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A UNIFORM ALGEBRA OF ANALYTIC FUNCTIONS ON A BANACH SPACE

T. K. CARNE, B. COLE AND T. W. GAMELIN

ABSTRACT. Let $A(B)$ be the uniform algebra on the unit ball of a dual Banach space $\mathcal{X} = \mathcal{Y}^*$ generated by the weak-star continuous linear functionals. We focus on three related problems: (i) to determine when $A(B)$ is a tight uniform algebra; (ii) to describe which functions in $H^\infty(B)$ are approximable pointwise on B by bounded nets in $A(B)$; and (iii) to describe the weak topology of B regarded as a subset of the dual of $A(B)$. With respect to the second problem, we show that any polynomial in elements of \mathcal{Y}^{**} can be approximated pointwise on B by functions in $A(B)$ of the same norm. This can be viewed as a generalization of Goldstine's theorem. In connection with the third problem, we introduce a class of Banach spaces, called Λ -spaces, with the property that if $\{x_j\}$ is a bounded sequence in \mathcal{X} such that $P(x_j) \rightarrow 0$ for any m -homogeneous analytic function P on \mathcal{X} , $m \geq 1$, then $x_j \rightarrow 0$ in norm. We show for instance that a Banach space has the Schur property if and only if it is a Λ -space with the Dunford-Pettis property.

1. INTRODUCTION

We are interested in studying certain algebras of analytic functions on a Banach space. Our motivating problem is to determine when these algebras are tight. We obtain some partial results, and this leads along the way to a number of other problems involving Banach spaces and the analytic functions thereon.

We fix a (complex) Banach space \mathcal{Y} , with dual space $\mathcal{X} = \mathcal{Y}^*$. We will focus on one specific uniform algebra $A(B)$ associated with \mathcal{Y} , defined to be the algebra generated by the weak-star continuous linear functionals on the closed unit ball \bar{B} of \mathcal{X} . In other words, $A(B)$ is the uniform closure on \bar{B} of polynomials in the elements of \mathcal{Y} , regarded as functions on the weak-star compact set \bar{B} .

The functions in $A(B)$ are clearly analytic on B . It is easy to see that the spectrum of $A(B)$ coincides with \bar{B} . Arenson [1] has shown that the strong boundary points of $A(B)$ are precisely the complex extreme points of \bar{B} . Arenson also described the Gleason parts of $A(B)$, in a more general setting.

There are three types of questions which we shall focus on, and which are related. The first problem is to determine when $A(B)$ is a tight uniform algebra

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on \bar{B} . The notion of tightness was introduced in [2]. It corresponds to the solvability of a certain $\bar{\partial}$ -problem with sup-norm estimates and a mild gain in smoothness. We conjecture that if $A(B)$ is tight, then \mathcal{Z} is finite dimensional. We are able to show that $A(B)$ is not tight for many Banach spaces, such as the nonreflexive Banach spaces, and the L^p -spaces.

In developing criteria for tightness we are led to study the weak topology of B as a subset of the dual of $A(B)$, that is, the $A(B)^{**}$ -topology of B . One can describe the $A(B)^{**}$ -topology on B as the weak topology with respect to a certain algebra of bounded analytic functions on B , namely, the algebra $A_{pb}(B)$ of functions in $H^\infty(B)$ which can be approximated pointwise on B by bounded nets in $A(B)$. This leads to our second problem, to determine when $A_{pb}(B)$ coincides with $H^\infty(B)$. We will show that if \mathcal{Z} has the metric approximation property, then every $f \in H^\infty(B)$ is the pointwise limit of a net of functions in $A(B)$ with the same norm as f . Along the way we extend Goldstine's theorem on the approximation of linear functionals on the unit ball of a dual Banach space by weak-star continuous functionals, obtaining the analogous approximation result for polynomials in linear functionals by polynomials in weak-star continuous linear functionals.

Our third problem is to understand the $A(B)^{**}$ -topology of B . We show that the $A(B)^{**}$ -topology of B does not coincide with the norm topology unless \mathcal{Z} is finite dimensional. On the other hand, for certain Banach spaces \mathcal{Z} we show that the $A(B)^{**}$ -convergent sequences in B are norm convergent. This type of result can be regarded as a variant of Schur's theorem, that the weakly convergent sequences in l^1 are norm convergent.

The ideas related to the Schur property are treated in the context of an arbitrary Banach space. We define a Banach space \mathcal{Z} to be a Λ -space if whenever $\{x_j\}$ is a sequence in \mathcal{Z} such that $f(x_j) \rightarrow 0$ for every m -homogeneous analytic function f on \mathcal{Z} , for any $m \geq 1$, then $\|x_j\| \rightarrow 0$. The spaces l^p , $1 \leq p < \infty$, are Λ -spaces, as are the spaces $L^p[0, 1]$ for $2 \leq p < \infty$. The spaces $C(X)$ and $L^1[0, 1]$ are not Λ -spaces. We show that if \mathcal{Z} has the Dunford-Pettis property, then the weak topology of \mathcal{Z} coincides on any weakly compact set with the "weak polynomial topology", the topology of pointwise convergence for all m -homogeneous analytic functions, $n \geq 1$. It follows that \mathcal{Z} has the Schur property if and only if \mathcal{Z} is a Λ -space with the Dunford-Pettis property.

The paper is organized as follows. §§2 and 3 contain some preliminary observations and §4 is devoted to pointwise bounded approximation. The $A(B)^{**}$ -topology is discussed in §5. §§6 and 7 are devoted to Λ -spaces. In §8 we sharpen some results from [2] on tight algebras, and in §9 we consider the problem of when $A(B)$ is tight.

For background information on uniform algebras see [7], and for the bidual of a uniform algebra see [3] or [8]. Our reference for analytic functions on a Banach space is [11].

We will denote by \mathcal{X} an arbitrary Banach space. The open unit ball of \mathcal{X} will be denoted by $\text{ball } \mathcal{X}$, or by B when no confusion can arise. The closed unit ball is denoted by \bar{B} . The algebra of bounded analytic functions on B is denoted by $H^\infty(B)$. For $m \geq 1$, $\mathcal{P}_m(\mathcal{X})$ will denote the m -homogeneous analytic functions on \mathcal{X} . Thus $\mathcal{P}_1(\mathcal{X})$ is the dual \mathcal{X}^* of \mathcal{X} . We denote by $\mathcal{P}(\mathcal{X})$ the algebra of analytic polynomials, that is, finite linear combinations of elements of the \mathcal{P}_m 's. Finally, we will use upon occasion the pointed bracket $\langle \cdot, \cdot \rangle$ for the pairing between a Banach space and its dual, in no particular order.

2. PRELIMINARIES ON $H^\infty(B)$

In this section we note for future reference certain facts about the algebra $H^\infty(B)$ of bounded analytic functions on the unit ball B of an arbitrary Banach space \mathcal{X} .

The functions in $H^\infty(B)$ satisfy a Schwarz-type estimate of the form

$$(2.1) \quad |f(x) - f(y)| \leq 2\|f\|_B \|x - y\| / (1 - \|x\|), \quad x, y \in B.$$

Indeed, if g is analytic on a ball in \mathcal{X} with center x and radius $\delta > 0$, and $g(x) = 0$, then the Schwarz inequality yields

$$|g(y)| \leq \|g\| \|y - x\| / \delta, \quad \|y - x\| < \delta.$$

Applying this estimate to $f - f(x)$, with $\delta = 1 - \|x\|$, we obtain (2.1) above.

The estimate (2.1) shows in particular that the functions in $H^\infty(B)$ are uniformly continuous on any fixed subball rB , $0 < r < 1$, of B .

Let $f \in H^\infty(B)$. For each $x \in \mathcal{X}$, the function $f(\lambda x)$ has a power series expansion

$$(2.2) \quad f(\lambda x) = \sum_{m=0}^{\infty} f_m(x) \lambda^m,$$

converging normally for $|\lambda| < 1/\|x\|$. The coefficient functions f_m belong to $\mathcal{P}_m(\mathcal{X})$ and satisfy $\|f_m\|_B \leq \|f\|_B$. We refer to $f \sim \sum f_m$ as the *Taylor series expansion* of f . In view of the uniform continuity of f on each ball rB , $0 < r < 1$, we obtain easily the following lemma as a consequence of the corresponding results for analytic disks.

2.1. Lemma. *Let $f \in H^\infty(B)$ have power series expansion $f \sim \sum f_m$. Then the series $\sum f_m$ converges uniformly to f on each fixed ball rB , $0 < r < 1$. Furthermore, if $\{g_m\}$ is the sequence of Cesàro means of the partial sums of the series for f , then*

$$\|g_m\|_B \leq \|f\|_B,$$

and g_m converges uniformly to f on each ball rB , $0 < r < 1$.

3. THE BIDUAL OF $A(B)$

Now we turn to the fixed Banach space \mathcal{Y} and its dual \mathcal{Z} . The open unit ball of \mathcal{Z} will be denoted here by B . Recall that $A(B)$ is the uniform algebra

on \bar{B} generated by the weak-star continuous linear functionals, while $A_{pb}(B)$ is the algebra of functions on B which are approximable pointwise by bounded nets in $A(B)$. For $f \in A_{pb}(B)$ we define the triple-norm $|||f|||$ to be the infimum of constants $c > 0$ such that there is a net in $\text{cball } A(B)$ converging pointwise to f on B . Thus for each $f \in A_{pb}(B)$ there is a net $\{f_\alpha\}$ in $A(B)$ satisfying

$$\|f_\alpha\|_B \leq |||f|||, \quad f_\alpha(z) \rightarrow f(z), \quad z \in B.$$

On account of the uniform boundedness of the f_α 's, they converge normally on every analytic disk in B , so that f is analytic on any analytic disk in B . Since f is bounded, it is analytic on B , and

$$A_{pb}(B) \subseteq H^\infty(B).$$

The algebra $A_{pb}(B)$ is closely related to the bidual $A(B)^{**}$ of $A(B)$. Indeed, the point evaluations at points of B' extend weak-star continuously to $A(B)^{**}$, so that the functions $F \in A(B)^{**}$ determine point functions $F|_B$ on B . We claim that

$$(3.1) \quad A(B)^{**}|_B = A_{pb}(B),$$

and that

$$|||f||| = \inf\{\|F\| : F \in A(B)^{**}, F|_B = f\}, \quad f \in A_{pb}(B).$$

Indeed by Goldstine's theorem any $F \in A(B)^{**}$ is weak-star approximable by a net $\{f_\alpha\}$ in $A(B)$ satisfying $\|f_\alpha\|_B \leq \|F\|_{A(B)^{**}}$. The net $\{f_\alpha\}$ converges pointwise to $F|_B$ on B , so that $F|_B \in A_{pb}(B)$ and $|||F|_B||| \leq \|F\|_{A(B)^{**}}$. On the other hand, if $f \in H^\infty(B)$ is a pointwise limit of a net in $A(B)$ bounded by $|||f|||$, and if F is a weak-star adherent point of the net in $A(B)^{**}$, then $F|_B = f$, and $\|F\|_{A(B)^{**}} \leq |||f|||$.

Our main objective in this section is to record the following observation concerning the power series expansion of functions in $A_{pb}(B)$.

3.1. Theorem. *Suppose $f \in H^\infty(B)$ has Taylor series expansion $f \sim \sum f_m$. If f belongs to $A_{pb}(B)$, then so does f_m for all $m \geq 0$, and*

$$|||f_m||| \leq |||f|||, \quad m \geq 0.$$

Conversely, if each f_m belongs to $A_{pb}(B)$, and if the sequence $\{g_m\}$ of Cesàro means of the partial sums of $\sum f_m$ satisfies

$$\sup_m |||g_m||| < \infty,$$

then $f \in A_{pb}(B)$.

Proof. Let $f \in A_{pb}(B)$. Let $\{f_\alpha\}$ be a net in $A(B)$ converging pointwise to f on B , such that $\|f_\alpha\|_B \leq |||f|||$. Let f_α have Taylor expansion $\sum f_{\alpha m}$. Fix $x \in \mathcal{L}$ with $\|x\| \leq 1$. From the expansion (2.2) for $f(\lambda x)$, and the corresponding

expansion for $f_\alpha(\lambda x)$, we see that the coefficients $f_{\alpha m}(x)$ converge to $f_m(x)$ for each fixed $m \geq 0$. Furthermore,

$$|f_{\alpha m}(x)| \leq \|f_\alpha\|_B \leq \|f\|,$$

so that

$$\|f_{\alpha m}\|_B \leq \|f\|,$$

and

$$\|f_m\| \leq \|f\|.$$

This proves the first statement.

For the converse, suppose the Cesàro means g_m belong to $A_{pb}(B)$ and satisfy $\|g_m\| \leq c$, $m \geq 1$. Let $\epsilon > 0$ and let S be a finite subset of B . By choosing m so large that $|g_m - f| < \epsilon$ on S , and then finding $h_{S,\epsilon} \in A$ such that $\|h_{S,\epsilon}\| \leq c$ and $|h_{S,\epsilon} - g_m| < \epsilon$ on S , we obtain a net $\{h_{S,\epsilon}\}$ which converges pointwise boundedly to f on B . \square

There are two natural questions which arise. The first asks for which \mathcal{Z} is $A_{pb}(B) = H^\infty(B)$? The second is more technical, and asks when $H^\infty(B)$ is a direct summand of $A(B)^{**}$. In terms of the minimal idempotent P_B associated with the Gleason part B (cf. [8]), this problem boils down to determining when

$$P_B A(B)^{**} \cong H^\infty(B).$$

We address the first of these questions in the next section.

4. A GENERALIZATION OF GOLDSTINE'S THEOREM

In this section we aim to give a criterion under which $A_{pb}(B) = H^\infty(B)$. First we wish to establish the following approximation theorem, which can be viewed as an extension of Goldstine's theorem that ball \mathcal{Y} is weak-star dense in ball \mathcal{Y}^{**} .

4.1. Theorem. *Let B be the open unit ball of the dual Banach space $\mathcal{Z} = \mathcal{Y}^*$. Let f be a (finite) polynomial in elements of \mathcal{Y}^{**} . Then there is a net $\{f_\alpha\}$ of polynomials in elements of \mathcal{Y} such that*

$$\|f_\alpha\|_B \leq \|f\|_B, \quad \text{all } \alpha, \quad \lim f_\alpha(z) = f(z), \quad \text{all } z \in B.$$

The proof will be momentarily postponed. First we make some remarks.

Let $A_*(B)$ denote the algebra of uniform limits on B of polynomials in elements of \mathcal{Y}^{**} , i.e., $A_*(B)$ is the closed subalgebra of $H^\infty(B)$ generated by \mathcal{Y}^{**} . Because of the sharp estimate for the norms of the f_α 's, one can also approximate pointwise on B any function in $A_*(B)$. Thus Theorem 4.1 can be restated as asserting that ball $A(B)$ is dense in ball $A_*(B)$, in the topology of pointwise convergence on B . In particular, $A_*(B) \subseteq A_{pb}(B)$.

From the proof of Theorem 4.1 it will be seen that the f_α 's can be chosen to have the same degree as f . This can also be deduced directly from the statement of Theorem 4.1, by passing to the Cesàro means of partial sums associated with an arbitrary approximating net.

If f is homogeneous, then the f_α 's can be chosen to be homogeneous of the same degree as f . Again this can be seen from the proof, or it can be deduced directly from Theorem 4.1 by taking the appropriate term in the Taylor series of an arbitrary approximating net. Thus for instance if f is linear, then the f_α 's can be chosen to be linear, and the conclusion of Theorem 4.1 reduces to be Goldstine's theorem.

The proof of Theorem 4.1 will depend on the weak-star density of the unit ball of $\mathcal{B}(E, \mathcal{X})$ in that of $\mathcal{B}(E, \mathcal{X}^{**})$, whenever E is finite dimensional. This is Goldstine's theorem, applied to $\mathcal{B}(E, \mathcal{X})$, together with the equation

$$\mathcal{B}(E, \mathcal{X})^{**} \cong \mathcal{B}(E, \mathcal{X}^{**}).$$

For a proof of this equation, see the paper of Dean [4], where a short proof of the related "principle of local reflexivity" is given based on the equation. (See also [9, 13].) A dual version of the density statement is as follows.

4.2. Lemma. *Let \mathcal{X} be a Banach space, and let E be a finite dimensional Banach space. Let T be a continuous linear operator from \mathcal{X}^* to E . Then there is a net $\{T_\alpha\}$ of continuous linear operators from \mathcal{X}^* to E such that*

- (i) $T_\alpha(z) \rightarrow T(z)$ for all $z \in \mathcal{X}^*$,
- (ii) $T_\alpha(\text{ball } \mathcal{X}^*) \subseteq T(\text{ball } \mathcal{X}^*)$ for all α ,
- (iii) each T_α is continuous with respect to the weak-star topology of \mathcal{X}^* .

Proof. Replacing E by the range of T , we can assume that $T(\mathcal{X}^*) = E$. Replacing the norm on E by an equivalent norm, we can assume that $T(\text{ball } \mathcal{X}^*)$ coincides with ball E , so that in particular $\|T\| = 1$. The statement (ii) then becomes

$$(ii)' \quad \|T_\alpha\| \leq 1, \text{ all } \alpha.$$

Now we apply Goldstine's theorem, to approximate $T^* \in \mathcal{B}(E^*, \mathcal{X}^{**}) \cong \mathcal{B}(E^*, \mathcal{X})^{**}$ by a net $\{S_\alpha\}$ in $\mathcal{B}(E^*, \mathcal{X})$, so that $\|S_\alpha\| \leq 1$ and $\langle S_\alpha \varphi, z \rangle$ tends to $\langle T^* \varphi, z \rangle$ for all $\varphi \in E^*$ and $z \in \mathcal{X}^*$. Set $T_\alpha = S_\alpha^*$, regarded as an operator from \mathcal{X}^* to E . Then (ii)' and (iii) are valid, and furthermore

$$\lim \langle \varphi, T_\alpha z \rangle = \langle \varphi, Tz \rangle, \quad \varphi \in E^*, \quad z \in \mathcal{X}^*,$$

from which (i) follows. \square

Proof of Theorem 4.1. By hypothesis, f is a finite linear combination of monomials of the form $\varphi_1(z) \dots \varphi_k(z)$, where $\varphi_1, \dots, \varphi_k \in \mathcal{X}^{**}$. Let $\varphi_1, \dots, \varphi_N$ be the aggregate of all the functionals in \mathcal{X}^{**} appearing in the expression for f , and define a linear operator T from \mathcal{X}^* to \mathbb{C}^N by

$$T(z) = (\varphi_1(z), \dots, \varphi_N(z)), \quad z \in \mathcal{X}^*.$$

Let p be the polynomial in the coordinates of \mathbb{C}^N associated with f , so that

$$f(z) = p(T(z)), \quad z \in \mathcal{X}^*.$$

Evidently

$$|p(w)| \leq \|f\|_B, \quad w \in T(\text{ball } \mathcal{X}^*).$$

Let $\{T_\alpha\}$ be a net of linear operators from \mathcal{Z}^* to \mathbb{C}^N as in Lemma 4.2, and define

$$f_\alpha(z) = p(T_\alpha(z)), \quad z \in \mathcal{Z}^*.$$

From (4.1) and Lemma 4.2(ii) we obtain $|f_\alpha(z)| \leq \|f\|_B$ for all $z \in B$. By Lemma 4.2(i), $f_\alpha(z)$ tends to $f(z)$ for all $z \in B$.

Since T_α is continuous with respect to the weak-star topology of \mathcal{Z}^* , there are $x_\alpha^1, \dots, x_\alpha^N$ in \mathcal{Z} such that

$$T_\alpha(z) = (\langle z, x_\alpha^1 \rangle, \dots, \langle z, x_\alpha^N \rangle), \quad z \in \mathcal{Z}^*.$$

Then f_α is a polynomial in the weak-star continuous functionals on \mathcal{Z}^* corresponding to the x_α^j 's. \square

We wish to give now a concrete criterion for pointwise bounded approximation. Recall that a Banach space \mathcal{Z} has the *metric approximation property* if there is a net $\{T_\alpha\}$ of finite-rank operators on \mathcal{Z} such that $\|T_\alpha\| \leq 1$ and $T_\alpha(x) \rightarrow x$ for all $x \in \mathcal{Z}$. All reasonable Banach spaces, such as L^p -spaces and the spaces $C(X)$, have the metric approximation property.

4.3. Lemma. *Suppose that the Banach space \mathcal{Z} has the metric approximation property. Then any bounded analytic function f on ball \mathcal{Z} can be approximated there pointwise by a net $\{f_\beta\}$ of (finite) polynomials in elements of \mathcal{Z}^* , satisfying*

$$\|f_\beta\|_{\text{ball } \mathcal{Z}} \leq \|f\|_{\text{ball } \mathcal{Z}}.$$

Proof. Let the finite-rank operators T_α be as above. For each α , we can find a sequence $\{g_{\alpha,m}\}$ of analytic polynomials on the range of T_α , such that $g_{\alpha,m}(x) \rightarrow f(x)$ for all x in the range of T_α , and such that $|g_{\alpha,m}|$ is bounded on ball $T_\alpha(\mathcal{Z})$ by the supremum norm of f on ball \mathcal{Z} . Then the compositions $f_{\alpha,m} = g_{\alpha,m} \circ T_\alpha$ are the desired pointwise approximators to f . \square

The criterion we have been aiming at, guaranteeing that $A_{pb}(B) = H^\infty(B)$, now follows immediately upon combining Theorem 4.1 and Lemma 4.3.

4.4. Theorem. *Suppose that the dual Banach space \mathcal{Z} has the metric approximation property. Then for every $f \in H^\infty(B)$, there is a net $\{f_\alpha\}$ of polynomials in weak-star continuous linear functionals (i.e., $f_\alpha \in A(B)$) such that*

$$\|f_\alpha\|_B \leq \|f\|_B, \quad f_\alpha(z) \rightarrow f(z), \quad z \in B.$$

5. TOPOLOGIES FOR B ASSOCIATED WITH $A(B)$

We may regard the open unit ball B of \mathcal{Z} as a subset of the dual $A(B)^*$ of the uniform algebra $A(B)$, by identifying $z \in B$ with the evaluation functional φ_z at z . As such B inherits three topologies from $A(B)^*$ which are of concern to us: the Gleason topology (the relative norm topology from $A(B)^*$), the weak topology, or $A(B)^*$ -topology; and the Gelfand topology (the weak-star topology of $A(B)^*$). The Gelfand topology of B coincides with the relative weak-star

topology it inherits as a subset of the dual Banach space \mathcal{Z} . The Gleason topology of B is identified in the following lemma.

5.1. Lemma. *The Gleason topology of B (induced by the norm of $A(B)^*$) coincides with the norm topology of B (induced by the norm of \mathcal{Z}).*

Proof. The estimate (2.1) yields

$$\|\varphi_x - \varphi_y\|_{A(B)^*} \leq 2\|x - y\|/(1 - \|x\|), \quad x, y \in B.$$

On the other hand, the Hahn-Banach theorem provides us with $L \in \mathcal{Z}^*$ satisfying $\|L\| = 1$ and $L(x - y) = \|x - y\|$. The function $F(z) = L(z - y)$ belongs to $A(B)^{**}$ and satisfies $F(x) = \|x - y\|$, $F(y) = 0$, and $\|F\| \leq 2$. It follows that

$$\frac{1}{2}\|x - y\| \leq \|\varphi_x - \varphi_y\|_{A(B)^*},$$

so that the two topologies are indeed equivalent, with uniform Lipschitz estimates on each fixed subball rB , $0 < r < 1$. \square

In anticipation of our work on tight algebras, we wish now to investigate in more detail the $A(B)^{**}$ -topology of B . In view of (3.1), a net $\{z_\alpha\}$ in B converges to z in the $A(B)^{**}$ -topology if and only if $f(z_\alpha) \rightarrow f(z)$ for all $f \in A_{pb}(B)$.

5.2. Lemma. *Fix $0 < r < 1$, and let $\{z_\alpha\}$ be a net in B such that $\|z_\alpha\| \leq r$ for all α . Then z_α converges to z in the $A(B)^{**}$ -topology if and only if for all $m \geq 1$, $f(z_\alpha) \rightarrow f(z)$ for every m -homogeneous analytic function f in $A_{pb}(B)$.*

Proof. This follows from (3.1), and the fact (Lemma 2.1 and Theorem 3.1) that any function in $A_{pb}(B)$ can be approximated uniformly on rB by linear combinations of such f 's. \square

5.3. Theorem. *Suppose that \mathcal{Z} is infinite dimensional. Then for any $0 < r < 1$, the $A(B)^{**}$ -closure of the sphere $\{\|z\| = r\}$ coincides with the closed ball $r\bar{B}$. In particular, the $A(B)^{**}$ -topology for B is different from the Gleason topology.*

Proof. It is clear that the closure of the sphere is contained in the ball. Suppose $z_0 \in B$ satisfies $\|z_0\| < r$. Let $\alpha = \{f_1, \dots, f_k\}$ be any finite collection of homogeneous functions whose restrictions to B belong to $A_{pb}(B)$. For such an α determined by k functions, let \mathcal{Z}_α be any subspace of \mathcal{Z} of dimension $k + 1$ containing z_0 . Then the set

$$\{f_1 = f_1(z_0)\} \cap \dots \cap \{f_k = f_k(z_0)\} \cap \mathcal{Z}_\alpha$$

is a complex analytic variety in \mathcal{Z}_α of codimension at least 1 containing z_0 , with no compact branches. By the maximum principle there is then a point $y_\alpha \in \mathcal{Z}_\alpha$ such that $\|y_\alpha\| = r$ and y_α lies in the variety. The y_α 's form a net, with the ordering of the indices by inclusion. By construction $f(y_\alpha) = f(z_0)$ eventually, for any m -homogeneous function f in $A_{pb}(B)$. By Lemma 5.2, y_α converges to z_0 in the $A(B)^{**}$ -topology. \square

The same proof shows that if \mathcal{X} is any infinite dimensional Banach space and $0 < r < 1$, then the H^∞ (ball \mathcal{X})-closure of the sphere $\{\|x\| = r\}$ coincides with r ball \mathcal{X} .

From Theorem 5.3 one can construct a net $\{z_\alpha\}$ in B such that $\|z_\alpha\| \rightarrow 1$, while z_α converges to 0 in the $A(B)^{**}$ -topology. On the other hand, for sequences a version of the uniform boundedness principle holds.

5.4. Lemma. *If $\{z_j\}_{j=1}^\infty$ is a sequence in B which converges in the $A(B)^{**}$ -topology, then there is $0 < r < 1$ such that $\|z_j\| \leq r$ for all j .*

Proof. Suppose the lemma fails. Passing to a subsequence, we can assume that $\|z_j\| \rightarrow 1$. Then

$$\lim_{j \rightarrow \infty} \|\varphi_{z_j} - \varphi_0\|_{A(B)^*} = 2,$$

where φ_0 is the evaluation functional at 0, φ_{z_j} at z_j . By a general result on uniform algebras, to follow, the z_j 's contain an interpolating subsequence for $A(B)^{**}$, and consequently they cannot converge in the $A(B)^{**}$ -topology. \square

The general result for function algebras, appealed to above, is as follows.

5.5. Theorem. *Let A be a uniform algebra on a compact space X , let $x \in X$, and let $\{x_j\}$ be a sequence in X lying in the same Gleason part as x , such that*

$$\lim_{j \rightarrow \infty} \|\varphi_{x_j} - \varphi_x\| = 2,$$

the norm of the evaluation functionals being that of A^ . Then the sequence $\{x_j\}$ has an interpolating subsequence for A^{**} .*

Proof. Passing to a subsequence, we can assume that $\|\varphi_{x_j} - \varphi_x\|$ converges very rapidly to 2. In view of the relation between the Gleason metric (norm metric of A^*) and Harnack's inequality (cf. [7]), we can find functions $g_j \in A$ such that $\operatorname{Re} g_j > 0$, $g_j(x) = 1/2^j$, and $\operatorname{Re} g_j(x_j) > j$. Let ϕ be the conformal map of the right half-plane onto the open unit disk such that $\phi(1) = 0$ and $\phi(\infty) = 1$. Then

$$f_m = \phi(g_1 + \cdots + g_m)$$

belongs to A , $|f_m| \leq 1$, and $f_m(x) \rightarrow 0$. Let $F \in A^{**}$ be any weak-star accumulation point of the sequence $\{f_m\}$. Then $|F| \leq 1$, and $F(x) = 0$. Since

$$\operatorname{Re}(g_1 + \cdots + g_m)(x_j) > j, \quad m \geq j,$$

we have

$$f_m(x_j) \in \phi(\{\operatorname{Re} \lambda \geq j\}), \quad m \geq j,$$

and consequently $F(x_j)$ belongs to the closure of the image of the half-plane $\{\operatorname{Re} z \geq j\}$ under ϕ . It follows that

$$\lim_{j \rightarrow \infty} F(x_j) = 1.$$

Note that $|F(x_j)| < 1$, since x_j is in the same Gleason part as x . Passing to a subsequence, we can assume that $\{F(x_j)\}$ is an interpolating sequence for the

algebra $H^\infty(\{|\lambda| < 1\})$ of bounded analytic functions on the open unit disk. By composing such functions with F , we see that $\{x_j\}$ is then interpolating for A^{**} . \square

In the next section we wish to show that for some specific Banach spaces \mathcal{X} , the sequences in B which converge in the $A(B)^{**}$ -topology are precisely those which converge in norm. Instead of dealing with a dual Banach space, it will be convenient to work with an arbitrary Banach space \mathcal{X} and the algebra $H^\infty(\text{ball } \mathcal{X})$.

6. Λ -SPACES

We say that a net $\{x_\alpha\}$ in \mathcal{X} converges in the *weak polynomial topology* to x if $f(x_\alpha) \rightarrow f(x)$ for all analytic polynomials $f \in \mathcal{P}(\mathcal{X})$. If $\{x_\alpha\}$ converges in norm to x , then $\{x_\alpha\}$ is weak polynomial convergent to x . In turn, weak polynomial convergence implies weak convergence of a net, since $\mathcal{P}_1(\mathcal{X}) = \mathcal{X}^* \subseteq \mathcal{P}(\mathcal{X})$.

We define a Banach space \mathcal{X} to be a Λ -space if whenever $\{x_j\}_{j=1}^\infty$ is a sequence in \mathcal{X} which is weak polynomial convergent to 0, then $\|x_j\| \rightarrow 0$.

Recall that a Banach space has the *Schur property* if whenever $\{x_j\}$ is a sequence in \mathcal{X} which converges weakly to 0, then $\|x_j\| \rightarrow 0$. Any Banach space with the Schur property is evidently a Λ -space. Finite dimensional spaces, and the sequence space l^1 , all have the Schur property, so that these spaces are all Λ -spaces. Since the restriction of any polynomial on \mathcal{X} to a subspace is polynomial, we obtain immediately the following.

6.1. Theorem. *Any closed subspace of a Λ -space is a Λ -space.*

The Banach space c_0 of null sequences is not a Λ -space. Indeed, let $\{e_j\}$ be the sequence of standard basis vectors in c_0 . By the Littlewood-Pelczynski Theorem [10, 12] any $f \in \mathcal{P}(c_0)$ can be approximated uniformly on the unit ball of c_0 by finite polynomials in the coordinate functions. It follows that $f(e_j) \rightarrow 0$ for all $f \in \mathcal{P}(c_0)$. Since the e_j 's have unit norm, c_0 fails to be a Λ -space. Combining this with Theorem 6.1, we obtain the following useful criterion.

6.2. Theorem. *If the Banach space \mathcal{X} has a (closed) subspace isomorphic to c_0 , then \mathcal{X} is not a Λ -space.*

From this criterion, it follows for instance that no infinite dimensional uniform algebra can be a Λ -space. Indeed, if A is an infinite dimensional uniform algebra, and if $\varepsilon > 0$, then there is a linear isomorphism T of c_0 into A such that

$$(1 - \varepsilon)\|Tx\| \leq \|x\| \leq (1 + \varepsilon)\|Tx\|, \quad x \in c_0.$$

The proof of this well-known fact runs along the following lines. Let $\{U_j\}$ be a sequence of nonempty disjoint open subsets of the Shilov boundary of A .

Choose $g_j \in A$ such that $\|g_j\| = 1$, and $|g_j| < \varepsilon/2^{j+1}$ off U_j for $j \geq 1$. Define T on c_0 by

$$Tx = \sum x_j g_j, \quad x \in c_0.$$

One checks easily that T has the required properties.

Our aim in the remainder of this section and the next is to establish the following assertions:

(6.1) l^p is a Λ -space for $1 \leq p < \infty$.

(6.2) $L^p(\mu)$ is a Λ -space for $2 \leq p < \infty$.

(6.3) $L^1[0, 1]$ is not a Λ -space.

We do not know whether $L^p(\mu)$ is a Λ -space, for $1 < p < 2$.

The following theorem is a sharp version of (6.1).

6.3. Theorem. *Let $1 \leq p < \infty$, and let N be an integer satisfying $N \geq p$. Suppose $\{f_j\}_{j=1}^\infty$ is a sequence in l^p such that $f_j \rightarrow 0$ weakly, and $F(f_j) \rightarrow 0$ for every $f \in \mathcal{P}_N(l^p)$. Then $\|f_j\|_p \rightarrow 0$.*

Proof. The f_j 's are bounded, so we assume that $\|f_j\|_p \leq 1$ for all j . Note that for purposes of proving that $f_j \rightarrow 0$ in norm, we can pass to subsequences whenever convenient. We can also replace f_j by any sequence g_j which is asymptotic to f_j , in the sense that $\|f_j - g_j\|_p \rightarrow 0$.

Since $f_j \rightarrow 0$ weakly, also $f_j \rightarrow 0$ pointwise. We can then, by passing to a subsequence, assume that there are blocks $E_j = \{m_j \leq k \leq n_j\}$ of integers so that the blocks E_j are pairwise disjoint, and so that $\|f_j \chi_{E_j} - f_j\|_p < 1/2^j$. Replacing f_j by $f_j \chi_{E_j}$, we can assume then that $f_j = 0$ off E_j . Now define the N -homogeneous function F on l^p by

$$F(f) = \sum \langle f, \bar{f}_j |f_j|^{p-2} \rangle^N,$$

where

$$\langle f, g \rangle = \sum_{k=1}^\infty f(k)g(k).$$

Let $q = p/(p - 1)$ be the conjugate index to p . Since the l^p -norms of the f_j 's are bounded by 1, the l^q -norms of $\bar{f}_j |f_j|^{p-2}$ are also bounded by 1. Suppose $f \in l^p$ satisfies $\|f\|_p \leq 1$. Then using Hölder's inequality we obtain

$$\begin{aligned} \sum |\langle f, \bar{f}_j |f_j|^{p-2} \rangle|^N &\leq \sum_j \|f \chi_{E_j}\|_p^N \\ &= \sum_j \left[\sum_{k \in E_j} |f(k)|^p \right]^{N/p} \leq \sum_j \sum_{k \in E_j} |f(k)|^p \leq 1. \end{aligned}$$

This shows that the series defining $F(f)$ converges absolutely, and that F is bounded by 1 on the unit ball of l^p . By hypothesis, $F(f_j) \rightarrow 0$. Since

$$F(f_j) = \langle f_j, \overline{f_j} |f_j|^{p-2} \rangle^N = \|f_j\|_p^{pN},$$

we obtain $\|f_j\|_p \rightarrow 0$, as required. \square

Note that the condition $N \geq p$ in Theorem 6.3 is sharp. This follows from Pitt's theorem [14], which asserts that if $1 \leq N < p$, then every continuous N -linear form in l^p can be approximated uniformly by N -linear forms depending on only a finite number of coordinates. Indeed, suppose $1 < p < \infty$, and let $\{e_j\}$ be the sequence of basis vectors in l^p , so that $\|e_j\| = 1$ while $e_j \rightarrow 0$ weakly. If $N < p$, then by Pitt's theorem $F(e_j) \rightarrow 0$ for every $F \in \mathcal{P}_N$, and Theorem 6.3 fails for $N < p$.

Next we consider spaces of the form $L^p(\mu)$. Since Theorem 6.3 is valid for l^2 , it is valid for any Hilbert space, hence for $L^2(\mu)$. For other L^p -spaces we begin with the following sharper version of (6.2).

6.4. Theorem. *Let μ be a positive measure on some measure space, let $2 \leq p < \infty$, and let N be an integer satisfying $N \geq p$. Let $\{f_j\}$ be a sequence in $L^p(\mu)$ such that $f_j \rightarrow 0$ weakly, and $F(f_j) \rightarrow 0$ for every $F \in \mathcal{P}_N(L^p(\mu))$. Then $\|f_j\|_p \rightarrow 0$.*

Proof. Since the f_j 's are all carried by some σ -finite set, we can assume that the measure μ is σ -finite. Replacing μ by a mutually absolutely continuous measure, we can assume that μ is a probability measure. We further assume that the f_j 's are bounded by 1 in the norm of $L^p(\mu)$. Then they are also bounded by 1 in the norm of $L^2(\mu)$, and $f_j \rightarrow 0$ weakly in $L^2(\mu)$.

If G is an N -homogeneous analytic function bounded on ball $L^2(\mu)$, then G restricts to an N -homogeneous analytic function bounded on ball $L^p(\mu)$. Consequently $G(f_j) \rightarrow 0$. By the version of the result already proved for Hilbert space, we obtain $\|f_j\|_2 \rightarrow 0$. Passing to a subsequence, we can assume that f_j converges almost uniformly to 0.

Now we proceed as in the proof for the l^p -spaces. Passing to a subsequence, we construct by induction pairwise disjoint measurable sets E_j such that $\|f_j \chi_{E_j} - f_j\|_p \rightarrow 0$. Replacing f_j by $f_j \chi_{E_j}$, we can then assume that f_j is carried by E_j . Now define

$$\begin{aligned} F(f) &= \sum \langle f, \overline{f_j} |f_j|^{p-2} \rangle^N \\ &= \sum \left[\int_{E_j} f(t) \overline{f_j(t)} |f_j(t)|^{p-2} d\mu(t) \right]^N. \end{aligned}$$

As before the series converges absolutely to an $F \in \mathcal{P}_N$. By hypothesis, $F(f_j) \rightarrow 0$. Since $F(f_j) = \|f_j\|_p^{pN}$, we obtain $\|f_j\|_p \rightarrow 0$, as required. \square

If $L^p(\mu)$ is infinite dimensional, the condition $N \geq p$ is sharp. This can be seen by embedding l^p isometrically into $L^p(\mu)$ and applying Pitt's theorem as before.

Next we aim to verify (6.3), that $L^1[0, 1]$ is not a Λ -space. We will actually establish the following.

6.5. Theorem. *The following are equivalent, for a measure μ :*

- (i) $L^1(\mu)$ is a Λ -space.
- (ii) $L^1(\mu)$ has the Schur property.
- (iii) μ is atomic.

Proof. Schur's theorem asserts that l^1 has the Schur property, and the straightforward proof shows that $L^1(\mu)$ has the Schur property whenever μ is atomic. On the other hand, in the case μ is not atomic it is easy to construct a sequence in $L^1(\mu)$ which converges weakly to 0 but not strongly. In fact, any such $L^1(\mu)$ contains an isometric copy of $L^1[0, 1]$, where the functions $\sin(2\pi mt)$ converge weakly but not strongly to 0 as $m \rightarrow \infty$.

If a Banach space has the Schur property, then trivially it is a Λ -space. On the other hand, using the representation of the projective tensor product of L^1 -spaces as an L^1 -space of the corresponding product measure, it is a straightforward matter to check that if a sequence in $L^1(\mu)$ tends to zero weakly, then it is in fact a weak polynomial null sequence. Thus if the Schur property fails in $L^1(\mu)$, then $L^1(\mu)$ is not a Λ -space.

Rather than give more details here, we will prove in the next section a general result for spaces with the Dunford-Pettis property, which yields as a corollary the equivalence of weak and weak polynomial convergence of sequences in $L^1(\mu)$. \square

7. RELATION TO THE DUNFORD-PETTIS PROPERTY

Recall that the Banach space \mathcal{X} has the *Dunford-Pettis property* if whenever $\{x_j\}$ is a weak null sequence in \mathcal{X} and $\{\varphi_j\}$ is a weak null sequence in \mathcal{X}^* , then $\varphi_j(x_j) \rightarrow 0$. For a discussion of the Dunford-Pettis property, see [6]. Spaces which enjoy the Dunford-Pettis property are $C(X)$ and $L^1(\mu)$. Our main aim in this section is to establish the following theorem.

7.1. Theorem. *Suppose that \mathcal{X} has the Dunford-Pettis property. Then the restriction of any polynomial on \mathcal{X} to a weakly compact set is weakly continuous. In particular, the weak topology coincides with the weak polynomial topology on weakly compact sets.*

For the proof, we begin with the following simple consequence of the Eberlein-Schmulian theorem.

7.2. Lemma. *Let E be a weakly compact subset of a Banach space \mathcal{X} , and let h be a function from E to a Banach space \mathcal{Y} . Suppose that h is sequentially*

continuous with respect to the weak topologies of \mathcal{X} and \mathcal{Y} , that is, if $\{x_n\}$ is a sequence in E converging weakly to x , then $h(x_n)$ converges weakly in \mathcal{Y} to $h(x)$. Then h is continuous with respect to the weak topologies.

Proof. It suffices to show that the graph Γ of h is weakly compact in $\mathcal{X} \oplus \mathcal{Y}$.

Let $\{(x_n, h(x_n))\}$ be a sequence in Γ . Since E is weakly compact, there is by the Eberlein-Schmulian theorem a subsequence $\{x_{n_j}\}$ which converges weakly to $x \in E$. By the assumption on h , $h(x_{n_j})$ converges weakly in \mathcal{Y} to $h(x)$. Hence $\{(x_{n_j}, h(x_{n_j}))\}$ converges weakly in $\mathcal{X} \oplus \mathcal{Y}$ to $(x, h(x)) \in \Gamma$, and Γ is weakly sequentially compact in $\mathcal{X} \oplus \mathcal{Y}$. By the Eberlein-Schmulian theorem, Γ is weakly compact. \square

Note that the preceding lemma shows in particular that a scalar-valued function on the weakly compact set E is weakly continuous if and only if it is weakly sequentially continuous.

7.3. Lemma. *The following are equivalent, for a Banach space \mathcal{X} .*

- (i) *Each weakly compact subset of \mathcal{X} is weakly polynomially compact.*
- (ii) *The restriction of any polynomial to a weakly compact subset of \mathcal{X} is weakly continuous.*
- (iii) *Any polynomial on \mathcal{X} is weakly sequentially continuous.*
- (iv) *If $\{x_k\}_{k=1}^\infty$ is a weak null sequence in \mathcal{X} , then for any $m \geq 1$, $\{x_k \otimes \cdots \otimes x_k\}$ is a weak null sequence in the projective tensor product $\mathcal{X} \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{X}$ (m times).*
- (v) *For any $m \geq 1$, the function*

$$\theta: \mathcal{X} \rightarrow \mathcal{X} \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{X} \text{ (} m \text{ times)}$$

defined by $\theta(x) = x \otimes \cdots \otimes x$ is sequentially continuous, from the weak topology of \mathcal{X} to the weak topology of $\mathcal{X} \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{X}$.

- (vi) *The restriction of θ to any weakly compact subset of \mathcal{X} is continuous with respect to the weak topologies of \mathcal{X} and of $\mathcal{X} \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{X}$.*

Proof. The equivalence of (i) and (ii) is clear. That (iii) implies (ii) follows from Lemma 7.2, so that (ii) and (iii) are also equivalent. The equivalence of (v) and (vi) also follows from Lemma 7.2.

Suppose that (v) holds, and let f be an m -homogeneous analytic function on \mathcal{X} . Then f is the restriction to the diagonal of a continuous m -linear form F on \mathcal{X} . The space of such forms is equivalent to the dual of the m -fold projective tensor product $\mathcal{X} \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{X}$, via the correspondence $F \leftrightarrow L$, where $L(x_1 \otimes \cdots \otimes x_m) = F(x_1, \dots, x_m)$. Now f is the composition $L \circ \theta$ of the weakly sequentially continuous functions L and θ . It follows that f is weakly sequentially continuous, and (iii) is valid.

Suppose next that (iii) holds. Suppose $\{x_k\}$ is a sequence in \mathcal{X} converging weakly to x . Let $L \in (\mathcal{X} \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{X})^*$, and let F be the corresponding continuous m -linear form on \mathcal{X} . By (iii), $F(x_k, \dots, x_k) = L(\theta(x_k))$ converges

to $F(x, \dots, x) = L(\theta(x))$. Thus $\theta(x_k)$ converges weakly to $\theta(x)$, and (v) is valid.

Since (v) implies (iv) trivially, it remains only to show that (iv) implies (v).

Suppose (iv) holds, and fix $m \geq 1$. Let $\{x_k\}$ be a sequence in \mathcal{X} which converges weakly to x . Let $L \in (\mathcal{X} \widehat{\otimes} \dots \widehat{\otimes} \mathcal{X})^*$, and let F be the corresponding m -linear form, as above. Write $x_k = x + y_k$, where $\{y_k\}$ is a weak null sequence. Then

$$\begin{aligned} L(\theta(x_k)) &= F(x_k, \dots, x_k) = F(x + y_k, \dots, x + y_k) \\ &= F(x, \dots, x) + F(x, \dots, x, y_k) + \dots + F(y_k, \dots, y_k). \end{aligned}$$

Invoking the hypothesis (iv) for the q -fold projective tensor products of \mathcal{X} with itself, for $1 \leq q \leq m$, and noting that x is fixed, we see that each of the summands involving the y_k 's tends to 0. Hence $L(\theta(x_k)) \rightarrow F(x, \dots, x) = L(\theta(x))$, and (v) holds. \square

To prove Theorem 7.1, it suffices now to verify the condition (iv) of Lemma 7.3 in the case \mathcal{X} has the Dunford-Pettis property. In fact, the following is true.

7.4. Lemma. *Let $m \geq 1$, and let $\mathcal{X}_1, \dots, \mathcal{X}_m$ be Banach spaces, of which all but at most one have the Dunford-Pettis property. Let $\{x_{j,k}\}_{k=1}^\infty$ be a weak null sequence in \mathcal{X}_j , $1 \leq j \leq m$. Then $\{x_{1,k} \otimes \dots \otimes x_{m,k}\}_{k=1}^\infty$ is a weak null sequence in the projective tensor product $\mathcal{X}_1 \widehat{\otimes} \dots \widehat{\otimes} \mathcal{X}_m$.*

Proof. The lemma is trivial if $m = 1$. Suppose that $m \geq 2$, and that $\mathcal{X}_2, \dots, \mathcal{X}_m$ have the Dunford-Pettis property. Set $\mathcal{W} = \mathcal{X}_1 \widehat{\otimes} \dots \widehat{\otimes} \mathcal{X}_{m-1}$. By an induction hypothesis we may assume that the sequence $w_k = x_{1,k} \otimes \dots \otimes x_{m-1,k}$ is a weak null sequence in \mathcal{W} . Let $F \in (\mathcal{X}_1 \widehat{\otimes} \dots \widehat{\otimes} \mathcal{X}_m)^* = (\mathcal{W} \widehat{\otimes} \mathcal{X}_m)^*$. This dual space is isometric to the space of bilinear forms on $\mathcal{W} \times \mathcal{X}_m$. Now the bilinear forms on $\mathcal{W} \times \mathcal{X}_m$ correspond to operators $T: \mathcal{W} \rightarrow \mathcal{X}_m^*$ in such a way that F and T are related by

$$F(w \otimes x) = \langle T(w), x \rangle, \quad w \in \mathcal{W}, \quad x \in \mathcal{X}_m.$$

Now $T(w_k)$ is also a weak null sequence in \mathcal{X}_m^* . Since \mathcal{X}_m has the Dunford-Pettis property, we obtain

$$F(x_{1k} \otimes \dots \otimes x_{mk}) = \langle T(w_k), x_{m,k} \rangle \rightarrow 0.$$

This establishes the lemma, and the proof of Theorem 7.1 is thereby complete. \square

We remark that on the basis of Lemma 7.4, a stronger property than Lemma 7.3(vi) can be established for spaces with the Dunford-Pettis property. Indeed the same proof shows that the condition in Lemma 7.4 implies that the map

$$(x_1, \dots, x_m) \rightarrow x_1 \otimes \dots \otimes x_m$$

is weakly continuous from any weakly compact subset of $\mathcal{X}_1 \oplus \dots \oplus \mathcal{X}_m$ to the projective tensor product $\mathcal{X}_1 \widehat{\otimes} \dots \widehat{\otimes} \mathcal{X}_m$, equipped with the weak topology.

7.5. Theorem. *A Banach space \mathcal{X} has the Schur property if and only if \mathcal{X} is a Λ -space and \mathcal{X} has the Dunford-Pettis property.*

Proof. We have already observed that any Banach space with the Schur property is a Λ -space, and it is likewise trivial that the Schur property implies the Dunford-Pettis property. For the reverse implication, suppose that \mathcal{X} is a Λ -space with the Dunford-Pettis property. Let $\{x_k\}$ be a weak null sequence in \mathcal{X} . By Theorem 7.1, $\{x_k\}$ is a weak polynomial null sequence. By the definition of Λ -space, x_k converges to zero in norm, and consequently \mathcal{X} has the Schur property. \square

As remarked before, from Theorem 7.4 one sees that $C(X)$ and $L^1(\mu)$ are not Λ -spaces, when X is an infinite compact space and the measure μ is not atomic. Indeed, these spaces have the Dunford-Pettis property, while the Schur property is easily seen to fail.

8. TIGHT UNIFORM ALGEBRAS

In this section, we prove some preliminary lemmas on tight algebras and compactly tight algebras, in the context of an arbitrary uniform algebra A on a compact space X . The lemmas represent sharpened versions of some results in [2], designed to treat the algebras at hand.

For $g \in C(X)$ the Hankel-type operator S_g from A to $C(X)/A$ is defined by

$$S_g(f) = gf + A, \quad f \in A.$$

The algebra A is *tight* on X if the operators S_g are weakly compact, for all $g \in C(X)$. The algebra A is *compactly tight* if each S_g is a compact operator. We begin with the following.

8.1. Lemma. *Let A be a uniform algebra in a compact space X . Let $g \in C(X)$, and let $\{\nu_\alpha\}$ be a bounded net of measures on X such that $\nu_\alpha \perp A$ for all α , and ν_α converges weak-star to 0. If the operator S_g is weakly compact, then*

$$\lim_\alpha \int Fg \, d\nu_\alpha = 0$$

for all $F \in A^{**}$, that is, $g\nu_\alpha$ converges weakly to 0 in A^* . If S_g is compact, then the convergence is uniform over $F \in \text{ball}(A^{**})$, that is $g\nu_\alpha$ converges to 0 in the norm of A^* .

Proof. The adjoint of S_g is the operator

$$S_g^*: A^\perp \rightarrow A^*$$

given by the pairing

$$\langle S_g^*(\nu), f \rangle = \int fg \, d\nu, \quad f \in A.$$

If S_g is weakly compact, so is S_g^* , and the net $\{g\nu_\alpha\}$ is a weakly precompact subset of A^* . Since $\nu_\alpha \rightarrow 0$ weak-star, the only weak limit point of the net $\{g\nu_\alpha\}$ in A^* is 0, and consequently $\{g\nu_\alpha\}$ converges weakly to 0 in A^* .

If S_g is compact, then so is S_g^* , and $\{g\nu_\alpha\}$ is a norm compact subset of A^* . In this case we conclude that $\{g\nu_\alpha\}$ converges to 0 in the norm of A^* . \square

8.2. Lemma. *Let A be a uniform algebra on X , and let $\{x_\alpha\}$ be a net in X converging to $x_0 \in X$. Let μ_α be a representing measure on X for x_α such that the net $\{\mu_\alpha\}$ converges weak-star to a measure μ satisfying $\mu(\{x_0\}) = 0$. If A is tight on X , then*

$$\lim_{\alpha} F(x_\alpha) = F(x_0), \quad \text{all } F \in A^{**},$$

that is, $\{x_\alpha\}$ converges weakly to x_0 in A^* . If A is compactly tight on X , then the convergence is uniform over $F \in$ ball A^{**} , that is, $\{x_\alpha\}$ converges to x_0 in the norm of A^* .

Proof. Let δ_α be the point mass at x_α , δ_0 at x_0 . Then $\nu_\alpha = \mu_\alpha - \delta_\alpha$ converges weak-star to $\nu = \mu - \delta_0$. Let U be a neighborhood of x_0 such that $\mu(U) < \varepsilon$. Shrinking U , we can assume that $\mu_\alpha(U) < \varepsilon$ for α large, choose $g \in C(X)$ so that $0 \leq g \leq 1$, $g = 0$ on $X \setminus U$, and $g = 1$ in a neighborhood of x_0 . By Lemma 8.1,

$$(8.1) \quad \lim_{\alpha} \int F g d(\nu_\alpha - \nu) = 0, \quad F \in A^{**},$$

so that

$$\lim_{\alpha} \left[F(x_0) - F(x_\alpha) + \int gF d\mu_\alpha - \int gF d\mu \right] = 0, \quad F \in A^{**}.$$

Since $\|g\mu_\alpha\| \leq \varepsilon$ and $\|g\mu\| < \varepsilon$, we obtain

$$\limsup |F(x_\alpha) - F(x_0)| \leq \limsup \left| \int gF d\mu_\alpha - \int gF d\mu \right| \leq 2\varepsilon \|F\|.$$

This gives the desired pointwise convergence, with uniform convergence over ball A^{**} when the convergence in (8.1) is uniform. \square

An analytic disk in X will be for us a nonconstant map φ of the open unit disk Δ into X such that $f \circ \varphi$ is analytic on Δ for all $f \in A$. Then also $F \circ \varphi$ is analytic for all $F \in A^{**}$. Moreover, $\varphi(\Delta)$ is contained in a single Gleason part of A . As an application of the preceding lemma we give the following criterion for tightness, based on analytic disks.

8.3. Lemma. *Suppose A is tight on X . Let $\rho_\alpha: \Delta \rightarrow X$ be analytic disks, and suppose $\rho_\alpha(0) = x_\alpha$ converges to x_0 . Suppose also that for some $f \in A$ the net $f \circ \rho_\alpha$ converges to a nonconstant analytic function on Δ . Then $F(x_\alpha) \rightarrow F(x_0)$ for all $F \in A^{**}$. If moreover A is compactly tight, then this convergence is uniform over all $F \in$ ball A^{**} .*

Proof. Fix $r < 1$, and let σ_r be the normalized arc-length measure $d\theta/2\pi$ on the circle $\{|z| = r\}$ in Δ . Then $\nu_\alpha = \rho_\alpha^* \sigma_r$ is a representing measure on X

for x_0 . The ρ_α 's are locally equicontinuous. Let ρ be a weak limit of the ρ_α 's and let μ be a weak-star cluster point of the ν_α 's. Then $\mu = \rho^* \sigma_r$. Take $f \in A$ such that $f \circ \rho_\alpha$ does not converge to a constant. Then for large α , $f \circ \rho_\alpha \neq (f \circ \rho_\alpha)(0)$ a.e. $d\sigma_r$, so that $f \neq f(x_\alpha)$ a.e. $d\mu$, and $\mu(\{x_0\}) = 0$. Now we can apply Lemma 8.2. \square

Something can be recovered from the proof of Lemma 8.2 when the limit measure μ is allowed to have some mass at x_0 , as long as $\mu \neq \delta_0$. We will use M_A to denote the spectrum of A , and $M_{A^{**}}$ to denote the spectrum of the bidual algebra A^{**} .

8.4. Lemma. *Suppose that A is a tight uniform algebra in X . Let $\{x_\alpha\}$ be a net in X converging to $x_0 \in X$, and let μ_α be a representing measure on X for x_α such that $\{\mu_\alpha\}$ converges weak-star to μ . Suppose that $\mu(\{x_0\}) = c$, where $0 \leq c < 1$. Then the x_α 's eventually lie in the same Gleason part of M_A as x_0 . Any weak-star adherent point φ in $M_{A^{**}}$ of the net $\{x_\alpha\}$, regarded as weak-star continuous homomorphisms in $M_{A^{**}}$, belongs to the same Gleason part of $M_{A^{**}}$ as x_0 and satisfies $\|\varphi - x_0\| \leq 2c$, the norm being that of the dual space of A^{**} . Furthermore, if A is compactly tight, then*

$$\limsup_\alpha \|x_\alpha - x_0\| \leq 2c.$$

Proof. The first statement follows from [2, Theorem 10.2], that the operator $\lambda \rightarrow F\lambda$, $\lambda \in A^\perp$, is weak-star to weak-star continuous for each fixed $F \in A^{**}$. If we apply this to the idempotent $P \in A^{**}$ corresponding to the Gleason part of x_0 , we see that $P(\mu_\alpha - \delta_\alpha)$ converges a weak-star to $P(\mu - \delta_0)$. Now $P(\mu - \delta_0) = \mu - \delta_0$ is nonzero by hypothesis, while $P(\mu_\alpha - \delta_\alpha)$ is $\mu_\alpha - \delta_\alpha$ or 0 depending respectively on whether x_α is in the same Gleason part as x_0 or not. From this the first assertion follows.

For the second statement, choose U and g as in the proof of Lemma 8.2, so that $\mu(U) < c + \varepsilon$ and $\mu_\alpha(U) < c + \varepsilon$. The estimate (8.1) then yields

$$\limsup |F(x_\alpha) - F(x_0)| \leq 2(c + \varepsilon)\|F\|, \quad F \in A^{**}.$$

Since $F(\varphi)$ is a cluster point of $\{F(x_\alpha)\}$, we obtain

$$|F(\varphi) - F(x_0)| \leq 2c\|F\|, \quad F \in A^{**}.$$

This yields the norm estimate for $\varphi - x_0$, and it shows that φ and x_0 are in the same Gleason part of $M_{A^{**}}$.

Suppose now that A is compactly tight. By Lemma 8.1, the norm of $g(\mu_\alpha - \delta_\alpha - \mu + \delta_0)$ in A^* tends to 0. Since $g(x_\alpha) = 1$ for α large, we obtain for α large that

$$\sup_{F \in \text{ball } A^{**}} \left| \int Fg d\mu_\alpha - \int Fg d\mu - F(x_\alpha) + F(x_0) \right| < \varepsilon.$$

Now both $\int Fg d\mu_\alpha$ and $\int Fg d\mu$ are bounded in modulus by $\|F\|\mu_\alpha(U) \leq (c + \varepsilon)\|F\|$. From this we obtain

$$|F(x_\alpha) - F(x_0)| \leq \varepsilon + 2(c + \varepsilon)\|F\|$$

for α large. Hence

$$\|x_\alpha - x_0\| \leq 2c + 3\epsilon$$

for α large, and we obtain the final statement of the lemma. \square

A standard example from rational approximation theory (the roadrunner set) shows that we cannot expect x_α to converge to x_0 in Lemma 8.4.

9. WHEN IS $A(B)$ TIGHT?

Now we return to the uniform algebra $A(B)$ generated by the weak-star continuous linear functionals on the closed unit ball \bar{B} of the dual Banach space \mathcal{Z} . The preceding lemmas give us some information on \mathcal{Z} in the case in which $A(B)$ is tight on \bar{B} .

9.1. Theorem. *If $A(B)$ is tight on \bar{B} , then \mathcal{Z} is reflexive. Furthermore for each $0 < r < 1$ the $A(B)^{**}$ -topology on rB coincides with the weak-star topology on rB .*

Proof. Let $\{x_\alpha\}$ be a net in rB which converges weak-star to x . Fix $z \in \mathcal{Z}$ such that $0 < \|z\| < 1 - r$. Then the maps

$$\rho_\alpha(\lambda) = x_\alpha + \lambda z, \quad |\lambda| \leq 1,$$

are analytic disks in B , which converge weak-star to

$$\rho(\lambda) = x + \lambda z, \quad |\lambda| \leq 1.$$

If L is any weak-star continuous linear functional on \mathcal{Z} for which $L(z) \neq 0$, then $L \circ \rho$ is nonconstant. By Lemma 8.3, $\{x_\alpha\}$ converges to x in the $A(B)^{**}$ -topology. Hence the $A(B)^{**}$ -topology coincides with the weak-star topology on rB . Since the weak topology on rB is between the weak-star and the $A(B)^{**}$ -topologies, the weak-star and weak topologies coincide on rB . Thus by Alaoglu's theorem closed balls in B are weakly compact, and consequently \mathcal{Z} is reflexive. \square

9.2. Theorem. *If $A(B)$ is compactly tight, then \mathcal{Z} is finite dimensional.*

Proof. Let $\{x_\alpha\}$ be as above. Then $F(x_\alpha) \rightarrow F(x)$ uniformly over bounded subsets of $A(B)^{**}$, hence uniformly for all $F \in \mathcal{Z}^*$. Hence $\|x_\alpha - x\| \rightarrow 0$, and the weak-star topology coincides with the norm topology on rB . It follows that \bar{B} is norm compact, so \mathcal{Z} is finite dimensional. \square

The following criterion covers the ℓ^p spaces for $1 \leq p < \infty$.

9.3. Theorem. *Suppose that the dual Banach space \mathcal{Z} is a Λ -space, and that \mathcal{Z} has the metric approximation property. If $A(B)$ is tight, then \mathcal{Z} is finite dimensional.*

Proof. Suppose that \mathcal{Z} is infinite dimensional. By the Josefson-Nissenzweig theorem (see [5]), there is then a sequence $\{z_j\}_{j=1}^\infty$ in \mathcal{Z} such that $\|z_j\| = 1/2$, $j \geq 1$, and z_j converges weak-star to 0. By Theorem 9.1, z_j converges

to 0 in the $A(B)^{**}$ -topology. By Theorem 3.1, $A(B)$ is pointwise boundedly dense in $H^\infty(B)$, so that $f(z_j) \rightarrow 0$ for all $f \in H^\infty(B)$. Since \mathcal{Z} is a Λ -space, $\|z_j\| \rightarrow 0$. This contradiction establishes the theorem. \square

The following criterion covers the spaces of the form $L^p(\mu)$, $1 < p < \infty$.

9.4. Theorem. *Suppose that for some $1 < p < \infty$, the dual Banach space \mathcal{Z} has a quotient space isomorphic to ℓ^p . Then $A(B)$ is not tight.*

Proof. In view of Theorem 9.1, we can assume that \mathcal{Z} is reflexive. Let T be a continuous linear operator from \mathcal{Z} onto ℓ^p , such that $\|T\| = 1$. The j th component of $T(x)$ will be denoted by $L_j(x)$, so that $T(x) = \sum L_j(x)e_j$. Define

$$f(x) = \sum_{j=1}^{\infty} L_j(x)^N,$$

where N is an integer satisfying $N \geq p$. One checks that f is a pointwise limit of its partial sums, so that $f \in A_{pb}(B)$. On account of the open mapping principle, there exists $M > 0$ so that each e_j is of the form $T(z_j)$ for some $z_j \in \mathcal{Z}$ satisfying $\|z_j\| \leq M$. Set $y_j = z_j/(2M)$. Then $f(y_j) = (1/2M)^N$ is bounded away from 0. On the other hand, if z is any weak limit point of the sequence $\{y_j\}$, then $T(z)$ is a weak limit point of the sequence $e_j/(2M)$, so that $T(z) = 0$, and $f(z) = 0$. Thus f is not continuous with respect to the weak topology on $\frac{1}{2}B$, and by Lemma 8.2 $A(B)$ is not tight. \square

In closing, we mention the following criterion.

9.5. Theorem. *If $A(B)$ is tight on \overline{B} , then each point of the unit sphere $\overline{B} \setminus B$ of \mathcal{Z} is a (generalized) peak point for $A(B)$.*

Proof. Let $z \in \overline{B} \setminus B$, and let μ be a representing measure on \overline{B} for z . Then $\mu_r(E) = \mu(rE)$ defines a representing measure μ_r for rz , which converges weak-star to μ as $r \uparrow 1$. By [2], or Lemma 8.4, either $\mu = \delta_z$ or else rz lies in the same Gleason part as z eventually. The latter case is excluded, since B forms by itself a Gleason part for $A(B)$. Hence the only representing measure for z is the point mass δ_z , and z is a (generalized) peak point for A . \square

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