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REGULARITY AND ALGEBRAS OF ANALYTIC FUNCTIONS IN INFINITE DIMENSIONS

R. M. ARON, P. GALINDO, D. GARCÍA, AND M. MAESTRE

ABSTRACT. A Banach space E is known to be Arens regular if every continuous linear mapping from E to E' is weakly compact. Let U be an open subset of E, and let $H_b(U)$ denote the algebra of analytic functions on U which are bounded on bounded subsets of U lying at a positive distance from the boundary of U. We endow $H_b(U)$ with the usual Fréchet topology. $M_b(U)$ denotes the set of continuous homomorphisms $\phi : H_b(U) \to \mathbb{C}$. We study the relation between the Arens regularity of the space E and the structure of $M_b(U)$.

INTRODUCTION

For an open subset U of a complex Banach space E, let $H_b(U)$ denote the algebra of holomorphic functions f on U such that for all bounded subsets $B \subset U$ satisfying $\operatorname{dist}(B, E\backslash U) > 0$, $||f||_B = \sup_{x\in B} |f(x)| < \infty$. There is a countable family of such bounded sets B which exhausts U, and thus there is a countable family of norms $(|| \ ||_B)$ which determines a metric on $H_b(U)$. It is easy that with this metric $H_b(U)$ is a Fréchet algebra. In Sections 1 and 2, we will be interested in the set $M_b(U)$ of continuous homomorphisms $\phi : H_b(U) \to \mathbb{C}$. The most obvious homomorphism on $H_b(U)$ is point evaluation at a point $a \in U$, δ_a, and it is classical that in the finite dimensional setting U is a domain of holomorphy if and only if the only homomorphisms are such evaluations. The situation for infinite dimensional Banach spaces is usually quite different, even for the case $U = E$. In [2], for example, it was shown that there is a continuous, linear, multiplicative, mapping from $H_b(E)$ to $H_b(E'')$ which extends each function $f \in H_b(E)$ to $\tilde{f} \in H_b(E'')$. (The extension technique, which has been considerably studied and refined by [8], [11], [14], and [20], will be outlined in Section 1.) Thus, to each point a'' in E'' we can associate a homomorphism $\tilde{\delta}_{a''}$, by $\tilde{\delta}_{a''}(f) = \tilde{f}(a'')$. At this point, it is natural to ask whether this process could continue. Specifically, starting with $f \in H_b(E)$, could we not "doubly extend" f to $\tilde{\tilde{f}} \in H_b(E^{iv})$, and obtain a "new" homomorphism $\tilde{\tilde{\delta}}_{a^{iv}}$ corresponding to each point of the fourth dual E^{iv} of E, and so on? As we will

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see in Section 1, *Arens regularity* [1] plays a crucial role in determining whether or not elements of the fourth dual produce new homomorphisms. In Section 2, we will see that $M_b(U)$ has a natural analytic structure precisely when E is symmetrically regular. (This answers a problem which was discussed in [4].)

The algebra $H_b(E)$ was studied in [4], where it was found useful in studying the Banach algebra $H^\infty(B)$ of bounded analytic functions on the open unit ball B of E. In a natural way, $M_b(E)$ can be made into a convolution algebra, making it a semi-group with identity. In Section 3, we examine this convolution product, giving a partial solution to a question in [4] .

For background information on holomorphic functions in infinite dimensions, we refer to [10] or to [21]. Our general notation will follow [4]. In order to deal with many duals of a Banach space E and many transposes of linear maps $T : E \to F$, we have chosen the following convenient, albeit inconsistent, notation: E', E'', E''' denote the first, second, and third duals of E, respectively, while the fourth, etc., duals will be denoted E^{iv}, etc. Similarly, T^t, T^{tt}, T^{ttt} denote the first, second, and third transposes of T, respectively, while the fourth, etc., transpose of T will be denoted T^{iv}, etc.

1. Regularity and E^{iv}

Given a continuous, $n-$linear mapping $A : E \times \cdots \times E \to \mathbb{C}$, A can be extended to a continuous, $n-$linear mapping $\tilde{A} : E'' \times \cdots \times E'' \to \mathbb{C}$ by

$$\tilde{A}(x''_1, ..., x''_n) = \lim_{\alpha_1} \ldots \lim_{\alpha_n} A(x_{\alpha_1}, ..., x_{\alpha_n}),$$

where for each j, (x_{α_j}) is a net in E converging weak-* to x''_j. It will be important for us that, in general, $\tilde{A}$ will not be symmetric even if A is [1]. However, for every $x_1, \ldots, x_{p-1} \in E$ and $x''_{p+1}, \ldots, x''_n \in E''$ the mapping

$$x'' \in E'' \rightsquigarrow \tilde{A}(x_1, \ldots, x_{p-1}, x'', x''_{p+1}, \ldots, x''_n)$$

is $w(E'', E')-$continuous. Recalling that an n-homogeneous continuous polynomial $P : E \to \mathbb{C}$ is defined by the relation $P(x) = A(x, ..., x)$ for a unique symmetric continuous n-linear form A (see, e.g. [10]), we see that each such P has an extension, $\tilde{P} : E'' \to \mathbb{C}$. Moreover, $||P|| = ||\tilde{P}||$ ([8]) and so we may extend any $f \in H_b(E)$ to $\tilde{f} \in H_b(E'')$. It is routine that the extension mapping taking f to $\tilde{f}$ is continuous, linear, and multiplicative between the Fréchet algebras $H_b(E)$ and $H_b(E'')$. As a result, for each $x'' \in E''$, we can define an element $\tilde{\delta}_{x''} \in M_b(E)$ by $\tilde{\delta}_{x''}(f) = \tilde{f}(x'')$.

Of course, we can continue this procedure to obtain extensions of the original function to any even dual of E. In particular, for each $x^{iv} \in E^{iv}$ we can obtain an element $\tilde{\tilde{\delta}}_{x^{iv}} \in M_b(E)$, given by $\tilde{\tilde{\delta}}_{x^{iv}}(f) = \tilde{\tilde{f}}(x^{iv})$ for all $f \in H_b(E)$. However, by so doing, have we actually obtained a new element of $M_b(E)$? That is, is there some $x'' \in E''$ such that $\tilde{\delta}_{x''} = \tilde{\tilde{\delta}}_{x^{iv}}$? We will show in this section that the answer to this question is closely connected with the question of whether E is regular. Specifically, we show that if E is symmetrically regular, in a sense to be defined below, then every homomorphism obtained by evaluating a doubly-extended function $\tilde{\tilde{f}}$ at a point of the fourth dual of E is, in fact, equal to a homomorphism obtained by evaluating $\tilde{f}$ at a point of E''. On the other hand, we show that if E fails to be symmetrically regular then there are points of E^{iv} which yield "new" homomorphisms. Recall that E is said to be *regular* if every continuous bilinear $A : E \times E \to \mathbb{C}$ is Arens regular, (see, e.g. [1], [23]) i.e., the following two extensions of A to $E'' \times E''$ coincide:

$$(x'', y'') \to \lim_\alpha \lim_\beta A(x_\alpha, y_\beta)$$

and

$$(x'', y'') \to \lim_\beta \lim_\alpha A(x_\alpha, y_\beta),$$

(where (x_α), resp. (y_β), is a net in E converging weak-* to x'', resp. y''). Equivalently, E is regular if every bounded linear mapping from E to E' is weakly compact (see, e.g. [23]). E will be said to be *symmetrically regular* if the above extensions coincide for every continuous bilinear symmetric A. Equivalently, E is symmetrically regular if every continuous symmetric linear mapping $T : E \to E'$ is weakly compact. Recall that T is *symmetric* means that $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in E$.

We begin with the following simple but useful result. Observe that to each continuous bilinear form $A : E \times E \to \mathbb{C}$ one can associate a continuous linear mapping $T : E \to E'$ by $\langle Tx, y\rangle = A(x, y)$.

Lemma 1.1. *Let $A : E \times E \to \mathbb{C}$ be a continuous symmetric bilinear form, with associated linear mapping T, and let $x'' \in E''$ be fixed. Then $T^{tt}(x'') \notin E'$ if and only if there exists $y'' \in E''$ such that $\tilde{A}(x'', y'') \neq \tilde{A}(y'', x'')$.*

Proof. First, observe that if (y_β) converges weak-* to y'', then for any $x_\alpha \in E$ the following holds:

$$A(x_\alpha, y_\beta) = \langle Tx_\alpha, y_\beta\rangle = \langle Ty_\beta, x_\alpha\rangle = \langle y_\beta, T^t x_\alpha\rangle,$$

using the symmetry of A, so that by the weak-* density of E in E'',

$$\langle Tx_\alpha, y''\rangle = \langle T^{tt} y'', x_\alpha\rangle = \langle y'', T^t x_\alpha\rangle. \tag{1}$$

Suppose that there exists a y'' satisfying the condition of the lemma. Let (x_α) tend weak-* to x'' and let (y_β) tend weak-* to y''. Then

$$\tilde{A}(x'', y'') = \lim_\alpha \lim_\beta A(x_\alpha, y_\beta) = \lim_\alpha \lim_\beta \langle Tx_\alpha, y_\beta\rangle = \lim_\alpha \langle Tx_\alpha, y''\rangle. \tag{2}$$

Since $T^{tt} : (E'', \text{weak-*}) \to (E''', \text{weak-*})$ is continuous, it follows from (2) that

$$\tilde{A}(x'', y'') = \lim_\alpha \langle T^{tt} x_\alpha, y''\rangle = \langle T^{tt} x'', y''\rangle. \tag{2'}$$

Also, by (1),

$$\begin{aligned} \tilde{A}(y'', x'') &= \lim_\beta \lim_\alpha A(y_\beta, x_\alpha) \\ &= \lim_\beta \lim_\alpha \langle Ty_\beta, x_\alpha\rangle = \lim_\beta \langle Ty_\beta, x''\rangle = \lim_\beta \langle T^{tt} x'', y_\beta\rangle \end{aligned} \tag{3}$$

If $T^{tt}x''$ were an element of E', then (3)$= \lim_\beta \langle T^{tt} x'', y_\beta\rangle = (2')$, which is a contradiction.

The converse argument is similar. If $T^{tt}x'' \notin E'$, then there is $y'' \in E''$ and a net $(y_\beta) \subset E$ which tends weak-* to y'', but such that $\langle T^{tt}x'', y_\beta\rangle \not\to \langle T^{tt}x'', y''\rangle$. By (2′) , $\tilde{A}(x'', y'') = \langle T^{tt}x'', y''\rangle$, which by (3) and the above observation is different from $\tilde{A}(y'', x'') = \lim_\beta \langle T^{tt}x'', y_\beta\rangle$. Q.E.D.

Let $\rho : E^{iv} \to E''$ denote the natural restriction mapping of an element $x^{iv} \in (E''')'$ to E', so that every $x^{iv} \in E^{iv}$ can be written uniquely as $x^{iv} = \rho(x^{iv}) + w$, where w annihilates E'. As motivation for the following results, consider a continuous 2−homogeneous polynomial $P : E \to \mathbb{C}$. If E is symmetrically regular, then the associated linear mapping $T : E \to E'$ is weakly compact and, consequently, $T^{tt} : E'' \to E'''$, $T^{iv} : E^{iv} \to E^{v}$, etc. all map into E'. Because of this, it should not be surprising that the extension $\tilde{\tilde{P}}$ should in fact be equal to $\tilde{P} \circ \rho$.

Lemma 1.2. *If E is symmetrically regular, then for every symmetric continuous m-linear mapping $A : E \times \cdots \times E \to C$, $\tilde{\tilde{A}}(x_1^{iv}, \ldots, x_m^{iv}) = \tilde{A}(\rho(x_1^{iv}), \ldots, \rho(x_m^{iv}))$. In particular $\tilde{\tilde{A}}(x_1^{iv}, \ldots, x_m^{iv}) = 0$ if $x_j^{iv} \in E^{iv}$ is in $(E')^{\perp}$ for some $j = 1, ..., m$.*

Proof. First, recall (see, e.g., 8.3 Thm. [4]) that under the assumption of symmetric regularity, the extension of an arbitrary symmetric multilinear form in E to E'' is separately weak-*continuous and symmetric.

Fix $(x_1^{iv}, \ldots, x_m^{iv}) \in E^{iv} \times ... \times E^{iv}$. By definition, for every $z_1, \ldots, z_{m-1} \in E''$

$$\tilde{\tilde{A}}(z_1, \ldots, z_{m-1}, x_m^{iv}) = \lim_{x''_m \to x_m^{iv}} \tilde{A}(z_1, \ldots, z_{m-1}, x''_m),$$

where $x''_m \to x_m^{iv}$ in the $w(E^{iv}, E''')$-topology. But then $x''_m \to \rho(x_m^{iv})$ in the $w(E'', E')$-topology so by the weak-* separate continuity of $\tilde{A}$,

$$\tilde{\tilde{A}}(z_1, \ldots, z_{m-1}, x_m^{iv}) = \tilde{A}(z_1, \ldots, z_{m-1}, \rho(x_m^{iv})).$$

Next,

$$\begin{aligned}\tilde{\tilde{A}}(z_1, \ldots, z_{m-2}, x_{m-1}^{iv}, x_m^{iv}) &= \lim_{x''_{m-1} \to x_{m-1}^{iv}} \tilde{\tilde{A}}(z_1, \ldots, z_{m-2}, x''_{m-1}, x_m^{iv}) \\ &= \lim_{x''_{m-1} \to x_{m-1}^{iv}} \tilde{A}(z_1, \ldots, z_{m-2}, x''_{m-1}, \rho(x_m^{iv})) = \tilde{A}(z_1, \ldots, z_{m-2}, \rho(x_{m-1}^{iv}), \rho(x_m^{iv})),\end{aligned}$$

again by the weak-* separate continuity of $\tilde{A}$. The conclusion follows by repeating this argument m times. Q.E.D.

Theorem 1.3. *Suppose that E is symmetrically regular. Then to every point $x^{iv} \in E^{iv}$ corresponds a point $x'' \in E''$ such that $\tilde{\tilde{\delta}}_{x^{iv}} = \tilde{\delta}_{x''}$. Conversely, if E is not symmetrically regular, then there exists a point $x^{iv} \in E^{iv}$ for which $\tilde{\tilde{\delta}}_{x^{iv}} \neq \tilde{\delta}_{x''}$ for any $x'' \in E''$.*

Proof. First, we note that any point $x^{iv} \in E^{iv}$ can be expressed as $x^{iv} = x'' + z$, where $x'' = \rho(x^{iv})$ and z annihilates E'. If $\tilde{\tilde{\delta}}_{x^{iv}} = \tilde{\delta}_{y''}$ for some $y'' \in E''$, then necessarily $\tilde{\tilde{\phi}}(x^{iv}) = \tilde{\phi}(y'')$ for all $\phi \in E'$, so that $x''(\phi) + z(\tilde{\phi}) = y''(\phi)$ for all ϕ. Since $z(\tilde{\phi}) = 0$, the Hahn-Banach theorem implies that $x'' = y''$. In other words, given $x^{iv} \in E^{iv}$, the only possible point in E'' which can yield the same homomorphism is the point x'' arising from the decomposition of E^{iv}.

If E is symmetrically regular, it follows by Lemma 1.2 that for all n-homogeneous polynomials $P : E \to \mathbb{C}$ and all $x^{iv} \in E^{iv}$, we have $\tilde{\tilde{P}}(x^{iv}) = \tilde{P}(\rho(x^{iv}))$. Therefore, $\tilde{\tilde{\delta}}_{x^{iv}} = \tilde{\delta}_{\rho(x^{iv})}$ because the set of all continuous polynomials is dense in $H_b(E)$.

Conversely, suppose that E is not symmetrically regular, let $T : E \to E'$ be a continuous symmetric, non-weakly compact linear mapping, and let $P : E \to \mathbb{C}$ be the 2-homogeneous polynomial given by $P(x) = \langle Tx, x\rangle$. By ([12], Thm. VI.4.2) it follows that there is $x_0'' \in E''$ such that $T^{tt}(x_0'') \in E'''\backslash E'$. Hence there is $w \in (E')^{\perp} \subset E^{iv}$ such that $\langle T^{tt}x_0'', w\rangle \neq 0$. We will prove that there exists $\alpha \in \mathbb{C}$ such that $\tilde{\tilde{\delta}}_{x_0''+\alpha w}(P) \neq \tilde{\delta}_{x_0''}(P)$. To do this, first define $B : E \times E \to \mathbb{C}$ by $B(x,y) = \langle Tx, y\rangle$ and note that $\tilde{B}(x'', y'') = \langle T^{tt}x'', y''\rangle$ for all x'' and y'' in E''. By Lemma 1.1, $\tilde{B}$ is not symmetric. Define $C : E'' \times E'' \to \mathbb{C}$ to be the symmetrization of B, $C(x'', y'') = \frac{1}{2}[\tilde{B}(x'', y'') + \tilde{B}(y'', x'')]$. A calculation ([4], Lemma 8.1) then shows that the bilinear mapping $\tilde{C} : E^{iv} \times E^{iv} \to \mathbb{C}$ is given by $\tilde{C}(x^{iv}, y^{iv}) = \frac{1}{2}[\langle T^{iv}x^{iv}, y^{iv}\rangle + \langle \pi T^{iv}x^{iv}, y^{iv}\rangle]$, where $T^{iv} : E^{iv} \to E^{v}$ denotes the fourth transpose of T and $\pi : E^{v} \to E'''$ is the mapping which restricts an element of E^{v} to E''. In addition, we observe that $\tilde{P}(x'') = \tilde{B}(x'', x'') = C(x'', x'')$ and that $\tilde{\tilde{P}}(x^{iv}) = \tilde{C}(x^{iv}, x^{iv}) = \frac{1}{2}[\langle T^{iv}x^{iv}, x^{iv}\rangle + \langle \pi T^{iv}x^{iv}, x^{iv}\rangle]$.

Now,

$$\tilde{\tilde{P}}(x_0'' + \alpha w) = \frac{1}{2}[\langle T^{iv}x_0'', x_0''\rangle + \alpha\langle T^{iv}x_0'', w\rangle + \alpha\langle T^{iv}w, x_0''\rangle + \alpha^2\langle T^{iv}w, w, \rangle]$$

$$+\frac{1}{2}[\langle \pi T^{iv}x_0'', x_0''\rangle + \alpha\langle \pi T^{iv}x_0'', w\rangle + \alpha\langle \pi T^{iv}w, x_0''\rangle + \alpha^2\langle \pi T^{iv}w, w\rangle].$$

Note that $T^{iv}x_0'' = T^{tt}x_0''$, so that $\pi T^{iv}x_0'' = \pi T^{tt}x_0'' = T^{tt}x_0''$. In addition,

$$\langle T^{iv}w, x_0''\rangle \ = \langle w, T^{ttt}x_0''\rangle = \langle w, T^{t}x_0''\rangle = 0$$

since $w \in (E')^{\perp}$. Moreover, since $\pi T^{iv}w = T^{iv}w|_{E''}$, we also have that $\langle \pi T^{iv}w, x_0''\rangle = 0$. Therefore,

$$\begin{aligned}\tilde{\tilde{P}}(x_0'' + \alpha w) &= \tilde{P}(x_0'') + \alpha\langle T^{tt}x_0'', w\rangle + \frac{\alpha^2}{2}[\langle T^{iv}w, w > + < \pi T^{iv}w, w\rangle] \\ &= \tilde{\delta}_{x_0''}(P) + \alpha\langle T^{tt}x_0'', w\rangle + \alpha^2\tilde{\tilde{\delta}}_{w}(P),\end{aligned}$$

which is not equal to $\tilde{\delta}_{x_0''}(P)$ for all except at most two values of $\alpha \in \mathbb{C}$. Q.E.D.

Remark 1.4. (a). For certain Banach spaces, there is a continuous symmetric non-weakly compact operator $T : E \to E'$ such that $T^{tt}(E''\backslash E) \subset E'''\backslash E'$. For such a *Tauberian* [13] symmetric operator T, choose $x_0'' \in E''\backslash E, w \in (E')^{\perp}$, and $\alpha \in C$ such that $\tilde{\tilde{P}}(x_0'' + \alpha w) - \tilde{P}(x_0'') \neq 0$. Since the function $\theta_{\alpha,w} : x'' \in E'' \to \tilde{\tilde{P}}(x'' + \alpha w) - \tilde{P}(x'')$ is analytic, it follows that $\theta_{\alpha,w}$ is non-zero on a dense open subset of E''. However, we do not know if $\theta_{\alpha,w}(0) \neq 0$ or even if $\theta_{\alpha,w}|_E$ can be identically 0.

(b). Note that the argument in Theorem 1.3 only shows that if E fails to be symmetrically regular then $\tilde{\tilde{\delta}}_{x^{iv}}$ is a "new" homomorphism for *some* $x^{iv} \in E^{iv}\backslash E''$. We do not know of conditions which imply that *every* element of $E^{iv}\backslash E''$ yields such a homomorphism. For example, not every point of $(\ell_1)^{iv}$ yields a new homomorphism. Indeed, if this were not the case, then the mapping

$$x^{iv} \in (\ell_1)^{iv} \to (\tilde{\tilde{f}}(x^{iv}))_{f\in H_b(\ell_1)} \in \prod_{f\in H_b(\ell_1)} \mathbb{C}$$

would be one-to-one. However, since ℓ_1 is separable, the space $H_b(\ell_1)$ has cardinality c and, consequently, the cardinality of $\Pi_{f\in H_b(\ell_1)}\mathbb{C}$ is 2^c. Since the cardinality of ℓ_1^{iv} is at least 2^{2^c} ([9], p. 211), we have a contradiction. This type of cardinality argument can be extended to certain non-regular Banach spaces E, to show that in general not every element of every even dual of E yields a new homomorphism.

(c). Even if it were true that every element $x^{iv} \in E^{iv}\backslash E''$ yielded a homomorphism $\tilde{\tilde{\delta}}_{x^{iv}} \notin \{\tilde{\delta}_{x''} \ : \ x'' \in E''\}$, it is not at all clear that the process could be continued to find points $x^{vi} \in E^{vi}\backslash E^{iv}$ which correspond to new homomorphisms. Indeed, since the elements in E' do not separate points in E^{iv}, the argument given at the start of the above proof, showing that there is at most one "candidate" in E'' corresponding to a point in E^{iv}, does not extend to this case.

(d). If E is symmetrically regular, then no point of any even dual of E yields a new homomorphism. In fact for E^{vi} we have that

$$\tilde{\tilde{\tilde{A}}}(x_1^{vi}, \ldots, x_m^{vi}) = \tilde{A}(\rho(x_1^{vi}), \ldots, \rho(x_m^{vi}))$$

where $\rho(x_j^{vi})$ denotes the restriction of x_j^{vi} to E'. The same argument used in Lemma 1.2 yields this for E^{vi}, and the same proof works for each succesive even dual space. Since each even dual of a C^*-algebra is also a C^*-algebra, this remark is also seen to be true, in the context of C^*-algebras, by repeated application of Theorem 1.3. We are grateful to Peter Harmand for pointing out that in general the bidual of a symmetrically regular Banach space may fail to be symmetrically regular and so this argument cannot be applied in general. Since by ([17], Chapter III Example 3.5,5) the Banach space $E = (c_0(\ell_1^n), \| \ . \ \|_\infty)$ has property (V) of Pełczyński, it follows by [15] (see also [14], 7.6. Example) that E is regular. However, E'' is isometrically isomorphic to the space $l_\infty(\ell_1^n)$ which contains as a complemented subspace a copy of ℓ_1, and so E'' is not symmetrically regular.

Whether or not E is symmetrically regular, there are in general (many) more homomorphisms in $M_b(E)$ than those obtained by evaluating at points of E'' :

Proposition 1.5 (cf. Theorem 7.2 [5])**.** *If E is a Banach space and there is a polynomial $P \in P(^nE)$ which is not weakly continuous on bounded sets, then there exists a homomorphism θ which does not belong to E''.*

Proof. We first observe that if $(x_i)_{i\in I}$ is any bounded subset of E and $\mathcal{U}$ an ultrafilter on I then the function defined as the limit of $(f(x_i))$ along the ultrafilter $\mathcal{U}$,

$$\begin{array}{rccl} \theta_{\mathcal{U}}: & \mathcal{H}_b(E) & \longrightarrow & C \\ & f & \rightsquigarrow & \theta_{\mathcal{U}}(f) := \lim_{\mathcal{U}} f(x_i), \end{array}$$

is an element of $M_b(E)$.

By hypothesis, there exists a bounded net $(x_d)_{d\in(D,\leq)}$ which is weakly convergent to a point $x_0 \in E$ such that $P(x_d)$ does not converge to $P(x_0)$ in $\mathbb{C}$. Considering a subnet and multiplying by a scalar, if necessary, we can suppose that $|P(x_d) - P(x_0)| \geq 1$ for every $d \in D$. On D we take the base of filter $\mathcal{B} := \{\{d \in D \ : \ d \geq d_0\}, \ d_0 \in D\}$. Let $\mathcal{U}$ be an ultrafilter such that $\mathcal{B} \subset \mathcal{U}$. We claim that $\theta_{\mathcal{U}} \notin E''$. To see this, suppose instead that there exists $x'' \in E''$ such that $\tilde{\delta}_{x''} = \theta_{\mathcal{U}}$. Since E' is a subset of $\mathcal{H}_b(E)$ we have

$$x''(x') = \tilde{\delta}_{x''}(x') = \theta_{\mathcal{U}}(x') \text{ for every } x' \in E'. \tag{4}$$

On the other hand, given $x' \in E'$ and $\epsilon > 0$ there exists $U \in \mathcal{U}$ verifying $|\theta_{\mathcal{U}}(x') - x'(x_d)| < \epsilon$ for every $d \in U$, and since (x_d) is weakly convergent to x_0, we can find a $d_0 \in D$ such that $\mid x'(x_d) - x'(x_0) \mid < \epsilon$ for every $d \geq d_0$, $d \in D$. But $U \cap \{d \in D \ : \ d \geq d_0\} \neq \emptyset$, hence $\mid \theta_{\mathcal{U}}(x') - x'(x_0) \mid < 2\epsilon$ for every $\epsilon > 0$. Thus

$$\theta_{\mathcal{U}}(x') = x'(x_0) \text{ for every } x' \in E'. \tag{5}$$

As a consequence of (1) and (2) and the fact that E' separates points of E'' we obtain that $x'' = x_0$. Hence, since we are assuming that $\theta_{\mathcal{U}} = \tilde{\delta}_{x''}$, $\theta_{\mathcal{U}}(P) = \delta_{x_0}(P) = P(x_0)$. But since $\theta_{\mathcal{U}}(P) = \lim_{\mathcal{U}} P(x_d)$ and $|P(x_d) - P(x_0)| \geq 1$ for every $d \in D$, we have $|\theta_{\mathcal{U}}(P) - P(x_0)| \geq 1$, which is a contradiction. Q.E.D.

Remark 1.6. The hypothesis in Proposition 1.5 is fulfilled if ℓ_2 is a quotient of E. To see this, take the $2-$homogeneous polynomial $P \circ q : E \to \mathbb{C}$, where q is the quotient map and $P(a) = \sum_{n=1}^{\infty} a_n^2$ for $a = (a_n) \in \ell_2$. Let $(x_n) \subset E$ be a bounded sequence such that $q(x_n) = e_n$ for every n. Then the set $\{x_n \ : n \in \mathbb{N}\}$ is weakly precompact, and therefore $0 \in E$ is a weak cluster point of $\{(x_n - x_m) \ : \ m \neq n, \ m, n \in \mathbb{N}\}$. If $P \circ q$ were weakly continuous on bounded sets, then it would follow that $0 = P \circ q(0)$ would be a cluster point of $\{P \circ q(x_n - x_m) \ : \ n \neq m\} = \{2\}$. In particular, the hypothesis holds whenever E contains an isomorphic copy of ℓ_1 [6].

We devote the final part of this section to studying the relationship between regularity and symmetric regularity.

Proposition 1.7. *Consider E and F two Banach spaces. $E \times F$ is a regular space if and only if every map in any of the following four spaces $\mathcal{L}(E, E')$, $\mathcal{L}(E, F')$, $\mathcal{L}(F, E')$, $\mathcal{L}(F, F')$ is weakly compact.*

Proof. Suppose that the four spaces of functions have the property. Let $T = (T_1, T_2) : E \times F \longrightarrow E' \times F'$ be a continuous linear mapping. Then

$$T(x, y) = (T_1(x, 0), 0) + (T_1(0, y), 0) + (0, T_2(x, 0)) + (0, T_2(0, y))$$

for every $(x, y) \in E \times F$. We define

$$R_1 : E \longrightarrow E' \text{ by } R_1(x) := T_1(x, 0), \qquad x \in E,$$

$$R_2 : F \longrightarrow E' \text{ by } R_2(y) := T_1(0, y), \qquad y \in F,$$

$$R_3 : E \longrightarrow F' \text{ by } R_3(x) := T_2(x, 0), \qquad x \in E,$$

$$R_4 : F \longrightarrow F' \text{ by } R_4(y) := T_2(0, y), \qquad y \in F.$$

Since R_j, $j = 1, 2, 3, 4$, are weakly compact operators and since

$$T(x, y) = (R_1(x), 0) + (R_2(y), 0) + (0, R_3(x)) + (0, R_4(y))$$

for every $(x, y) \in E \times F$, the conclusion holds.

The other implication is clear. Q.E.D.

Corollary 1.8. (1) *If E is regular, then $E \times E$ is regular too.*

(2) *If E is regular, then $E \times \mathbb{C}$ is regular too.*

(3) *If E is not reflexive, then $E \times E'$ is not regular.*

The proof of (3) above is immediate from the observation that the immersion of E into $E'' = (E')'$ cannot be weakly compact. In fact, $E \times E'$ is not even symmetrically regular, since the mapping $T : E \times E' \to (E \times E')'$ defined as $T(x, x') = (x', x)$ is easily seen to be symmetric and not weakly compact. So, for example, for the version of James space E which has the property that both it and its dual are regular ([18]) we have that the bidual E'' is regular because E has codimension one in E'', although by (3) $E \times E'$ is not (symmetrically) regular.

Proposition 1.9. (1) *If $E \times E$ is symmetrically regular, then E is regular.*

(2) *If E is a Banach space, $E \times E$ is regular if and only if it is symmetrically regular.*

Proof. (1) Let B be a continuous bilinear form in E. Define a continuous bilinear form C_B on $E \times E$ by $C_B((x,y),(u,v)) = B(x,v) + B(u,y)$. Since C_B is symmetric, its extension to $(E \times E)''$, $\tilde{C_B}$, is symmetric and weak-* separately continuous. Then it can be checked that $\tilde{B}(x'',y'') = \lim_\alpha \lim_\beta B(x_\alpha, y_\beta) = \lim_\alpha \lim_\beta C_B((x_\alpha,0),(0,y_\beta)) = \tilde{C_B}((x'',0),(0,y''))$. Therefore $\tilde{B}$ is weak-* separately continuous and the value of the iterated limit is independent of the order in which we take limits. As a result, E is regular.

The proof of (2) follows from (1) and from Corollary 1.8 (1). Q.E.D.

Since for many Banach spaces, E and $E \times E$ are isomorphic, this proposition shows that it is very often the case that E is regular if and only if E is symmetrically regular. We are grateful to Denny Leung [19] who has pointed out that the dual J' of James space is symmetrically regular although J' is not regular.

2. Analytic structure on $M_b(U)$

Let U be a domain in E. For each $\phi \in M_b(U)$, there is a bounded subset $U_r = \{x \in E : ||x|| \leq r \text{ and dist } (x, E\backslash U) > \frac{1}{r}\}$ such that $|\phi(f)| \leq ||f||_{U_r}$ for all $f \in H_b(U)$. The homomorphism ϕ lies in a fiber over a point w_0 of E'', which is defined by $w_0 = \pi(\phi) = \phi|_{E'}$. Following ([16], Ch. I, Sec. G) for each $w \in E''$ with $||w|| \leq \delta$, where $\delta < \frac{1}{r}$, define $\phi^w : H_b(U) \to \mathbb{C}$ by

$$\phi^w(f) = \sum_{n=0}^{\infty} \phi\left(\widetilde{\frac{d^n f(\cdot)}{n!}}(w)\right),$$

where $\sum_{n=0}^{\infty} \frac{d^n f(x)}{n!}$ is the Taylor series expansion of f about $x \in U$. Note that the function

$$x \in U \xrightarrow{\widetilde{\frac{d^n f(\cdot)}{n!}}(w)} \widetilde{\frac{d^n f(x)}{n!}}(w)$$

is indeed in $H_b(U)$, since

$$||\widetilde{\frac{d^n f(\cdot)}{n!}}(w)||_{U_m} \leq \sup_{x \in U_m} ||\widetilde{\frac{d^n f(x)}{n!}}|| \cdot ||w||^n = \sup_{x \in U_m} ||\frac{d^n f(x)}{n!}|| \cdot ||w||^n,$$

by [8] and $\sup_{x \in U_m} ||\frac{d^n f(x)}{n!}|| \leq K||f||_{U_p}$ for some $p \in \mathbb{N}$ and $K > 0$ by Cauchy's inequalities. Applying Cauchy's inequalities again,

$$|\sum_{n=0}^{\infty} \phi\left(\widetilde{\frac{d^n f(\cdot)}{n!}}(w)\right)| \leq \sum_{n=0}^{\infty} \sup_{x \in U_r} ||\frac{d^n f(x)}{n!}|| \cdot ||w||^n \leq C_\delta ||f||_{U_{r+1}}.$$

Thus, ϕ^w is a well-defined, continuous linear form on $H_b(U)$. In fact, $\phi^w \in M_b(U)$, since

$$\phi^w(fg) = \sum_{n=0}^{\infty} \phi\left(\widetilde{\frac{d^n fg(\cdot)}{n!}}(w)\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \phi\left(\widetilde{\frac{d^k f(\cdot)}{k!}}(w)\widetilde{\frac{d^{n-k} g(\cdot)}{(n-k)!}}(w)\right)$$
$$= \phi^w(f)\phi^w(g).$$

Also note that $\pi(\phi^w) = w + \pi(\phi)$.

We can attempt to put an analytic structure on $M_b(U)$ by defining a basic neighborhood about a homomorphism $\phi \in M_b(U)$ to be the set $V_{\phi,\frac{1}{m}} = \{\phi^w \ : \ w \in E'', \quad ||w|| < \frac{1}{m}\}$, where $m \in \mathbb{N}, \quad m > r$. As we will see, our attempt will succeed exactly when we have symmetric regularity. In the case when $U = E$, the definition of ϕ^w is considerably simpler. Here, for each $z \in E''$ define $\tau_z : E \to E''$ by $\tau_z(x) = x + z$. The mapping τ_z induces a type of adjoint $\tau_z^* : H_b(E) \to H_b(E)$ by $\tau_z^*(f) = \tilde{f} \circ \tau_z$. Now, since $\phi^w(f) = \phi\left(\sum_{n=0}^{\infty} \widetilde{\frac{d^n f(\cdot)}{n!}}(w)\right) = \phi\left(\tilde{f}(\cdot + w)\right)$, ϕ^w can be described as $\phi \circ \tau_w^*$. Thus the neighborhoods about $\phi \in M_b(E)$ have the form $\{\phi \circ \tau_z^* : z \in E'', \ ||z|| \leq \delta\}$.

Although the following equivalence is known ([4] Thm. 8.3 vii), we include a proof of one implication since the argument is essential to our next result.

Lemma 2.1. *E is symmetrically regular (if and) only if for all $v, w \in E''$, $\tau_{w+v}^*(P) = \tau_w^* \circ \tau_v^*(P)$ for all $P \in P(^nE)$, $n \in \mathbb{N}$.*

Proof. Suppose that E is symmetrically regular and let v and w be elements of E''. Fix $P \in P(^nE)$ and let A be the associated symmetric n-linear form. For $u \in E''$ and $x \in E$, $(\tilde{P} \circ \tau_u)(x) = \tilde{P}(x+u) = \tilde{A}(x+u, ..., x+u) = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!}\tilde{A}(x^k, u^{m-k})$. For each k, the k-homogeneous polynomial $x \in E \to \tilde{A}(x^k, u^{m-k})$ is associated to the symmetric k-linear form $(x_1, ..., x_k) \in E^k \to \tilde{A}(x_1, ..., x_k, u, ..., u)$. Because of the symmetric regularity of E, we see that the extension of this k-linear form to $(E'')^k$ must coincide with the mapping $(z_1, ..., z_k) \in (E'')^k \to \tilde{A}(z_1, ..., z_k, u, ..., u)$. Therefore, $\widetilde{\tilde{P} \circ \tau_u}(z) = \sum_{k=0}^{m} \binom{m}{k} \widetilde{\tilde{A}(z^k}, u^{m-k}) = \widetilde{\tilde{P}(z+u)}$. Hence, for all $x \in E$, $\tau_w^*\left(\tau_v^*(P)\right)(x) = \widetilde{\tilde{P} \circ \tau_v|_E} \circ \tau_w(x) = \widetilde{\tilde{P} \circ \tau_v|_E}(w+x) = \widetilde{\tilde{P} \circ \tau_v}(w+x)$, which by the above equality, $= \tilde{P}(w+x+v) = (\tilde{P} \circ \tau_{w+v})(x) = \tau_{w+v}^*(P)(x)$. Consequently, $\tau_{w+v}^* = \tau_w^* \circ \tau_v^*$. Q.E.D.

Theorem 2.2. *If E is a symmetrically regular Banach space, then for all open subsets U of E, the family $\mathcal{V} := \{V_{\phi,\epsilon} \ : \ \phi \in M_b(U) \ \ and\ \epsilon > 0\ chosen\ as\ above\}$ is a basic neighborhood system for a Hausdorff topology on $M_b(U)$.*

Proof. Fix $V_{\phi,\epsilon}$ and a point $\psi = \phi^w \in V_{\phi,\epsilon}$. The first part of the argument consists in showing that for some $\delta > 0$, $V_{\psi,\delta} \subset V_{\phi,\epsilon}$; that is, we must prove that every $\psi^v = (\phi^w)^v \in V_{\psi,\delta}$ is in $V_{\phi,\epsilon}$. For this, the symmetric regularity of E will be used to show that $(\phi^w)^v = \phi^{w+v}$.

Consider $v \in E''$, chosen so small that $(\phi^w)^v$ is well-defined, and fix $f \in H_b(U)$. In order to simplify the notation, we write $P_n(x)$ instead of $\frac{d^n f(x)}{n!}$ and $A_n(x)$ for its associated multilinear mapping. It is known [2] that by means of the Taylor series expansions f has an analytic extension $\tilde{f}$ to some open subset $U'' \subset E''$.

Let's first calculate the Taylor series expansion of the function $\tilde{P}_m(\cdot)(v)$ about a given point $x \in U$, where $\tilde{P}_m(\cdot)(v) : x \in U \rightsquigarrow \tilde{P}_m(x)(v)$. Let $n \in \mathbb{N}$ be such that $x \in U_n$ and $y \in E$. Since for any complex number μ,

$$\sum_{k=0}^{m} \mid \tbinom{m}{k} \mu^k \tilde{A}_m(x)(y^{m-k}, v^k) \mid \leq \sum_{k=0}^{m} \tbinom{m}{k} |\mu|^k \sup_{x \in U_n} \{|\tilde{A}_m(x)(y^{m-k}, v^k)|\}$$

$$\leq \sum_{k=0}^{m} \tbinom{m}{k} \frac{||y||^{m-k}||\mu v||^k}{\epsilon^m} \sup_{x \in U_n} \{||\tilde{A}_m(x)||_{\epsilon B}\} \leq \sum_{k=0}^{m} \tbinom{m}{k} \frac{||y||^{m-k}||\mu v||^k}{\epsilon^m} \frac{m^m}{m!} ||f||_{U_r + \epsilon B}$$

$$= (\frac{||y|| + ||\mu v||}{\epsilon})^m \frac{m^m}{m!} ||f||_{U_r + \epsilon B},$$

it turns out that the double series $\sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} \mu^k (\tilde{A}_m(x)(y^{m-k}, v^k))$ is absolutely convergent if $(||y|| + ||\mu v||) < \frac{\epsilon}{e}$. Choose $y \in E$ such that $||y|| < \frac{\epsilon}{e}$ and pick $\mu_0 > 0$ so that $(||y|| + ||\mu_0 v||) < \frac{\epsilon}{e}$. By our assumption of symmetric regularity, $\tilde{P}_m(x)(y + \mu v) = \sum_{k=0}^{m} \binom{m}{k} \mu^k \tilde{A}_m(x)(y^{m-k}, v^k)$ and

$$\sum_{n=0}^{\infty} \tilde{P}_n(x + y)(\mu v) = \tilde{f}(x + y + \mu v) = \sum_{m=0}^{\infty} \tilde{P}_m(x)(y + \mu v)$$

for all $|\mu| \leq \mu_0$. Hence,

$$\sum_{n=0}^{\infty} \tilde{P}_n(x + y)(\mu v) = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \tbinom{m}{k} \mu^k \tilde{A}_m(x)(y^{m-k}, v^k).$$

Since the double series is absolutely convergent for $|\mu| \leq \mu_0$, we may reverse the order of the summation to obtain

$$\sum_{k=0}^{\infty} \sum_{m=k}^{\infty} \tbinom{m}{k} \mu^k \tilde{A}_m(x)(y^{m-k}, v^k) = \sum_{n=0}^{\infty} \tilde{P}_n(x + y)(\mu v) = \sum_{k=0}^{\infty} \mu^k \tilde{P}_k(x + y)(v)$$

for all $|\mu| \leq \mu_0$. So by the uniqueness of the Taylor expansion, $\tilde{P}_k(x + y)(v) = \sum_{m=k}^{\infty} \binom{m}{k} \tilde{A}_m(x)(y^{m-k}, v^k)$, and therefore the Taylor series expansion of $\tilde{P}_k(\cdot)(v)$ at x must be $\tilde{P}_k(x + y)(v) = \sum_{m=k}^{\infty} \binom{m}{k} \tilde{A}_m(x)(y^{m-k}, v^k)$.

Next, we prove that $(\phi^w)^v = \phi^{w+v}$. Using the argument of the previous lemma, we see that

$$(\phi^w)^v(f) = \sum_{k=0}^{\infty} \phi^w(\tilde{P}_k(\cdot)(v)) = \sum_{k=0}^{\infty} \sum_{m=k}^{\infty} \tbinom{m}{k} \phi(\tilde{A}_m(\cdot)(w^{m-k}, v^k)).$$

Since $|\phi(g)| \leq ||g||_{U_r}$ for some $r \in \mathbb{N}$, we have

$$\sum_{k=0}^{m} |\tbinom{m}{k} \phi(\tilde{A}_m(\cdot)(w^{m-k}, v^k))| \ \leq \ \sum_{k=0}^{m} \tbinom{m}{k} \ \sup_{x \in U_r} \{|\tilde{A}_m(x)(w^{m-k}, v^k)|\}$$

$$\leq \sum_{k=0}^{m} \tbinom{m}{k} \frac{||w||^{m-k}||v||^k}{\epsilon^m} \sup_{x \in U_r} \{||\tilde{A}_m(x)||_{\epsilon B}\} \leq \sum_{k=0}^{m} \tbinom{m}{k} \frac{||w||^{m-k}||v||^k}{\epsilon^m} \frac{m^m}{m!} ||f||_{U_r + \epsilon B}$$

$$= (\frac{||w|| + ||v||}{\epsilon})^m \frac{m^m}{m!} ||f||_{U_r + \epsilon B}.$$

Therefore the double series $\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}\phi(\tilde{A}_m(\cdot)(w^{m-k},v^k))$ is absolutely convergent if $(||w||+||v||)<\frac{\epsilon}{e}$, so we may reverse the order of summation to obtain

$$(\phi^w)^v(f)=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}\phi(\tilde{A}_m(\cdot)(w^{m-k},v^k))=\sum_{m=0}^{\infty}\phi\{\sum_{k=0}^{m}\binom{m}{k}(\tilde{A}_m(\cdot)(w^{m-k},v^k))\}$$

$$=\sum_{m=0}^{\infty}\phi\left(\tilde{P}_m(\cdot)(w+v)\right)=\phi^{w+v}(f).$$

We should mention that in the case $U=E$, the proof of $\phi^{w+v}=(\phi^w)^v$ is immediate from the fact that the symmetric regularity assumption implies that $\tau^*_{w+v}=\tau^*_w\circ\tau^*_v$.

Now we verify that if $\psi\in V_{\phi,\epsilon}$, then for some $\delta>0$, $V_{\psi,\delta}\subset V_{\phi,\epsilon}$. Indeed, $\psi=\phi^w$ for some $w\in E''$ with $||w||<\epsilon$. If $\delta<\epsilon-||w||$ and δ is chosen so that $(\psi)^v$ is defined for every $v\in E''$ with $||v||<\delta$, then $\psi^v=(\phi^w)^v=\phi^{w+v}\in V_{\phi,\epsilon}$.

Finally, we establish that this is a Hausdorff topology. Take ϕ and $\psi\in M_b(U),\psi\neq\phi$. We distinguish two cases:

(i) $\pi(\psi)\neq\pi(\phi)$. We claim that if $r>0$ is chosen so that $r<\frac{||\pi(\psi)-\pi(\phi)||}{2}$ and $V_{\psi,r},\ V_{\phi,r}\in\mathcal{V}$, then $V_{\phi,r}\cap V_{\psi,r}=\emptyset$. Otherwise, there would exist $v,\ w\in E''$, $||v||$, $||w||<r$ such that $\psi^v=\phi^w$. Hence $\pi(\psi)+v=\pi(\psi^v)=\pi(\phi^w)=\pi(\phi)+w$, so that $||\pi(\psi)-\pi(\phi)||=||w-v||<2r$, which is a contradiction.

(ii) $\pi(\psi)=\pi(\phi)$. We claim that if there exist $V_{\phi,r},\ V_{\psi,s}\in\mathcal{V}$ such that $V_{\phi,r}\cap V_{\psi,s}\neq\emptyset$, then $\psi=\phi$. Given $\theta\in V_{\phi,r}\cap V_{\psi,s}$, there are $v,w\in E''$ with $||v||<r$ and $||w||<s$ satisfying $\phi^v=\theta=\psi^w$. Since $\pi(\psi)+v=\pi(\psi^v)=\pi(\phi^w)=\pi(\phi)+w$ we have $v=w$. Let $\rho=\min\{r,s\}$ and $\delta>0$ so that $(\psi^v)^{v'}=(\phi^v)^{v'}$ is well-defined for all v', $||v'||<\delta$. Given $f\in H_b(U)$, the functions $\alpha,\ \beta:\{w:||w||<\rho\}\to\mathbb{C}$ defined by $\alpha(w)=\psi^w(f)$, $\beta(w)=\phi^w(f)$ are both analytic, because $\phi(\widetilde{\frac{d^nf(\cdot)}{n!}}(w))$ is analytic for all n and the series $\sum_{n=0}^{\infty}||\frac{d^nf(\cdot)}{n!}||_{U_r}\rho^n$ is convergent as we have seen at the beginning of this section. Since $||v||<\rho$, α and β coincide on the non-empty open subset $\{v+v':||v'||<\delta\}\cap\{w:||w||<\rho\}$, so they coincide by the identity theorem. In particular, $\psi(f)=\alpha(0)=\beta(0)=\phi(f)$. Q.E.D.

Proposition 2.3. *If E is not symmetrically regular, the collection $\{V_{\phi,\epsilon}\ :\ \phi\in M_b(E)\}$ does not define a topology on $M_b(E)$.*

Proof. Because of the assumption, there exists a symmetric bilinear form $A:E\times E\to\mathbb{C}$ whose associated linear mapping T is not weakly compact. Thus there is $x''\in E''$ such that $T^{tt}x''\notin E''$, and by Lemma 1.1 there exists $y''\in E''$ such that $\tilde{A}(x'',y'')\neq\tilde{A}(y'',x'')$. Let P be the polynomial associated to A. For any $\mu\in\mathbb{C}$, consider the function $Q_\mu\in H_b(E)$ defined by $Q_\mu(x)=\tau^*_{\mu x''}(P)=\tilde{P}(\mu x''+x)$. A routine calculation shows that $Q_\mu(x)=P(x)+\mu^2\tilde{P}(x'')+\mu\tilde{A}(x,x'')+\mu\tilde{A}(x'',x)=P(x)+\mu^2\tilde{P}(x'')+2\mu\tilde{A}(x,x'')$, and so it follows that $\widetilde{Q_\mu}(z)=\tilde{P}(z)+\mu^2\tilde{P}(x'')+2\mu\tilde{A}(z,x'')$ for all $z\in E''$. Thus if λ and μ are nonzero complex numbers,

$$(\delta_0\circ\tau^*_{\lambda y''})\circ(\tau^*_{\mu x''})(P)=(\delta_0\circ\tau^*_{\lambda y''})Q_\mu=(\tilde{Q}_\mu\circ\tau_{\lambda y''})(0)=\widetilde{Q_\mu}(\lambda y'')$$
$$=\tilde{P}(\lambda y'')+\mu^2\tilde{P}(x'')+2\mu\tilde{A}(\lambda y'',x'')$$
$$\neq\tilde{P}(\mu x'')+\lambda^2\tilde{P}(y'')+\mu\lambda[(\tilde{A}(x'',y'')+\tilde{A}(y'',x'')]=\tilde{A}(\mu x''+\lambda y'',\mu x''+\lambda y'')$$
$$=\tilde{P}(\mu x''+\lambda y'')=(\delta_0\circ\tau^*_{\lambda y''+\mu x''})(P).$$

Therefore, $(\delta_0 \circ \tau^*_{\lambda y''}) \circ (\tau^*_{\mu x''}) \neq (\delta_0 \circ \tau^*_{\lambda y''+\mu x''})$ for all non-zero λ and μ.

Suppose that the collection $\{V_{\phi,\epsilon} \ : \ \phi \in M_b(E)\ \}$ defines a topology in $M_b(E)$. Then for sufficiently small $\lambda_0 > 0$, $V_{\delta_0,\epsilon}$ is a neighborhood of $\phi = (\delta_0 \circ \tau^*_{\lambda_0 y''})$. So for small enough $r > 0$, we have $V_{\phi,r} \subset V_{\delta_0,\epsilon}$. In particular, for a suitable choice of r_0, $\phi \circ \tau^*_{v_0 x''}$ belongs to $V_{\delta_0,\epsilon}$, i.e. there is $z \in E''$ so that $\phi \circ \tau^*_{r_0 x''} = \delta_0 \circ \tau^*_z$. Then $\pi(\phi \circ \tau^*_{r_0 x''}) = \pi(\delta_0 \circ \tau^*_z)$; that is, $\lambda_0 y'' + r_0 x'' = z$ and this contradicts the above inequality. Q.E.D.

Corollary 2.4. *If E is symmetrically regular, then π is a local homeomorphism over E'' and $M_b(U)$ has an analytic structure over E''.*

Proof. For every $\phi \in M_b(U)$, $\pi(\phi) = w_0$, the inverse mapping of $\pi_\phi : V_{\phi,\delta} \to E''$ is given by $\pi_\phi^{-1} : \{w + w_0 : ||w|| < \delta\} \to V_{\phi,\delta}$, $\pi_\phi^{-1}(w + w_0) = \phi^w$. Now for any $\psi \in M_b(U)$, $\pi_\psi \circ \pi_\phi^{-1}(w+w_0) = \pi(\phi)+w = w+w_0$ is obviously analytic. Q.E.D.

Given $\phi \in M_b(E)$, we consider the "sheet" $S(\phi) = \{\phi \circ \tau^*_v \ : \ v \in E''\}$, which is an open subset of $M_b(E)$ homeomorphic to E''. Since such sets $S(\phi)$ either coincide or are disjoint, $S(\phi)$ is also closed.

Corollary 2.5. *Assume that E is symmetrically regular. Then $M_b(E)$ is connected if and only if every homomorphism in $H_b(E)$ is an evaluation at some point of E''.*

Proof. The connectedness of $M_b(E)$ leads to $M_b(E) = S(\delta_0)$. Hence every $\psi \in M_b(E)$ has the form $\psi = \delta_0 \circ \tau^*_v$ for some $v \in E''$, i.e., $\psi = \tilde{\delta}_v$. Conversely, if $M_b(E) = E''$ then π is a global homeomorphism. Q.E.D.

3. The convolution operator on symmetrically regular spaces

Let us recall the necessary definitions. Given $f \in H_b(E)$ and $\phi \in M_b(E)$, the convolution $\phi * f$ is the element in $H_b(E)$ defined by $\phi * f(x) = \phi(f \circ \tau_x)$ for $x \in E$. Moreover the convolution $\phi * \psi$ of $\phi,\ \psi \in M_b(E)$ is defined by $\phi * \psi(f) = \phi(\psi * f)$ where $f \in H_b(E)$. It turns out that $\phi * \psi$ is in $M_b(E)$ and, in fact, that $\pi(\psi * \phi) = \pi(\psi) + \pi(\phi)$ (see §6 of [4]).

Proposition 3.1. *The following are equivalent:*

1. *Given $\phi,\ \psi \in M_b(E)$ and $v, w \in E''$ we have $(\psi\circ\tau^*_v)*(\phi\circ\tau^*_w) = (\psi*\phi)\circ\tau^*_{v+w}$.*
2. *E is symmetrically regular and the convolution operator $* : M_b(E)\times M_b(E) \to M_b(E)$ is analytic.*
3. *$\theta \circ \tau^*_z = \tilde{\delta}_z * \theta$ for all $\theta \in M_b(E)$ and $z \in E''$.*
$\tilde{\delta}_z * \theta = \theta * \tilde{\delta}_z$ for all $\theta \in M_b(E)$ and $z \in E''$.

Proof. $1 \Rightarrow 2$. Taking $\phi = \psi = \delta_0$, we have $\tilde{\delta}_v * \tilde{\delta}_w = \tilde{\delta}_{v+w}$ and this is equivalent to symmetric regularity ([4], Thm. 8.3). For the analyticity of $*$, let $\phi,\ \psi \in M_b(E)$. Our hypothesis implies that the convolution does indeed map $S(\psi) \times S(\phi)$ into $S(\psi * \phi)$. Since the atlas giving the structure in $M_b(E)$ is $\{\pi\}$, the proof follows by observing that the mapping

$$\begin{array}{ccccccc} E''\times E'' & \xrightarrow{(\pi^{-1},\pi^{-1})} & S(\psi)\times S(\phi) & \xrightarrow{*} & S(\psi*\phi) & \xrightarrow{\pi} & E'' \\ (v,w) & \rightsquigarrow & (\psi\circ\tau^*_v,\phi\circ\tau^*_w) & \rightsquigarrow & (\psi\circ\tau^*_v)*(\phi\circ\tau^*_w) & \rightsquigarrow & \pi(\psi)+v+\pi(\phi)+w \end{array}$$

is analytic.

$2 \Rightarrow 3$. Since the convolution product is continuous on $M_b(E)$, the mapping $\phi \in M_b(E) \to \phi*\theta \in M_b(E)$ is continuous for every $\theta \in M_b(E)$. $S(\delta_0)$ is connected since it is homeomorphic to E'', and therefore its image under the above mapping,

$\mathrm{Im}(S(\delta_0))$, is also connected. But $\theta = \delta_0 * \theta \in \mathrm{Im}(S(\delta_0))$, and so $S(\theta) \cap \mathrm{Im}(S(\delta_0))$ is non empty. Thus $\mathrm{Im}(S(\delta_0)) \subset S(\theta)$ because $S(\theta)$ is an open and closed set. Then $\tilde{\delta}_z * \theta \in S(\theta)$ for all $z \in E''$ which means that for every $z \in E''$, there exists $w \in E''$ such that $\tilde{\delta}_z * \theta = \theta \circ \tau_w^*$. Finally, $z + \pi(\theta) = \pi(\tilde{\delta}_z * \theta) = \pi(\theta \circ \tau_w^*) = \pi(\theta) + w$, so $z = w$ and $\tilde{\delta}_z * \theta = \theta \circ \tau_z^*$.

$3 \Rightarrow 4$. First, we show that E is symmetrically regular. Suppose that E is not symmetrically regular. Let x'', y'' and P be as in the proof of Proposition 2.3 and consider $\theta = \tilde{\delta}_{y''}$. For $Q = (\tau_{x''}^*)(P)$, we know that $\widetilde{Q}(z) = \tilde{P}(z) + \tilde{P}(x'') + 2\tilde{A}(z, x'')$ for all $z \in E''$. Thus, $(\tilde{\delta}_{y''} \circ \tau_{x''}^*)(P) = \tilde{Q}(y'') = \tilde{P}(y'') + \tilde{P}(x'') + 2\tilde{A}(y'', x'')$. On the other hand, $(\tilde{\delta}_{y''} * P)(x) = \tilde{P}(x + y'') = \widetilde{P(x) + P}(y'') + \tilde{A}(x, y'') + \tilde{A}(y'', x) = P(x) + \tilde{P}(y'') + 2\tilde{A}(x, y'')$. Then, $\tilde{\delta}_{x''} * \tilde{\delta}_{y''}(P) = (\widetilde{\delta_{y''} * P})(x'') = \tilde{P}(x'') + \tilde{P}(y'') + 2\tilde{A}(x'', y'')$. Therefore, $\tilde{\delta}_{y''} \circ \tau_{x''}^* \neq \tilde{\delta}_{x''} * \tilde{\delta}_{y''}$.

To finish the proof let us point out that if E is symmetrically regular then $\theta \circ \tau_z^* = \theta * \tilde{\delta}_z$ for every $\theta \in M_b(E)$ and every $z \in E''$. Indeed, given $P \in P(^nE)$, for $x \in E$ we have $(\tilde{\delta}_z * P)(x) = \tilde{\delta}_z(P \circ \tau_x) = \widetilde{P \circ \tau_x}(z) = (\tilde{P} \circ \tau_z)(x)$ as we have seen in Lemma 2.1 . Therefore, $(\theta * \tilde{\delta}_z)(P) = \theta(\tilde{P} \circ \tau_z) = (\theta \circ \tau_z^*)(P)$. Since $\theta * \tilde{\delta}_z = \theta \circ \tau_z^* = \tilde{\delta}_z * \theta$, the implication $3 \Rightarrow 4$ follows.

$4 \Rightarrow 1$. Since E is symmetrically regular ([4], Thm. 8.3) applying the argument in the preceding paragraph and the fact that $*$ is associative ([4], Lemma 6.5), we have $(\psi \circ \tau_v^*) * (\phi \circ \tau_w^*) = (\psi * \tilde{\delta}_v) * (\phi * \tilde{\delta}_w) = \psi * (\tilde{\delta}_v * \phi) * \tilde{\delta}_w = (\psi * \phi) * (\tilde{\delta}_v * \tilde{\delta}_w)$. But $(\tilde{\delta}_v * \tilde{\delta}_w) = \tilde{\delta}_{v+w}$, applying ([4], Thm. 8.3). Finally, $(\psi * \phi) * \tilde{\delta}_{v+w} = (\psi * \phi) \circ \tau_{v+w}^*$. Q.E.D.

As is pointed out in ([4], Thm. 8.3) symmetric regularity is a necessary condition for the commutativity of the convolution. Now we present a condition under which all the equivalent statements in Proposition 3.1 hold. In particular it yields a sufficient condition for commutativity of convolution on every sheet. This also gives a partial answer to the question of whether $\theta \circ \tau_z^* = \tilde{\delta}_z * \theta$ for all $\theta \in M_b(E)$ and $z \in E''$ (cf. §9 of [4]).

Let us introduce some convenient notation to be used in the next few results. Given $P \in P(^mE)$ and its symmetric associated m-linear mapping A, we may consider its first derivative $dP : E \to E'$, $dP(x)(u) = mA(u, x, \ldots, x)$, $x, u \in E$. In particular, if $m = 2$, the linear mapping $dP = 2T$ where T is the linear mapping of Section 1. The linear transpose of dP, $dP^t : E'' \to P(^{m-1}E)$, is an extension to E'' of the mapping $C_P : E \to P(^{m-1}E)$, given by $C_P(x)(u) = A(x, u^{m-1})$. Recall that a (vector valued) polynomial is called *weakly compact* if it maps the unit ball onto a weakly relatively compact set [22].

Proposition 3.2. *Suppose that for every $P \in P(^mE)$ and all $m \in \mathbb{N}$, $dP : E \to E'$ is weakly compact. Then $\phi \circ \tau_z^* = \tilde{\delta}_z * \phi$ for all $\phi \in M_b(E)$ and all $z \in E''$.*

Proof. It is enough to prove the equality for all $P \in P(^mE)$, that is, to prove that $(\tilde{\delta}_z * \phi)(P) = (\widetilde{\phi * P})(z)$ coincides with $(\phi \circ \tau_z^*)(P) = \phi(\tilde{P} \circ \tau_z)$ for every $z \in E''$. Note that the hypothesis implies that E is symmetrically regular.

For every $x,\ u \in E$,

$$(\phi * P)(x) = \phi(u \rightsquigarrow P(x + u))$$

$$= \phi(u \rightsquigarrow \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} A(x^k, u^{m-k})) = \sum_{k=0}^{m} \phi(u \rightsquigarrow \frac{m!}{k!(m-k)!} A(x^k, u^{m-k})).$$

For each k consider the k-linear form B_k on E defined by

$$B_k(x_1,\dots,x_k)=\phi(u\rightsquigarrow A(x_1,\dots,x_k,u^{m-k})).$$

Then $\tilde{B}_k(z_1,\dots,z_k)=\lim_{\alpha_1}\dots\lim_{\alpha_k}\phi(u\rightsquigarrow A(x_{\alpha_1},...,x_{\alpha_k},u^{m-k})),\quad z_1,\dots,z_k\in E''$, where $\{x_{\alpha_i}\}$ is a (bounded) net in E converging to z_i in the weak-* topology, $1\le i\le k$. For any $i=1,\dots,k$, let $z_1,\dots,z_k\in E''$ and $x_{\alpha_1},...,x_{\alpha_{i-1}}$ be fixed. Since $\tilde{A}$ is $w(E'',E')-$separately continuous (and symmetric), the net in $P(^{m-k}E)$

$$\{u\rightsquigarrow\tilde{A}(x_{\alpha_1},\dots,x_{\alpha_i},z_{i+1},\dots,z_k,u^{m-k})\}_{\alpha_i}$$

is pointwise convergent to the polynomial $u\rightsquigarrow\tilde{A}(x_{\alpha_1},\dots,x_{\alpha_{i-1}},z_i,\dots,z_k,u^{m-k})$. Define an $m-k+1$-homogeneous polynomial on E by

$$Q(u)=\tilde{A}(x_{\alpha_1},\dots,x_{\alpha_{i-1}},u,z_{i+1},\dots,z_k,u^{m-k}).$$

Then $dQ:E\to E'$ is given by

$$dQ(x)(u)=(m-k+1)\tilde{A}(x_{\alpha_1},...,x_{\alpha_{i-1}},u,z_{i+1},...,z_k,x^{m-k})$$

and, by assumption, dQ is weakly compact. Then applying ([22], Prop. 2.1), dQ^t is weakly compact, hence $C_Q:E\to P(^{m-k}E)$ is also weakly compact. Observe that $C_Q(x_{\alpha_i})(u)=A(x_{\alpha_1},\dots,x_{\alpha_i},z_{i+1},...,z_k,u^{m-k})$. Now, since $\{x_{\alpha_i}\}$ is a bounded net in E, the net $\{C_Q(x_{\alpha_i})\}_{\alpha_i}$ has a weakly convergent subnet in $P(^{m-k}E)$ which will also be pointwise convergent, and hence it must converge weakly to $u\rightsquigarrow\tilde{A}(x_{\alpha_1},\dots,x_{\alpha_{k-1}},z_i,\dots,z_k,u^{m-k})$. There is no loss of generality if we assume that this subnet is the whole net. Therefore, since ϕ is continuous on $P(^{m-k}E)$,

$$\begin{aligned}&\lim_{\alpha_i}\phi(u\rightsquigarrow\tilde{A}(x_{\alpha_1},...,x_{\alpha_i},z_{i+1},\dots,z_k,u^{m-k}))\\&\qquad=\phi(u\rightsquigarrow\tilde{A}(x_{\alpha_1},\dots,x_{\alpha_{i-1}},z_i,\dots,z_k,u^{m-k})).\end{aligned}$$

Hence,

$$\lim_{\alpha_1}\dots\lim_{\alpha_k}\phi(u\rightsquigarrow\tilde{A}(x_{\alpha_1},...,x_{\alpha_k},u^{m-k}))=\phi(u\rightsquigarrow\tilde{A}(z_1,...,z_k,u^{m-k})).$$

Consequently, $\widetilde{B_k}(z,\dots,z)=\phi(u\rightsquigarrow\tilde{A}(z,...,z,u^{m-k}))$ and recalling Lemma 2.1, $\widetilde{(\phi*P)}(z)=\phi(u\rightsquigarrow\tilde{P}(z+u))$ as we wanted. Q.E.D.

Remark 3.3. (a) *The hypothesis in Proposition* 3.2 *is satisfied by every reflexive space E and by every Banach space in which all continuous polynomials are weakly continuous on bounded sets.*

The first case is trivial. For the second, let $P:E\to\mathbb{C}$ be an $(n+1)-$homogeneous polynomial. Since P is weakly continuous on bounded sets, the associated mappings $C_P:E\to P(^nE)$, defined by $C_P(x)(u)=A(x,u,\dots,u)$, and $C_P':P(^nE)'\to E'$ are compact [7]. Moreover, the polynomial $\Delta:E\to P(^nE)'$ given by $\Delta(x)=\delta_x$ is continuous and $dP=nC_P'\circ\Delta$, so dP is compact.

(b) *If E satisfies the hypothesis in Proposition* 3.2 *and if E also has the Dunford-Pettis property, then E has the* (RP) *property of* [3]. Recall that the (RP) property means that whenever (u_j) and (v_j) are bounded sequences in E such that $\{P(u_j-v_j)\}$ converges to 0 for all polynomials P in E, then $\{P(u_j)-P(v_j)\}$ converges to 0. To see this, note that the sequence (u_j-v_j) is weakly null and since C_P is weakly compact, it follows from the Dunford-Pettis property that $\{C_P(u_j-v_j)\}$ is a norm null sequence in $P(^nE)$. This yields by the polarization formula that $\{P(u_j)-P(v_j)\}$ converges to 0.

As a result, since neither ℓ^∞ nor $C([0,1])$ has the (RP) property [3], neither of them satisfies the assumption of the former proposition.

The convolution product is not commutative in general. The following example shows that even under the (equivalent) conditions of Proposition 3.1, commutativity still may not hold. Our example is constructed in $M_b(\ell_2)$.

Example 3.4. Let Γ be an ultrafilter on $\mathbb{N}$ containing all sets of the form $E_n = \{2n, 2n+2, 2n+4, \dots\}$ and let Θ be an ultrafilter on $\mathbb{N}$ containing all sets of the form $O_n = \{2n+1, 2n+3, 2n+5, \dots\}$. For every $f \in H_b(\ell_2)$, we define $\gamma(f) = \lim_{n\in\Gamma} f(e_n)$ and $\theta(f) = \lim_{n\in\Theta} f(e_n)$. It is immediate that γ and θ belong to $M_b(\ell_2)$. Observe that $(\gamma * f)(x) = \lim_{n\in\Gamma} f(e_n + x)$ and that $(\theta * f)(x) = \lim_{n\in\Theta} f(e_n + x)$.

Let P be the continuous 4-homogeneous polynomial on ℓ_2 defined by $P((x_n)) = \sum_{j=1}^{\infty}(x_1^2 + x_3^2 + \cdots + x_{2j+1}^2)x_{2j}^2$. For every $k \in N$, we have $(\gamma * P)(e_{2k+1}) = \lim_{n\in\Gamma} P(e_n + e_{2k+1})$, but whenever n is an even number bigger than $2k+1$, $P(e_n + e_{2k+1}) = 1$, and therefore $(\gamma * P)(e_{2k+1}) = 1$. Thus

$$(\theta * \gamma)(f) = \lim_{n\in\Theta}(\gamma * P)(e_n) = 1$$

because every $O_n \in \Theta$. On the other hand, $(\theta * P)(e_{2k}) = \lim_{n\in\Theta} P(e_n + e_{2k})$, but $P(e_n + e_{2k}) = 0$ if $n > 2k+1$, so $(\theta * P)(e_{2k}) = 0$. Thus $(\gamma * \theta)(P) = \lim_{n\in\Gamma}(\theta * P)(e_n) = 0$ because every $E_n \in \Gamma$. Therefore $(\gamma * \theta) \neq (\theta * \gamma)$. Q.E.D.

We remark that the non-commuting homomorphisms γ and θ constructed above both lie in the same fiber $\pi^{-1}(0)$. We deal with γ and the same applies to θ. If $f \in (\ell_2)'$ then $\{f(e_n)\}$ is a null sequence, hence $\gamma(f) = \lim_{n\in\Gamma} f(e_n)$ must be 0 because it is the only cluster value of $f(e_n)$; so $\gamma(f) = 0$ and $\pi(f) = 0$.

Remark 3.5. The argument used in the above example can be adapted to show the following:

(a). If E is a Banach space with a normalized basis (x_j) and associated coefficient functionals (L_j), and if for some positive integer N, $\sum |L_j(x)|^N < \infty$ for all $x = \sum L_j(x)x_j$ in E, then the convolution product is not commutative on $M_b(E)$. This case includes the spaces ℓ_p, $L_p[0,1]$ for $1 \le p < \infty$.

(b). The completed projective tensor product, $\ell_2\hat{\otimes}_\pi\ell_2$, is not symmetrically regular. Indeed, let A be the bilinear form on $\ell_2\hat{\otimes}_\pi\ell_2$ given by:

$$A(\sum_n \lambda_n a_n \otimes b_n, \sum_m \mu_m c_m \otimes d_m) = \sum_{n,m} \lambda_n \mu_m A(a_n \otimes b_n, c_m \otimes d_m),$$

where $A(a\otimes b, c\otimes d) = \sum_{k=1}^{\infty}(\alpha_1\beta_1 + \cdots + \alpha_{2k+1}\beta_{2k+1})\gamma_{2k}\delta_{2k}$ $(a = (\alpha_1, \alpha_2, \dots) \in \ell_2$, etc.). A calculation shows that

$$\lim_{n\in\Theta}\lim_{m\in\Gamma} P(e_n\otimes e_n + e_m\otimes e_m) \neq \lim_{m\in\Gamma}\lim_{n\in\Theta} P(e_n\otimes e_n + e_m\otimes e_m),$$

where P is the 2−homogeneous polynomial on $\ell_2\hat{\otimes}_\pi\ell_2$ associated to A.

The next proposition shows that the above example is sharp in the sense that the convolution of two homomorphisms restricted to the polynomials of degree less than four is commutative.

Proposition 3.6. *A Banach space E is symmetrically regular if and only if $*$ is commutative when restricted to polynomials of degree two. Furthermore, $*$ is commutative for polynomials of degree three if it is analytic on $M_b(E)$.*

Proof. As usual, we let A denote the symmetric multilinear form associated with a homogeneous polynomial P. Recall that given $y \in E$, $x'' \in E''$ and A a continuous symmetric bilinear form

$$x''(v \in E \rightsquigarrow A(y,v)) = \tilde{A}(y,x'').$$

Hence given $\phi \in M_b(E)$ we have

$$\phi(v \in E \rightsquigarrow A(y,v)) = \tilde{A}(y,\pi(\phi)).$$

Similarly, if A is a continuous symmetric trilinear form

$$\phi(v \in E \rightsquigarrow A(y,y,v)) = \tilde{A}(y,y,\pi(\phi)).$$

Let ϕ and ψ be continuous homomorphisms on $\mathcal{H}_b(E)$. Fix $P \in P(^2E)$. For every $x \in E$,

$$\begin{aligned}(\psi * P)(x) &= \psi(P \circ \tau_x) = \psi(u \rightsquigarrow P(x+u)) = \psi(u \rightsquigarrow P(x) + P(u) + 2A(x,u)) \\ &= P(x) + \psi(P) + \psi(u \rightsquigarrow 2A(x,u)) = P(x) + \psi(P) + 2\tilde{A}(x,\pi(\psi)).\end{aligned}$$

Therefore, $(\phi * \psi)(P) = \phi(x \rightsquigarrow P(x) + \psi(P) + 2\tilde{A}(x,\pi(\psi)) = \phi(P) + \psi(P) + \phi(x \rightsquigarrow 2\tilde{A}(x,\pi(\psi)) = \phi(P) + \psi(P) + 2\tilde{A}(\pi(\phi),\pi(\psi))$, by the very definition of $\tilde{A}$. Thus,

$$(\psi * \phi)(P) = \phi(P) + \psi(P) + 2\tilde{A}(\pi(\phi),\pi(\psi)).$$

Since ϕ and ψ are arbitrary, the commutativity holds if and only if $\tilde{A}$ is symmetric for every continuous bilinear form A defined on E. That is, commutativity holds if and only if E is symmetrically regular.

Hence to finish the proof it is enough to check that if E satisfies the conditions of 3.1 the equality

$$(\phi\ * \psi)(P) = (\psi\ * \psi)(P)$$

holds for every $\phi\ , \psi \in M_b(E)$ and every continuous 3–homogeneous polynomial P. First of all let's suppose that $\pi(\phi) = \pi(\psi) = 0$. Let $P \in P(^3E)$. For every $x \in E$, $(\psi * P)(x) = \psi(P \circ \tau_x)=\psi(u \rightsquigarrow P(x+u)) = \psi(u \rightsquigarrow P(x) + P(u) + 3A(x,u,u) + 3A(x,x,u)) = P(x) + \psi(P) + \psi(u \rightsquigarrow 3A(x,u,u) + 3A(x,x,u)) = P(x) + \psi(P) + \psi(u \rightsquigarrow 3A(x,u,u)) + 3A(x,x,\pi(\psi)) = P(x) + \psi(P) + \psi(u \rightsquigarrow 3A(x,u,u))$. Therefore,

$$(\phi\ * \psi)(P) = \phi(x \rightsquigarrow P(x) + \psi(P)) + \phi(x \rightsquigarrow \psi(u \rightsquigarrow 3A(x,u,u)))$$

$$= \phi(P) + \psi(P) + \psi(u \rightsquigarrow 3A(u,u,\pi(\phi))) = \phi(P) + \psi(P)$$

because the mapping $x \rightsquigarrow \psi(u \rightsquigarrow 3A(u,u,x))$ is linear and continuous and $\pi(\phi) = 0 \in E$. The calculation of $(\psi * \phi)(P)$ would yield the same result.

Now let ϕ and ψ be arbitrary continuous homomorphisms on $H_b(E)$. Put $\pi(\phi) = v$ and $\pi(\psi) = w$ and suppose $P \in P(^3E)$. Then $F := \tilde{P} \circ \tau_{v+w}$ is a non-homogeneous polynomial on E. Since $\phi * \tilde{\delta}_{-v}$ and $\psi * \tilde{\delta}_{-w}$ lie in the fiber over $0 \in E$, we have

$$(\phi * \tilde{\delta}_{-v}) * (\psi * \tilde{\delta}_{-w})(F) = (\psi * \tilde{\delta}_{-w}) * (\phi * \tilde{\delta}_{-v})(F),$$

but by the associativity of $*$ and Proposition 3.1, we have $(\phi * \tilde{\delta}_{-v}) * (\psi * \tilde{\delta}_{-w})(F) = [(\phi * \psi) * \tilde{\delta}_{-v-w}](F) = (\phi * \psi)(F \circ \tau_{-v-w}) = (\phi * \psi)(P)$. Working in the same manner with the right side of the above identity, we reach the desired conclusion. Q.E.D.

Remark 3.7. *The above proof shows that if ϕ and ψ lie in fibers over points in E, $\phi * \psi(P) = \psi * \phi(P)$ for every polynomial P of degree less than or equal to three, regardless of the analyticity of $*$.*

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Department of Mathematics, Kent State University, Kent, Ohio 44242
E-mail address: `aron@mcs.kent.edu`

Departamento de Análisis Matemático, Universidad de Valencia, Doctor Moliner 50, 46100 Burjasot (Valencia), Spain
E-mail address: `galindo@vm.ci.uv.es`
E-mail address : `garciad@vm.ci.uv.es`
E-mail address : `maestre@vm.ci.uv.es`