

$$\Phi_k(w) = \left( 0, \dots, 0, \frac{\alpha_k - w_1}{1 - \alpha_k w_1}, \dots, \frac{\alpha_k - w_k}{1 - \alpha_k w_k}, 0, \dots \right), \quad w \in B^{**},$$

where the first block consists of  $m_k$  zeros. As in the proof of Lemma 6.2, the sequence  $\{\Phi_k\}$  has a pointwise limit along any fixed free ultrafilter  $\mathcal{U}$ , which determines an analytic map  $S$  of  $B^{**}$  into  $\mathfrak{M}$ . The map  $S$  is nonexpansive from the Gleason metric of  $B^{**}$  to the Gleason metric of  $\mathfrak{M}$ . Evidently the image of  $S$  annihilates  $l^1$ , and so is contained in the fiber  $\mathfrak{M}_0$  over 0.

Next observe that the Gleason metric  $\rho$  of  $B^{**}$  is expressed in terms of the Gleason metric  $\rho_\Delta$  of the open unit disk  $\Delta$  by

$$(6.1) \quad \rho(z, w) = \sup_j \rho_\Delta(z_j, w_j), \quad z, w \in B^{**}.$$

The estimate  $\rho_\Delta(z_j, w_j) \leq \rho(z, w)$  is obtained by considering functions which depend only on the  $j$ th coordinate of  $B^{**}$ . Taking the supremum over  $j$ , we obtain the inequality  $\geq$  in (6.1). To establish the reverse inequality, we can by composing each coordinate variable with an appropriate linear fractional transformation assume that  $z = 0$ . Consideration of the analytic disk

$$\lambda \rightarrow \frac{1}{\|w\|} (\lambda w_1, \lambda w_2, \dots), \quad |\lambda| < 1,$$

then easily yields the estimate  $\rho(0, w) \leq \rho_\Delta(0, \|w\|)$ , which coincides with the supremum of  $\rho_\Delta(0, w_j)$ . This establishes (6.1).

For  $N \geq i \geq 1$  define

$$g_{i,N}(z) = \prod_{j=N}^{\infty} \frac{\alpha_j - z_{i+m_j}}{1 - \alpha_j z_{i+m_j}}, \quad z \in B^{**}.$$

Our conditions on the  $\alpha_j$ 's guarantee that the product converges, and  $g_{i,N} \in H^\infty(B)$ . We compute that

$$g_{i,N}(\Phi_k(z)) = z_i \prod_{j \geq N, j \neq k} \alpha_j, \quad k \geq N.$$

Taking the limit as  $k$  tends to  $\infty$  through  $\mathcal{U}$ , we obtain

$$g_{i,N}(S(z)) = z_i \prod_{j \geq N} \alpha_j.$$

For  $z, w \in B^{**}$ , choose  $h \in H^\infty(\Delta)$  such that

$$|h| \leq 1 \quad \text{and} \quad \rho_\Delta(z_i, w_i) = |h(z_i) - h(w_i)|.$$

Then  $h \circ g_{i,N}$  belongs to  $H^\infty(B^{**})$ , and  $(h \circ g_{i,N})(S(z)) - (h \circ g_{i,N})(S(w))$  tends to  $\rho_\Delta(z_i, w_i)$  as  $N \rightarrow \infty$ . It follows that

$$\rho(S(z), S(w)) \geq \rho_\Delta(z_i, w_i), \quad i \geq 1.$$

Taking the supremum over  $i$  and using (6.1), we obtain

$$\rho(S(z), S(w)) \geq \rho(z, w), \quad z, w \in B^{**}.$$

Since  $S$  is nonexpansive, equality holds here, and the proof is complete.  $\square$

The analytic structure produced in Theorem 6.6 is by no means maximal. In fact, it is possible to fit  $S$  into a family  $\zeta \rightarrow S_\zeta$  of analytic injections, depending analytically on the parameter  $\zeta \in B^{**}$ , so that the image of  $B^{**}$  under  $S_\zeta$  is contained in the fiber over  $\zeta$ . To do this one simply introduces an analytic parameter  $\zeta$  in the definition of  $\Phi_k$ , setting

$$\Phi_k(\zeta, w) = \left( \zeta_1, \dots, \zeta_k, 0, \dots, 0, \frac{\alpha_k - w_1}{1 - \alpha_k w_1}, \dots, \frac{\alpha_k - w_k}{1 - \alpha_k w_k}, 0, \dots \right), \quad \zeta, w \in B^{**},$$

and one passes to the limit along the free ultrafilter as above, arriving thereby at the following result.

**6.7. THEOREM.** *Let  $\mathfrak{X} = c_0$ , and let  $B$  and  $B^{**}$  be the open unit balls of  $c_0$  and  $l^\infty$  respectively. Then there is a map*

$$\Phi: B^{**} \times B^{**} \rightarrow \mathfrak{M}$$

*such that:*

- (i)  $\Phi$  is continuous with respect to the norm topology on  $B^{**}$ ;
- (ii)  $f \circ \Phi$  is analytic on  $B^{**} \times B^{**}$  for all  $f \in H^\infty(B)$ ;
- (iii)  $\Phi(z, w) \in \mathfrak{M}_z$  for all  $z, w \in B^{**}$ ;
- (iv) the image of  $\Phi$  is disjoint from  $B^{**}$ ; and
- (v)  $\Phi$  is one-to-one.

### 7. Other Algebras Related to $H^\infty(B)$

To preserve consistency of point of view, we have limited our investigation to the algebra  $H^\infty(B)$ . Nonetheless, there are other natural algebras of analytic functions on  $B$ , and analytic structure in  $\mathfrak{M}$  is related to analytic structure in the maximal ideal spaces of these other algebras.



First of all, there is the algebra  $H_b(\mathfrak{X})$  (see [ACG]) which consists of all entire functions on  $\mathfrak{X}$  that are bounded on bounded sets and which has spectrum  $\mathfrak{M}_b(\mathfrak{X})$ . The restrictions of these entire functions to  $B$  generate a uniform algebra denoted  $H_{uc}^\infty(B)$ , the maximal ideal space of which can be identified with a compact subset of  $\mathfrak{M}_b$ . Lastly, there is  $A(B)$ , the infinite-dimensional analogue of the disk algebra, which is the uniformly closed algebra generated on  $B$  by  $\mathfrak{X}^*$ . It is easily seen that its maximal ideal space is just  $\bar{B}^{**}$ .

Dual to the inclusions  $A(B) \subset H_{uc}^\infty(B) \subset H^\infty(B)$ , there exist induced maps between the corresponding maximal ideal spaces:

$$\mathfrak{M} = \mathfrak{M}_{H^\infty(B)} \xrightarrow{\pi_1} \mathfrak{M}_{H_{uc}^\infty(B)} \xrightarrow{\pi_2} \mathfrak{M}_{A(B)} = \bar{B}^{**}.$$

Clearly,  $\pi_2 \circ \pi_1 : \mathfrak{M} \rightarrow \bar{B}^{**}$  is just the projection used elsewhere in this paper. It is important to look for analytic structure in the fibers of both  $\pi_1$  and  $\pi_2$ . A careful examination of the results given earlier in this paper shows that sometimes analytic structure lies in fibers of  $\pi_1$ , sometimes in those of  $\pi_2$ . For example, Corollary 5.2 admits the following refinement.

**7.1. THEOREM.** *Let  $\mathfrak{X}$  be an infinite-dimensional Banach space. Then  $\pi_1^{-1}(\zeta)$  contains an analytic disk for some  $\zeta \in \pi_2^{-1}(0)$ .*

*Proof.* As in the proof of Theorem 5.1, there exist a disk  $D \subset \mathbb{C}$  and analytic maps  $z_k : D \rightarrow B^{**}$  so that, for an appropriate subnet  $\{k_\alpha\}$ ,  $z = \lim_\alpha z_{k_\alpha}$  defines a one-to-one, analytic map  $z : D \rightarrow \mathfrak{M}_0$ . The one piece of new information we provide here is that  $f \circ z$  is constant on  $D$  for each  $f \in H_{uc}^\infty(B)$ . For that purpose, it suffices to assume  $f$  is homogeneous since the linear span of such functions is dense in  $H_{uc}^\infty(B)$ . In [DG] it is shown that the canonical extension of a homogeneous  $f$  is homogeneous and hence uniformly continuous with respect to the norm topology on  $B^{**}$ . By construction, the maps  $z_k$  also satisfy  $\|z_k(\lambda) - z_k(0)\| < c_k$  for  $\lambda \in D$  where  $c_k \rightarrow 0$ ; so uniform continuity implies that  $f \circ z$  is constant. Hence, we conclude that  $\pi_1 \circ z$  maps  $D$  to a single point  $\zeta \in \mathfrak{M}_{H_{uc}^\infty(B)}$ , where  $\pi_2(\zeta) = 0$ . □

In a very different direction, the results of Section 6 through Theorem 6.5 establish the existence of analytic maps into  $\mathfrak{M}_0$  by studying entire functions on  $\mathfrak{X}$ . For each of these theorems, we could simply replace  $\mathfrak{M}_0$  with  $\pi_2^{-1}(0)$ ; the proofs would remain unchanged. Consequently, these results give information about analytic structure in fibers of  $\pi_2$  and shed no additional light on analyticity in the fibers of  $\pi_1$ . Moreover, use of entire functions in Section 6 means that most of the arguments apply to the algebra  $H_b(\mathfrak{X})$  of entire functions as well as its spectrum  $\mathfrak{M}_b(\mathfrak{X})$ . For example, the proof to Theorem 6.3 shows the following.

**7.2. THEOREM.** *If  $\mathfrak{X}$  is a superreflexive Banach space, then a nonseparable Hilbert space injects into  $\mathfrak{M}_b(\mathfrak{X})$  via an analytic map.*

Finally, for  $\mathfrak{X} = c_0$ , it is known (see [ACG]) that  $H_{uc}^\infty(B) = A(B)$ . Hence the fibers of  $\pi_2$  are trivial, and the analytic structure described in Theorem 6.6 actually lies in  $\pi_1^{-1}(0)$ .

In conclusion, we mention that we know of no nontrivial restriction on the analytic structure in  $\mathfrak{M}_0$ . There are a number of test questions which can be raised. For instance, does there always exist infinite-dimensional analytic structure in  $\mathfrak{M}_0$ , or better yet, in  $\pi_1^{-1}(0)$ ? More specifically, does the unit ball  $\Delta^\infty$  of  $l^\infty$  inject analytically into  $\mathfrak{M}_0$  via a uniformly bicontinuous mapping when  $\mathfrak{X} = l^2$ ? A negative answer to the latter would suggest a real connection between the function theory and the geometry of the Banach space.

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