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# Entire Functions on $c_0$

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It is shown that every holomorphic function on $c_0$ which is bounded on weakly compact sets is bounded on bounded sets.

In [9] Valdivia proved the following result (see also [8]):

PROPOSITION 1. *A Banach space E is reflexive if and only if every weakly continuous function on E is bounded on bounded subsets of E.*

An infinite dimensional differentiable version of this proposition is proved in [5]. In [2] Aron *et al.* study various types of continuous and holomorphic functions on Banach spaces and in the course of their investigations ask if a holomorphic analog of Proposition 1 is valid. By considering holomorphic functions on $c_0$ we provide (Theorem 7) a negative answer to their question. This theorem is the main result of this paper. Information on the significance of this result is given in [2]. We refer to [4] for the theory of holomorphic functions on Banach spaces.

## 1

Let $E$ be a Banach space over the complex numbers. $P({}^nE)$ is the space of all continuous $n$-homogeneous polynomials on $E$. $H(E)$ is the space of all ($C$-valued) holomorphic functions on $E$, $H_{\text{wb}}(E)$ is the subspace of $H(E)$ consisting of all holomorphic functions which are bounded on weakly compact subsets of $E$ and $H_{\text{b}}(E)$ is the subspace of $H_{\text{wb}}(E)$ consisting of all holomorphic functions which are bounded on the bounded subsets of $E$.

If $E$ is reflexive, then $H_{\text{wb}}(E) = H_{\text{b}}(E)$ and in this article we show $H_{\text{wb}}(c_0) = H_{\text{b}}(c_0)$. A deep result of Josefson [6] states that $H_{\text{b}}(E) \neq H(E)$ for any infinite dimensional Banach space. In certain cases (e.g., $E = l_1$) we may have $H(E) = H_{\text{wb}}(E)$.



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The elements of the above spaces of holomorphic functions may also be described in a useful fashion by using Taylor series expansions. This description is as follows.

If $(P_n)_{n=0}^{\infty}$ is a sequence of continuous homogeneous polynomials, $P_n$ being $n$-homogeneous, then

(i) $\sum_{n=0}^{\infty} P_n \in H(E)$ if and only if $\lim_{n\to\infty} \|P_n\|_K^{1/n} = 0$ for every compact subset $K$ of $E$, (1.1)

(ii) $\sum_{n=0}^{\infty} P_n \in H_{\mathrm{wb}}(E)$ if and only if $\lim_{n\to\infty} \|P_n\|_W^{1/n} = 0$ for every weakly compact subset W of $E$, (1.2)

(iii) $\sum_{n=0}^{\infty} P_n \in H_{\mathrm{b}}(E)$ if and only if $\lim_{n\to\infty} \|P_n\|_B^{1/n} = 0$ for every bounded subset $B$ of $E$. (1.3)

## 2

$c_0$ will denote the space of all null sequences of complex numbers endowed with the usual sup norm topology. For $n$ a positive integer $q^n$ and $q_n$ will denote respectively the projections in $c_0$ onto the first $n$ and all but the first $n$ coordinates. For $n$ and $m$ positive integers with $m \leqslant n$ we let $q_m^n = q^n - q^m$. If $I$ is the identity mapping on $c_0$ and 0 is the zero mapping we let $q_0 = I$, $q^0 = 0$, $q_0^n = q^n - q^0 = q^n$ for any positive integer $n$, $q^\infty = I$, $q_\infty = 0$, $q_m^\infty = q^\infty - q^m = I - q^m = q_m$ for any positive integer $m$ and $q_0^\infty = q^\infty - q^0 = I - 0 = I$.

The following proposition may be deduced from results in [1–3, 7] but we include a proof for the sake of completeness.

PROPOSITION 2. *Continuous polynomials on $c_0$ are weakly continuous on weakly compact subsets of $c_0$.*

*Proof.* It suffices to consider homogeneous polynomials of degree $\geqslant 1$. We first show that every continuous homogeneous polynomial is weakly sequentially continuous at zero. Let $P$ be a continuous $n$-homogeneous polynomial on $c_0$ and let $(x_n)_n$ be a weak null sequence in $c_0$. Suppose $P(x_n) \not\to 0$ as $n \to \infty$. By taking a subsequence if necessary and on multiplying each $x_n$ by a scalar we may suppose there exists $\delta > 0$ and $(y_n)_n$ a sequence in $c_0$ such that

$$\|y_n\| = 1 \qquad \text{for all } n, \quad y_n \to 0 \text{ weakly as } n \to \infty$$

and

$$|P(y_n)| \geqslant \delta \qquad \text{for all } n.$$

Let $n_0 = 0$ and $N_1 = 1$.

Choose $n_1$, a positive integer, such that $|P(q^{n_1}(y_1))| \geqslant \delta/2$. Since $y_n \to 0$ weakly as $n \to \infty$ we can choose $N_2$ such that $\|q^{n_1}(y_n)\| \leqslant 1/2^2$ for all $n \geqslant N_2$. By induction we now choose two strictly increasing sequences of positive integers $(n_j)_j$ and $(N_j)_j$ such that

$$|P(q^{n_j}(y_{N_j}))| \geqslant \frac{\delta}{2} \qquad \text{for all} \quad j \geqslant 1 \tag{2.1}$$

and

$$\|q^{n_j}(y_{N_{j+1}})\| \leqslant \frac{1}{2^j} \qquad \text{for all} \quad j \geqslant 1. \tag{2.2}$$

By using the inequality

$$\sup_{\substack{|z|=1\\ z\in C}} \left| \sum_{j=0}^{n} a_j z^j \right| \geqslant \sum_{j=0}^{n} |a_j|^2,$$

(2.1), and induction we can choose for any positive integer $l$ a sequence of scalar $(\lambda_j)_{j=1}^{2^l}$ such that

$$|\lambda_j| \leqslant 1 \qquad \text{for all } j \tag{2.3}$$

and

$$\left| P\left( \sum_{j=1}^{2^l} \lambda_j q^{n_j}(y_{N_j}) \right) \right| \geqslant \frac{\delta}{2} (\sqrt{2})^l. \tag{2.4}$$

By (2.2) and (2.3)

$$\begin{aligned} \left\| \sum_{j=1}^{2^l} \lambda_j q^{n_j}(y_{N_j}) \right\| &= \left\| \sum_{j=1}^{2^l} \lambda_j q^{n_{j-1}}(y_{N_j}) + \sum_{j=1}^{2^l} \lambda_j q^{n_j}_{n_{j-1}}(y_{N_j}) \right\| \\ &\leqslant \sum_{j=1}^{2^l} \frac{1}{2^j} + \sup_j \|y_{N_j}\| \leqslant 2. \end{aligned} \tag{2.5}$$

Hence (2.4) and (2.5) imply

$$\sup_{\|x\| \leqslant 2} |P(x)| \geqslant \sup_l \frac{\delta}{2} (\sqrt{2})^l = \infty.$$

Since $P$ is continuous this is impossible and hence every continuous polynomial on $c_0$ is weakly sequentially continuous at zero.

Let $Q$ be a continuous $m$-homogeneous polynomial on $c_0$ $(m \geqslant 1)$ and let $A$ denote the symmetric $m$ linear form associated with $Q$. Let $W$ be a weakly compact subset of $c_0$. Since $c_0' = l_1$ is separable $W$ is metrizable and hence to complete the proof if suffices to show $Q|_W$ is weakly sequentially continuous.

Let $x_n \in W \to x$ weakly as $n \to \infty$. Now

$$Q(x_n) - Q(x) = Q(x + x_n - x) - Q(x)$$
$$= \sum_{j=0}^{m-1} \binom{m}{j} A(x)^j (x_n - x)^{m-j}.$$

For each integer $j$, $0 \leqslant j \leqslant m-1$, the mapping

$$y \in c_0 \to \binom{m}{j} A(x)^j (y)^{m-j}$$

is a homogeneous polynomial of degree $\geqslant 1$. Since $x_n - x \to 0$ weakly as $n \to \infty$ the first part of our proof shows that

$$\binom{m}{j} A(x)^j (x_n - x)^{m-j} \to 0 \quad \text{as} \quad n \to \infty \quad \text{for all } j, \quad j \leqslant m-1.$$

Hence $Q(x_n) \to Q(x)$ as $n \to \infty$. Hence $Q$ is continuous on $W$ and this completes the proof.

PROPOSITION 3. *If $f \in H_{\text{wb}}(c_0)$ and $W$ is a weakly compact subset of $c_0$, then $f|_W$ is weakly continuous.*

*Proof.* Since $\sum_{n=0}^{\infty} \| \hat{d}^n f(0)/n! \|_W < \infty$ we see that $f|_W$ is the uniform limit of a sequence of weakly continuous functions. Hence $f|_W$ is continuous. This completes the proof.

## 2

In proving the remainder of our results we shall frequently need to take subsequences. An examination of our proofs shows that in all the cases we consider we may assume without loss of generality that the subsequence is in fact the original sequence. Hence when we use the phrase "without loss of generality" it shall include, where appropriate, the operation of taking subsequences. This convention helps to reduce the number of subscripts and superscripts we shall need.

If $\alpha = (\alpha_j)_{j=1}^{\infty} \in c_0$ we let $\tilde{\alpha} = \{(\beta_j)_j;\ |\beta_j| \leqslant |\alpha_j|\ \forall j\}$. Also $\alpha$ is a compact polydisc in $c_0$. If $\Omega \subset c_0$ we let $\tilde{\Omega} = \bigcup_{\alpha \in \Omega} \tilde{\alpha}$. We call $\tilde{\Omega}$ the solid hull of $\Omega$ and $\Omega$ is said to be solid of $\Omega = \tilde{\Omega}$.

LEMMA 4. *If $\Omega$ is a bounded (resp. compact, weakly compact) subset of $c_0$, then $\tilde{\Omega}$ is bounded (resp. compact, weakly compact).*

*Proof.* First suppose $\sup\{\|\alpha\|;\ \alpha \in \Omega\} = M < \infty$. If $\beta = (\beta_n) \in \Omega$, then there exists $\alpha = (\alpha_n)_n \in \Omega$ such that $|\beta_n| \leqslant |\alpha_n|$ for all $n$. Hence $\sup_n |\beta_n| = \|\beta\| \leqslant \sup_n |\alpha_n| = \|\alpha\| \leqslant M$ and $\sup\{\|\beta\|;\ \beta \in \Omega\} = M < \infty$. Hence $\tilde{\Omega}$ is bounded whenever $\Omega$ is bounded. Now suppose $\Omega$ is compact. Let $x_n \in \tilde{\Omega}$ for each $n$. For each $n$ suppose $x_n \in \tilde{y}_n$, where $y_n \in \Omega$. By hypothesis and without loss of generality we may suppose there exists $y = (y^m)_{m=1}^{\infty} \in \Omega$ such that $\|y_n - y\| \to 0$ as $n \to \infty$. For each positive integer $n$ there exists $\theta_n = (\theta_n^m)_{m=1}^{\infty} \in D^N$ ($D$ is the unit dist in $C$) such that $x_n = \theta_n y_n$ (multiplication is coordinatewise). Now $D^N$ is a compact metric space when endowed with the product topology and hence we may suppose without loss of generality that $\theta_n \to \theta = (\theta^m)_{m=1}^{\infty} \in D^N$ as $n \to \infty$. Since $y \in \Omega$, $\theta y \in \tilde{\Omega}$ and to complete the proof we show $\|\theta_n y_n - \theta y\| \to 0$ as $n \to \infty$.

Let $\varepsilon > 0$ be arbitrary. Choose $n_1$ such that $|y^m| \leqslant \varepsilon$ for all $m \geqslant n_1$. Next choose $n_2$ such that $\|y_n - y\| \leqslant \varepsilon$ for all $n \geqslant n_2$ and $|\theta_n^m - \theta^m| \leqslant \varepsilon/(1 + \|y\|)$ for all $n \geqslant n_2$ and $1 \leqslant m \leqslant n_1$. Now for $n \geqslant n_2$ we have

$$\begin{aligned}
\|\theta_n y_n - \theta y\| &\leqslant \|\theta_n y_n - \theta_n y\| + \|\theta_n y - \theta y\| \\
&\leqslant \|y_n - y\| \cdot \sup_{n,m} |\theta_n^m| + \sup_m |y^m| \cdot \sup_{m \leqslant m_1} |\theta_n^m - \theta^m| \\
&\quad + \sup_{n,m}(|\theta_n^m| + |\theta^m|) \cdot \sup_{m \geqslant n_1} |y^m| \\
&\leqslant \varepsilon + \|y\| \cdot \frac{\varepsilon}{1 + \|y\|} + 2\varepsilon \leqslant 4\varepsilon.
\end{aligned}$$

Hence $\theta_n y_n \to \theta y$ as $n \to \infty$ and $\tilde{\Omega}$ is compact. Now suppose $\Omega$ is weakly compact. To show $\Omega$ is weakly compact it suffices to show $\theta_\alpha y_\alpha \to \theta y$ weakly as $\alpha \to \infty$, where $y_\alpha \in \Omega \to y \in \Omega$ weakly as $\alpha \to \infty$ and $\theta_\alpha \in D^N \to \theta \in D^N$ as $\alpha \to \infty$. Since $\Omega$ is bounded $\theta_\alpha y_\alpha \to \theta y$ weakly as $\alpha \to \infty$ if and only if $\theta_\alpha y_\alpha \to \theta y$ in each coordinate as $\alpha \to \infty$ and this follows immediately from our hypothesis. This completes the proof.

If $P \in P({}^m c_0)$ and $A$ is the associated symmetric $m$-linear form, $(n_j)_{j=1}^{l-1}$ is a strictly increasing sequence of positive integers, $(k_j)_{j=1}^{l}$ are nonnegative integers with $\sum_{j=1}^{l} k_j = m$ and $\lambda \in C$, $|\lambda| \leqslant 1$, then for any $x$ in $c_0$ we let

$$\begin{aligned}
&(\lambda(P)[k_1,\ldots,k_l;\, n_1,\ldots,n_{l-1}])(x) \\
&\quad = \lambda \binom{m}{k_1,\ldots,k_l} A(q^{n_1}(x))^{k_1}(q_{n_1}^{n_2}(x))^{k_2} \cdots (q_{n_{l-1}}(x))^{k_l},
\end{aligned}$$

where $\lambda(P)[k_1,\ldots,k_l;\, n_1,\ldots,n_{l-1}]$ is called a *modification* of $P$ and belongs to $P(^m c_0)$.

If $n_l > n_{l-1}$ and $0 \leqslant s \leqslant k_l$, then

$$\beta(\lambda(P))[k_1,\ldots,k_l; n_1,\ldots,n_{l-1}])[s, k_l - s; n_l]$$
$$= \lambda\beta(P)[k_1,\ldots,k_{l-1}, s, k_l - s; n_1,\ldots,n_l].$$

LEMMA 5. *Let $P \in P(^m c_0)$ and $\Omega \subset c_0$. If $Q$ is a modification of $P$, then $\|Q\|_{\tilde{\Omega}} \leqslant \|P\|_{\Omega}$.*

*Proof.* Let $Q = \lambda(P)[k_1,\ldots,k_l;\, n_1,\ldots,n_{l-1}]$ and let $n_0 = 0$ and $n_l = \infty$. If $x \in \Omega$, let $x_j = q_{n_{j-1}}^{n_j} x$ for $1 \leqslant j \leqslant l$. Note that $x = \sum_{j=1}^{l} x_j$ and $\sum_{j=1}^{l} \lambda_j x_j \in \tilde{\Omega}$ for all $\lambda_i \in C$, $|\lambda_i| \leqslant 1$. Now

$$P = \sum_{\substack{0 \leqslant k_i \leqslant m \\ \sum k_i = m}} (P)[k_1,\ldots,k_l; n_1,\ldots,n_{l-1}].$$

Hence

$$P\left(\sum_{j=1}^{l} \lambda_j x_j\right) = \sum_{\substack{0 \leqslant k_i \leqslant m \\ \sum k_i = m}} \lambda_1^{k_1}\lambda_2^{k_2}\cdots\lambda^{k_l}(P)[k_1,\ldots,k_l; n_1,\ldots,n_{l-1}](x).$$

By Parseval's inequality we have

$$|Q(x)|^2 = |\lambda(P)[k_1,\ldots,k_l; n_1,\ldots,n_{l-1}](x)|^2$$
$$\leqslant \sum_{\substack{0 \leqslant k_i \leqslant m \\ \sum k_i = m}} |(P)[k_1,\ldots,k_l; n_1,\ldots,n_{l-1}](x)|^2$$
$$\leqslant \sup_{|\lambda_i| \leqslant 1} \left| P\left(\sum_{i=1}^{l} \lambda_i x_i\right)\right|^2 \leqslant \|P\|_{\tilde{\Omega}}^2.$$

Hence $\|Q\|_{\tilde{\Omega}} \leqslant \|P\|_{\tilde{\Omega}}$ and this completes the proof.

If $f = \sum_{n=0}^{\infty} P_n \in H(c_0)$ and $g = \sum_{n=0}^{\infty} Q_n$, where each $Q_n$ is a modification of $P_n$, then we call the formal series $\sum_{n=0}^{\infty} Q_n$ a modification of $f$.

LEMMA 6. *If $f \in H(c_0)$ (resp. $H_b(c_0)$, $H_{wb}(c_0)$) and $g$ is a modification of $f$, then $g \in H(c_0)$ (resp. $H_b(c_0)$, $H_{wb}(c_0)$).*

*Proof.* It suffices to use (1.1)–(1.3), lemmata 4 and 5 to prove this result.

THEOREM 7. $H_{wb}(c_0) = H_b(c_0)$.

*Proof.* Suppose the result is not true. Then there exists an $f \in \sum_{n=0}^{\infty} P_n \in$

$H_{wb}(c_0)$, where $P_n$ is a $k_n$-homogeneous polynomial for all $n$ and $(k_n)_n$ is a strictly increasing sequence of positive integers, and a sequence of unit vectors in $c_0$, $(x_n)_{n=1}^{\infty}$, such that $|P_n(x_n)| > 1$ for all $n$. By taking a modification of $f$ if necessary we may suppose without loss of generality that

$$\frac{k_n}{\log(k_n + 1)} \geqslant n \qquad \text{for all } n. \tag{3.1}$$

Equation (3.1) implies $k_n \geqslant n \log(k_n + 1) = \log(k_n + 1)^n$ and hence $\exp(k_n) \geqslant (k_n + 1)^n$ for all $n$. Hence

$$\left(\frac{1}{(k_n + 1)^n}\right)^{1/k_n} \geqslant \left(\frac{1}{\exp(k_n)}\right)^{1/k_n} \geqslant \frac{1}{e} \qquad \text{for all } n. \tag{3.2}$$

Note if we take a strictly increasing sequence of positive integers $(n_j)_j$ and let $Q_j = P_{n_j}$ and $y_j = x_{n_j}$, then $g = \sum_{j=1}^{\infty} Q_j \in H_{wb}(c_0)$, $|Q_j(y_j)| > 1$ for all $j$ and

$$\frac{\deg(Q_j)}{\log(\deg(Q_j) + 1)} = \frac{\deg(P_{n_j})}{\log(\deg(P_{n_j}) + 1)} = \frac{k_{n_j}}{\log(k_{n_j} + 1)} \geqslant n_j > j$$

and hence (3.1) is preserved.

Our aim now is to construct a sequence $(\delta_n)_n$ of positive numbers with certain properties and then to show that the existence of the sequence $(\delta_n)_n$ leads to a contradiction. The construction of $(\delta_n)_n$ is rather technical and is by induction. We first show how to obtain $\delta_1$ and similar construction is used to obtain $\delta_n$.

First choose a positive integer $n_0$ such that

$$|P_1(q^j(x_1))| > 1 \qquad \text{for all } \ j \geqslant n_0. \tag{3.3}$$

For any positive integers $l$ and $n$ we have

$$\left| \sum_{j=0}^{k_n} (P_n)[j, k_n - j; l](x_n) \right| = |P_n(x_n)| > 1.$$

Hence we can choose for any integers $l$ and $n$ an integer $j_{n,l}$ such that $0 \leqslant j_{n,l} \leqslant k_n$ and

$$|(P_n)[j_{n,l}, k_n - j_{n,l}; l](x_n)| > \frac{1}{k_n + 1}. \tag{3.4}$$

We claim there exists an integer $n_1 \geqslant n_0$ and a choice of $(j_{n,n_1})_{n=2}^{\infty}$ satisfying (3.4) such that

$$\liminf_{n \to \infty} \frac{j_{n,n_1}}{k_n} > 0.$$

Suppose otherwise. Then without loss of generality we can choose a sequence of positive integers $(j_n)_{n\geqslant n_0}$ such that $0 \leqslant j_n \leqslant k_n$ for all $n$, $j_n/k_n \to 0$ as $n \to \infty$ and $|(P_n)[j_n,\ k_n - j_n;\ n](x_n)| > 1/(k_n+1)$ all $n \geqslant n_0$. Let $\alpha_n = \exp(-(k_n/j_n)^{1/2})$ for $n \geqslant n_0$. Then $\alpha_n \to 0$ as $n \to \infty$ and

$$\alpha_n^{j_n/k_n} = \exp\left(-\left(\frac{j_n}{k_n}\right)^{1/2}\right) \to 1 \qquad \text{as} \quad n \to \infty.$$

Now $g = \sum_{n\geqslant n_0} (P_n)[j_n, k_n - j_n; n]$ is a modification of $f$ and hence belongs to $H_{\mathrm{wb}}(c_0)$.

We have

$$\|\alpha_n q^n(x_n) + q_n(x_n)\| \leqslant \sup(|\alpha_n|, \|x_n\|) \leqslant 1$$

and so $\{\alpha_n q^n(x_n) + q_n(x_n)\}_{n\geqslant n_0}$ is a bounded subset of $c_0$. Since $(\alpha_n)_n \in c_0$ each coordinate of $\alpha_n q^n(x_n) + q_n(x_n)$ tends to zero as $n \to \infty$ and hence $\alpha_n q^n(x_n) + q_n(x_n) \to 0$ weakly as $n \to \infty$.

For each $n$ let $A_n$ be the symmetric $k_n$-linear form associated with $P_n$.

Since

$$\begin{aligned} &|(P_n)[j_n, k_n - j_n; n](\alpha_n q^n(x_n) + q_n(x_n))|^{1/k_n} \\ &\quad = \left| \binom{k_n}{j_n} A_n(\alpha_n q^n(x_n))^{j_n} (q_n(x_n))^{k_n - j_n} \right|^{1/k_n} \\ &\quad \geqslant \alpha_n^{j_n/k_n} \left(\frac{1}{k_n+1}\right)^{1/k_n} \to 1 \qquad \text{as} \quad n \to \infty \end{aligned}$$

we have reached a contradiction.

Hence we may suppose, without loss of generality, that there exist $n_1 \geqslant n_0$, $\delta_1 > 0$, and $(j_{n,n_1})_{n=2}^{\infty}$ such that

$$0 \leqslant j_{n,n_1} \leqslant k_n \qquad \text{for all} \quad n \geqslant 2, \tag{3.5}$$

$$\frac{j_{n,n_1}}{k_n} > \frac{\delta_1}{2} \qquad \text{for all} \quad n \geqslant 2, \tag{3.6}$$

$$\lim_{n\to\infty} \frac{j_{n,n_1}}{k_n} = \delta_1, \tag{3.7}$$

and

$$|(P_n)[j_{n,n_1}, k_n - j_{n,n_1}; n_1](x_n)| > \frac{1}{k_n+1} \qquad \text{for all} \quad n \geqslant 2. \tag{3.8}$$

By (3.5) and (3.7) we always have $0 < \delta_1 \leqslant 1$ and we now show that we also have $\delta_1 < 1$.

Suppose otherwise. Then $\lim_{n\to\infty}(k_n - j_{n,n_1})/k_n = 0$. Let $\beta_n = \exp(-(k_n/(k_n - j_{n,n_1}))^{1/2})$. Then $(\beta_n)_n \in c_0$ and $\beta_n^{k_n - j_{n,n_1}/k_n} \to 1$ as $n \to \infty$. The sequence $(\beta_n q_{n_1}(x_n))_{n=2}^{\infty}$ is a null sequence in $c_0$ and $(q^{n_1}(x_n))_{n=2}^{\infty}$ is a finite dimensional subset of $c_0$. Hence $\{(q^{n_1}(x_n) + \beta_n q_{n_1}(x_n))\}_{n=2}^{\infty}$ is a relatively compact subset of $c_0$.

Now $\sum_{n=2}^{\infty}(P_n)[j_{n,n_1}, k_n - j_{n,n_1}; n_1] \in H(c_0)$ and

$$\begin{aligned}
&|(P_n)[j_{n,n_1}, k_n - j_{n,n_1}; n_1](q^{n_1}(x_n) + \beta_n q_{n_1}(x_n))|^{1/k_n}\\
&= \left|\binom{k_n}{j_n} A_n(q^{n_1}(x_n))^{j_{n,n_1}}(\beta_n q_{n_1}(x_n))^{k_n - j_{n,n_1}}\right|^{1/k_n}\\
&= \beta_n^{k_n - j_{n,n_1}/k_n}\left|\binom{k_n}{j_n} A_n(q^{n_1}(x_n))^{j_{n,n_1}}(q_{n_1}(x_n))^{k_n - j_{n,n_1}}\right|^{1/k_n}\\
&= \beta_n^{k_n - j_{n,n_1}/k_n}|(P_n)[j_{n,n_1}, k_n - j_{n,n_1}; n_1](x_n)|^{1/k_n}\\
&\geqslant \beta_n^{k_n - j_{n,n_1}/k_n}\left(\frac{1}{k_n + 1}\right)^{1/k_n} \to 1 \qquad \text{as} \quad n \to \infty.
\end{aligned}$$

This is impossible and hence we may suppose $0 < \delta_1 < 1$. This completes the first step in our induction. Now suppose we are given for the positive integer $l$, $(\delta_i)_{i=1}^{l}$, $(n_i)_{i=1}^{l}$, and $(j_{n,n_i})_{n=i+1}^{\infty}$, $1 \leqslant i \leqslant l$, such that

$$0 < \delta_i < 1 \qquad \text{for all } i \quad \text{and} \quad \sum_{i=1}^{l} \delta_i < 1; \tag{3.9}$$

the sequence $(n_i)_{i=1}^{l}$ is a strictly increasing sequence of positive integers; (3.10)

$$\frac{j_{n,n_i}}{k_n} \geqslant \frac{\delta_i}{2} \text{ for all } n \geqslant i+1 \qquad \text{and} \qquad \lim_{n\to\infty}\frac{j_{n,n_i}}{k_n} = \delta_i \text{ for all } i; \tag{3.11}$$

$$0 \leqslant \sum_{i=1}^{l} j_{n,n_i} \leqslant k_n \text{ for all } n \geqslant l+1 \quad \text{and} \quad 0 \leqslant \sum_{i=1}^{n-1} j_{n,n_i} \leqslant k_n \text{ for } 2 \leqslant n \leqslant l. \tag{3.12}$$

To define the remaining conditions we need some notation. Let

$$\begin{aligned}
k_{n,i} &= k_n - \sum_{s=1}^{i} j_{n,n_s} \qquad \text{if} \quad i \leqslant l < n,\\
&= k_n - \sum_{s=1}^{n-1} j_{n,n_s} \qquad \text{if} \quad i = n \leqslant l.
\end{aligned}$$

Let

$$a_{k,i} = 1 \qquad \text{if} \quad k=1$$
$$= \frac{1}{k_i+1} \qquad \text{if} \quad k=2.$$
$$= \frac{1}{k_i+1} \frac{1}{k_i+1-j_{i,n_1}} \frac{1}{k_i+1-j_{i,n_1}-j_{i,n_2}} \cdots \frac{1}{k_i+1-\sum_{s=1}^{k-2} j_{i,n_s}} \qquad \text{if} \quad k>2.$$

We require

$$|P_1[0;0](q^j(x_1)| = |P_1(q^j(x_1))| > 1 \qquad \text{for all } j > n_1,$$

and

$$|(P_i)[j_{i,n_1}, j_{i,n_2}, \ldots, j_{i,n_{i-1}}, k_{i,i}; n_1, \ldots, n_{i-1}](q^j(x_i))| > a_{i,i} \tag{3.13}$$
$$\text{for} \quad 2 \leqslant i \leqslant l \quad \text{and} \quad j \geqslant n_i$$

and

$$|(P_n)[j_{n,n_1}, j_{n,n_2}, \ldots, j_{n,n_i}, k_{n,i}; n_1, \ldots, n_i](x_n)| > a_{i+1,n} \tag{3.14}$$

for each integer $i$, $1 \leqslant i \leqslant l$ and $n > l$. The triple $(n_1, \delta_1, (j_{n,n_1})_{n=2}^{\infty})$ satisfy (3.9) to (3.14), where $l = 1$.

We now construct $\delta_{l+1}$, $n_{l+1}$ and $(j_{n,n_{l+1}})_{n=l+1}^{\infty}$. By (3.14)

$$|(P_{l+1})[j_{l+1,n_1}, \ldots, j_{l+1,n_l}, k_{l+1,l}; n_1, \ldots, n_l](x_{l+1})| > a_{l+1,l+1}.$$

Hence there exists a positive integer $N_1$ such that

$$|(P_{l+1})[j_{l+1,n_1}, \ldots, j_{l+1,n_l}, k_{l+1,l}; n_1, \ldots, n_l](q^j(x_{l+1}))| > a_{l+1,l+1}$$

for all $j \geqslant N_1$.

Hence any choice of $n_{l+1} \geqslant N_1$ will satisfy (3.13). Since

$$\sum_{j=0}^{k_{n,l}} (P_n)[j_{n,n_1}, \ldots, j_{n,n_l}, j, k_{n,l} - j; n_1, \ldots, n_{l+1}]$$
$$= (P_n)[j_{n,n_l}, \ldots, j_{n,n_l}, k_{n,l}; n_1, \ldots, n_l]$$

Equation (3.14) implies that we can choose for each $n > l+1$ a $j_{n,n_{l+1}}$ such that $0 \leqslant j_{n,n_{l+1}} \leqslant k_{n,l}$ and

$$|(P_n)[j_{n,n_1},\dots,j_{n,n_l},j_{n,n_{l+1}},k_{n,l+1};n_1,\dots,n_{l+1}](x_n)|$$
$$> \frac{a_{l+1,n}}{k_{n,l}+1} = \frac{a_{l+1,n}}{k_n - \sum_{s=1}^{l} j_{n,n_s}} = a_{l+2,n}.$$

Hence (3.14) is satisfied and so also is (3.12).

It remains to show that the sequence $(j_{n,n_{l+1}})_{n>l+1}$ can be chosen so that (3.9) and (3.11) are satisfied. First suppose that any choice of $(j_{n,n_{l+1}})_{n>l+1}$ implies

$$\liminf_{n\to\infty} \frac{j_{n,n_{l+1}}}{k_n} = 0.$$

We may then suppose, without loss of generality, that for all $n \geqslant N_1$ there exists $j_n$, $0 \leqslant j_n \leqslant k_{n,l}$, such that $\lim_{n\to\infty} j_n/k_n = 0$ and

$$|(P_n)[j_{n,n_1},\dots,j_{n,n_l},j_n,k_{n,l}-j_n;n_1,\dots,n_l,n](x_n)| > a_{l+2},n.$$

Let $\alpha_n = \exp(-(k_n/j_n)^{1/2})$ for $n \geqslant N_1$. Then $(\alpha_n)_n \in c_0$ and $\alpha_n^{j_n/k_n} \to 1$ as $n \to \infty$. Now

$$h = \sum_{n \geqslant N_1}^{\infty} (P_n)[j_{n,n_1},j_{n,n_2},\dots,j_{n,n_l},j_n,k_{n,l}-j_n;n_1,\dots,n_l,n]$$

is a modification of $f$ and hence belongs to $H_{wb}(c_0)$. The sequence $\{\alpha_n q^n_{n_l}(x_n) + q_n(x_n)\}_{n \geqslant N_1}$ is a weak null sequence in $c_0$ since $(\alpha_n)_n \in c_0$ and $\|x_n\| = 1$ all $n$. Since the sequence $\{q^{n_l}(x_n)\}_{n \geqslant N_1}$ is contained in a finite dimensional compact subset of $c_0$ the sequence $\{q^{n_l}(x_n) + \alpha_n q^n_{n_l}(x_n) + q_n(x_n)\}_{n \geqslant N_1}$ is a weakly relatively compact subset of $c_0$.

On the other hand

$$|(P_n)[j_{n,n_1},\dots,j_{n,n_l},j_n,k_{n,l}-j_n;n_1,\dots,n_l](q^{n_l}(x_n) + \alpha_n q^n_{n_l}(x_n) + q_n(x_n))|^{1/k_n}$$
$$= \alpha_n^{j_n/k_n} |(P_n)[j_{n,n_1},\dots,j_{n,n_l},j_n,k_{n,l}-j_n;n_1,\dots,n_l,n](x_n)|^{1/k_n}$$
$$\geqslant \alpha_n^{j_n/k_n} a_{l+2,n}^{1/k_n}.$$

Since

$$\alpha_n^{j_n/k_n} \to 1 \quad \text{as } n \to \infty \qquad \text{and} \qquad a_{l+2,n}^{1/k_n} \geqslant \left(\frac{1}{2k_n}\right)^{l+1/k_n} \to 1 \quad \text{as } n \to \infty.$$

this is impossible.

Hence we may suppose without loss of generality that there exists $n_{l+1} > n_l$, $\delta_{l+1} > 0$ and $(j_{n,n_{l+1}})_{n=l+2}^{\infty}$ such that (3.10)–(3.14) are satisfied and also $\sum_{i=1}^{l+1} \delta_i \leqslant 1$.

We now show that $\sum_{i=1}^{l+1} \delta_i < 1$. Suppose otherwise. Then $\lim_{n\to\infty}(k_{n,n_{l+1}}/k_n) = 1 - \sum_{i=1}^{l+1} \delta_i = 0$. Let $\theta_n = \exp(-(k_n/k_{n,n_{l+1}})^{1/2})$ for all $n \geqslant l+2$. Then $(\theta_n)_n \in c_0$ and $\theta_n^{k_{n,l+1}/k_n} \to 1$ as $n \to \infty$. The sequence $\{q^{n_{l+1}}(x_n)\}_{n \geqslant j+2}$ is a relatively compact subset of $c_0$ and the sequence $\{\theta_n q_{n_{l+1}}(x_n)\}_{n \geqslant l+2}$ is a null sequence in $c_0$. Hence the sequence $\{q^{n_{l+1}}(x_n) + \theta_n q_{n_{l+1}}(x_n)\}_{n \geqslant l+2}$ is a relatively compact sequence in $c_0$. Now $g = \sum_{n_{l+2}}^{\infty} (P_n)[j_{n,n_1}, \ldots, j_{n,n_{l+1}}, k_{n,l}; n_1, \ldots, n_{l+1}]$ is a modification of $f$ and hence is a holomorphic function on $c_0$. Since

$$
\begin{aligned}
&|(P_n)[j_{n,n_1}, \ldots, j_{n,n_{l+1}}, k_{n,l}; n_1, \ldots, n_{l+1}](q^{n_{l+1}}(x_n) + \theta_n q_{n_{l+1}}(x_n)|^{1/k_n} \\
&\quad = \theta_n^{k_{n,l+1}/k_n} |(P_n)[j_{n,n_1}, \ldots, j_{n,n_{l+1}}, k_{n,l}; n_1, \ldots, n_{l+1}](x_n)|^{1/k_n} \\
&\quad \geqslant \theta_n^{k_{n,l+1}/k_n} \cdot a_{l+2,n}^{1/k_n} \geqslant \theta_n^{k_{n,l+1}/k_n} \left(\frac{1}{2k_n}\right)^{l+1/k_n} \\
&\quad \to 1 \qquad \text{as} \quad n \to \infty
\end{aligned}
$$

we have a contradiction and hence $\sum_{i=1}^{l+1} \delta_i < 1$. By induction we have shown the existence of $\{\delta_i, n_i, (j_{n,n_i})_{n=i+1}^{\infty}\}_{i=1}^{\infty}$ satisfying (3.9)–(3.14). Since $\sum_{n=1}^{\infty} \delta_n \leqslant 1$ we can choose $(\psi_n)_{n=1}^{\infty}$ such that $\psi_n > 0$ all $n$, $\psi_n \to +\infty$ as $n \to \infty$ and $\sum_{n=1}^{\infty} \delta_n \psi_n \leqslant \log 2$.

Let $\varepsilon_n = \exp(-\psi_n)$ for all $n$. Then $\varepsilon_n < 1$ for all $n$ and $(\varepsilon_n)_n \in c_0$. Since $\sum_{n=1}^{\infty} \delta_n \log 1/\varepsilon_n = \sum_{n=1}^{\infty} -\delta_n \log \varepsilon_n = \sum_{n=1}^{\infty} \delta_n \psi_n \leqslant \log 2$ we have for any integer $l$

$$
\begin{aligned}
&\textstyle\sum_{n=1}^{l} \delta_n \log \varepsilon_n = -\sum_{n=1}^{l} \delta_n \log 1/\varepsilon_n \geqslant -\log 2 = \log \tfrac{1}{2}, \\
&\text{i.e., } \log(\varepsilon_1^{\delta_1} \varepsilon_2^{\delta_2} \cdots \varepsilon_l^{\delta_l}) \geqslant \log \tfrac{1}{2} \text{ and hence} \\
&\varepsilon_1^{\delta_1} \varepsilon_2^{\delta_2} \cdots \varepsilon_l^{\delta_l} \geqslant \tfrac{1}{2} \text{ for all } l.
\end{aligned} \tag{3.15}
$$

Let $g = \sum_{l=2}^{\infty} (P_l)[j_{l,n_1}, \ldots, j_{l,n_{l-1}}, k_{l,l}; n_1, \ldots, n_{l-1}]$. Since $g$ is a modification of $f$ it belongs to $H_{wb}(c_0)$. Let

$$
\begin{aligned}
w_n &= \varepsilon_1 && \text{for} \quad j \leqslant n_1 \\
&= \varepsilon_{l+1} && \text{for} \quad n_{l-1} < j \leqslant n_l \quad \text{all } l > 1.
\end{aligned}
$$

Since $(\varepsilon_n)_n \in c_0$ we have $w = (w_n)_n \in c_0$. The solid hull of $w$, $\tilde{w}$, is a compact subset of $c_0$. The sequence $q^{n_1}(x_1), q_{n_1}^{n_2}(x_2), \ldots, q_{n_{l-1}}^{n_l}(x_l), \ldots,$ is a weak null sequence in $c_0$ and hence its closed convex hull $L$ is a weakly compact subset of $c_0$. Thus the set $\tilde{w} + L = W$ is also a weakly compact subset of $c_0$. For any integer $l$ the vector

$$
y_l = \varepsilon_1 q^{n_1}(x_l) + \varepsilon_2 q_{n_1}^{n_2}(x_l) + \cdots + \varepsilon_{l-1} q_{n_{l-2}}^{n_{l-1}}(x_l) + q_{n_{l-1}}^{n_l}(x_l)
$$

belongs to $W$ since $\varepsilon_1 q^{n_1}(x_l) + \cdots + \varepsilon_{l-1} q^{n_{l-1}}_{n_{l-2}}(x_l) \in \tilde{w}$ and $q^{n_l}_{n_{l-1}}(x_l)$ belongs to $L$.

Now $(a_{l,l})^{-1}$ is the product of $l-1$ integers each of which is less than $k_l + 1$. Hence

$$a_{l,l} \geqslant \left(\frac{1}{k_l+1}\right)^{l-1} \geqslant \left(\frac{1}{k_l+1}\right)^{l}$$

and so by (3.2)

$$a_{l,l}^{1/k_l} \geqslant \left(\frac{1}{k_l+1}\right)^{l/k_l} \geqslant \frac{1}{e}. \tag{3.16}$$

Hence

$$\begin{aligned}
&|(P_l)[j_{l,n_1},\dots, j_{l,n_{l-1}}, k_{l,l}; n_1,\dots, n_{l-1}](y_l)|^{1/k_l} \\
&= \left| \binom{k_l}{j_{l,n_1},\dots, j_{l,n_{l-1}}, k_{l,l}} A_l(\varepsilon_1 q^{n_1}(x_l))^{j_{l,n_1}} \cdots (\varepsilon_{l-1} q^{n_{l-1}}_{n_{l-2}}(x_l))^{j_{l,n_{l-1}}} (q^{n_l}_{n_{l-1}}(x_l))^{k_{l,l}} \right|^{1/k_l} \\
&= [\varepsilon_1^{j_{l,n_1}} \varepsilon_2^{j_{l,n_2}} \cdots \varepsilon_{l-1}^{j_{l,n_{l-1}}}]^{1/k_n} \\
&\quad \times \left| \binom{k_l}{j_{l,n_1},\dots, j_{l,n_{l-1}}, k_{l,l}} A_l(q^{n_1}(x_l))^{n_{l,n_1}} \cdots (q^{n_{l-1}}_{n_{l-2}}(x_l))^{j_{l,n_{l-1}}} (q^{n_l}_{n_{l-1}}(x_l))^{k_{l,l}} \right|^{1/k_l} \\
&= \varepsilon_1^{j_{l,n_1}/k_l} \cdots \varepsilon_{l-1}^{j_{l,n_{l-1}}/k_l} |(P_l)[j_{l,n_1},\dots, j_{l,n_{l-1}}, k_{l,l}; n_1,\dots, n_{l-1})(x_l)|^{1/k_l} \\
&\geqslant \varepsilon_1^{\delta_1/2} \cdots \varepsilon_{l-1}^{\delta_{l-1}/2} (a_{l,l})^{1/k_l} \qquad \text{(by 3.11 and 3.13)} \\
&\geqslant \frac{1}{\sqrt{2}} \cdot \frac{1}{e} \qquad \text{(by 3.15 and 3.16).}
\end{aligned}$$

This contradicts the fact that $g \in H_{wb}(c_0)$ and completes the proof.

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