Geometry and analytic boundaries of Marcinkiewicz sequence spaces

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Abstract

We investigate the geometric structure of the unit ball of the Marcinkiewicz sequence space m_Ψ^0, giving characterisations of its real and complex extreme points and of the exposed points in terms of the symbol Ψ. Using our knowledge of the geometry of $B_{m_\Psi^0}$ we then give necessary and sufficient conditions for a subset of $B_{m_\Psi^0}$ to be a boundary for $\mathcal{A}_u(B_{m_\Psi^0})$, the algebra of functions which are uniformly continuous on $B_{m_\Psi^0}$ and holomorphic on the interior of $B_{m_\Psi^0}$. We show that it is possible for the set of peak points of $\mathcal{A}_u(B_{m_\Psi^0})$ to be a boundary for $\mathcal{A}_u(B_{m_\Psi^0})$ yet for $\mathcal{A}_u(B_{m_\Psi^0})$ not to have a Šilov boundary in the sense of Globevnik.

Introduction

This paper examines the interaction between geometry of Banach spaces and algebras of holomorphic functions. Specifically, we will show that, for a large class of Banach spaces with an unconditional basis, the set of complex extreme points of the unit ball coincides with the peak points of an algebra of holomorphic functions defined on its ball. This allows us to give a complete characterisation of the complex extreme points in terms of its coordinates. It also enables us to give necessary and sufficient conditions on a subset of the unit sphere to be a boundary for the algebra of functions which are continuous on the closed unit ball and holomorphic on its interior. The connection between geometry of Banach spaces and holomorphic functions is not without precedent. Complex extreme points were introduced by Thorp and Whitley, [18], which allowed them to establish the Strong Maximum Modulus Principle while Globevnik, [12], showed that peak points for the ball algebra over E are complex extreme points of the unit ball of E.

Given a complex Banach space E we denote by $\mathcal{A}_b(B_E)$ the Banach algebra of all functions which are continuous and bounded on B_E, the closed unit ball of E, and holomorphic on the interior of B_E. By $\mathcal{A}_u(B_E)$ we denote the Banach

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algebra of functions in $\mathcal{A}_b(B_E)$ which are uniformly continuous on B_E. We note that $\mathcal{A}_u(B_E) = \mathcal{A}_b(B_E)$ if and only if E is finite dimensional. We use Δ to denote the open unit disc in $\mathbb{C}$.

A subset B of B_E is said to be a boundary for $\mathcal{A}_b(B_E)$ (resp. $\mathcal{A}_u(B_E)$) if $\|f\| = \sup_{z\in B} |f(z)|$ for all f in $\mathcal{A}_b(B_E)$ (resp. $\mathcal{A}_u(B_E)$). When E is finite dimensional the intersection of all closed boundaries for $\mathcal{A}_u(B_E)$ is also a closed boundary for $\mathcal{A}_u(B_E)$. Globevnik, [12], calls a minimal closed boundary for $\mathcal{A}_u(B_E)$ a Šilov boundary of $\mathcal{A}_u(B_E)$. For E an infinite dimensional Banach space the intersection of all closed boundaries of $\mathcal{A}_u(B_E)$ needs not be a boundary (see [11]) and therefore $\mathcal{A}_u(B_E)$ may not have a Šilov boundary in the sense of Globevnik. The first example of such a situation is due to Globevnik himself, [11], who shows that $\mathcal{A}_u(B_{c_0})$ does not have a Šilov boundary. Since then further examples have followed. Aron, Choi, Lourenço and Paques, [5], show that c_0 may be replaced by ℓ_∞. In [16], Moraes and Romero Grados proved that $\mathcal{A}_u(B_{G_p})$ has no Šilov boundary in the sense of Globevik for $1 \le p < \infty$ where each G_p is the predual of a Lorentz sequence space modeled on that constructed by Gowers in [13]. In [17] they establish the same result for $\mathcal{A}_b(B_{G_p})$. Recently, Choi and Han, [8], extended the results of the paper of Moraes and Romero Grados to an even larger class of preduals of Lorentz spaces which includes both c_0 and the space G_p of Moraes and Romero Grados, [16], while Acosta, Moraes and Romero Grados, [4], give a characterisation of boundaries of preduals of Lorentz spaces in terms of the distance to the set of strong peak points. Acosta, [1], shows that there is no Šilov boundary in the sense of Globevik for $\mathcal{A}_u(B_{C(K)})$ when K is infinite, compact and Hausdorff. This result extended that of Choi, García, Kim and Maestre, [7], who have the additional assumption that K is scattered. In [2], Acosta and Lourenço prove that the Schreier space and the space of compact operators, $K(\ell_p, \ell_q)$, $1 \le p \le q < \infty$, both fail to have Šilov boundaries in the sense of Globevnik. On the positive side however, it is shown in [5] that the unit sphere is a Šilov boundary for $\mathcal{A}_u(B_{\ell_p})$ when $1 \le p < \infty$ while Acosta and Lourenço, [2], prove that the Lorentz space $d(w, 1)$ and the space of trace class operators both possess Šilov boundaries in the sense of Globevnik. An excellent survey article on some of the results mentioned above can be found in [3].

A point x in B_E is said to be a peak point for $\mathcal{A}_b(B_E)$ (resp. $\mathcal{A}_u(B_E)$) if there is f in $\mathcal{A}_b(B_E)$ (resp. $\mathcal{A}_u(B_E)$) such that $|f(y)| < f(x)$ for all y in $B_E \setminus \{x\}$. The set of all peak points of $\mathcal{A}_b(B_E)$ (resp. $\mathcal{A}_u(B_E)$) is called the Bishop boundary of $\mathcal{A}_b(B_E)$ (resp. $\mathcal{A}_u(B_E)$).

Using a 1959 result of Bishop, [6], it can be shown that if E is finite dimensional the Bishop boundary of $\mathcal{A}_u(B_E)$ is equal to its Šilov boundary. When E is infinite dimensional, like the Šilov boundary, the Bishop boundary of $\mathcal{A}_u(B_E)$ may be empty.

The class of spaces which we choose to work with in this paper are a class of canonical *'prebiduals'* of the Marcinkiewicz sequence spaces. This class will include c_0 and all the spaces which are discussed in both [16] and [8]. We shall examine both their geometric and analytic structure. In the first section we will characterise their complex extreme, real extreme and exposed points while in Section 2 we will give necessary and sufficient conditions on a subset of the unit ball of the space, E, to be a boundary for the $A_u(B_E)$. This allows us to show that in many cases $A_u(B_E)$ does not have a Šilov boundary in the sense of Globevnik.

Given a bounded sequence $z = (z_n)_n$ we define the distribution of z, d_z, by $d_z(s) = \text{card}\{k \in \mathbb{N} : |z_k| > s\}$ for $s \geq 0$. Setting $z_n^* = \inf\{s > 0 : d_z(s) \leq n\}$ we obtain the decreasing rearrangement, z^*, of z.

Let us recall the definition and some elementary facts about Marcinkiewicz sequence spaces. We let $\Psi = (\Psi(n))_{n=0}^{\infty}$ be an increasing sequence of nonnegative real numbers with $\Psi(0) = 0$ and $\Psi(n) > 0$ if $n \geq 1$. Such functions will be called symbols. The Marcinkiewicz sequence space associated to the symbol Ψ, m_Ψ, is the vector space of all bounded sequences $(z_n)_n$ such that

$$\|z\| := \sup_{k \geq 1} \frac{\sum_{j=1}^{k} z_j^*}{\Psi(k)} < \infty,$$

where $z^* = (z_n^*)$ is the decreasing rearrangement of $(z_n)_n$. We denote by m_Ψ^0 the subspace of m_Ψ consisting of all z such that

$$\lim_{k \to \infty} \frac{\sum_{j=1}^{k} z_j^*}{\Psi(k)} = 0.$$

To avoid the case where $m_\Psi^0 = \{0\}$ we shall assume that $\lim_{n \to \infty} \Psi(n) = \infty$.

We assume without loss of generality that $\Psi(1) = 1$. This condition is equivalent to the assumption that $\|e_j\| = 1$ for all j in $\mathbb{N}$. It follows from [14] that we can also assume that $(\Psi(n)/n)$ is decreasing. From this it follows that if $z \in m_\Psi^0$ then $\lim_n |z_n| = 0$ and $\|z\|_\infty \leq \|z\|$. Thus $m_\Psi^0 \hookrightarrow c_0$ and the standard unit vectors $(e_j)_j$ form an unconditional basis for m_Ψ^0. However, this will also follow if we relax the condition and require that $(\Psi(n)/n)$ is eventually decreasing allowing a wider class of symbols to which our results apply.

Choi and Han, [8], examine the boundaries $\mathcal{A}_u(B_{m_\Psi^0})$ when Ψ is strictly increasing. Here we will investigate the geometric structure of the unit ball of m_Ψ^0 along with the boundaries of $\mathcal{A}_u(B_{m_\Psi^0})$ for arbitrary Ψ.

Given a Banach space E, a point z in B_E is said to be a real extreme point of B_E if z is not the midpoint of any line segment which is contained in B_E. When E is a complex Banach space we shall say that z in B_E is a complex extreme point of B_E if $\|z + \lambda y\| \leq 1$ for all $\lambda \in \overline{\Delta}$ implies that $y = 0$. The real extreme points of B_E are denoted by $\text{Ext}_{\mathbb{R}}(B_E)$ while the complex extreme points are denoted by $\text{Ext}_{\mathbb{C}}(B_E)$.

Let E be a complex Banach space. A point z in E is said to be exposed point of the unit ball of E if there is a linear function, $\varphi \in E'$, such that $\varphi(z) = 1$ and $Re(\varphi(y)) < 1$ for all $y \in B_E$, $y \neq z$. A unit vector z is strongly exposed if there is a unit vector $\varphi \in E'$ so that $\varphi(z) = 1$ and given any sequence $(z_k) \subseteq B_E$ with $\varphi(z_k) \to 1$ we can conclude that z_k converges to z in norm. We will say that φ strongly exposes B_E at z. We denote the set of exposed points of the unit ball of E by $\text{Exp}(B_E)$ and the set of strongly exposed points by $\text{St} - \text{Exp}(B_E)$.

1 Geometry of m_Ψ^0

In order to study the geometry of m_Ψ^0 we introduce the notion of the torus of level n for any $n \in \mathbb{N}$.

Definition 1 *The n-torus, $\mathcal{T}_n$, is the set of all $z \in m_\Psi^0$ such that*

(Tn1) *z has support of length n,*

(Tn2) $\sum_{j=1}^n z_j^* = \Psi(n)$,

(Tn3) $\sum_{j=1}^k z_j^* \leq \Psi(k)$, *for all $k < n$.*

Since $(\Psi(n))_{n\geq 0}$ is an increasing sequence, for any symbol Ψ, $\mathcal{T}_n$ is actually a subset of the unit sphere of m_Ψ^0.

We say that the support of z is $\sigma = \{j_1, \dots, j_n\}$ if $z = \sum_{k=1}^n z_{j_k} e_{j_k}$, where each z_{j_k} is nonzero and $j_1 < \cdots < j_n$. By m_Ψ^σ we understand the finite dimensional subspace of m_Ψ^0 given by $\Pi_\sigma(m_\Psi^0)$ where Π_σ is the continuous linear operator on m_Ψ^0 defined by $\Pi_\sigma(z) = \sum_{k=1}^n z_{j_k} e_{j_k}$. We endow m_Ψ^σ with the norm induced from that on m_Ψ^0. We denote by m_Ψ^n the space associated to the initial set $\sigma = \{1, \dots, n\}$ and use Π_n to denote the projection of m_Ψ or m_Ψ^0 onto m_Ψ^n which sends z to $(z_j)_{j=1}^n$.

Proposition 2 *Let Ψ be a symbol. Let $z \in \mathcal{T}_n$ and let σ be the support of z. Then $\Pi_\sigma(z)$ is a peak point for $\mathcal{A}_u(B_{m_\Psi^\sigma})$.*

Proof: When $n = 1$ $\mathcal{A}_v(B_{m_\Psi^1})$ is the disc algebra $\mathcal{A}(\Delta)$ and every point of $\mathcal{T}_1$, the unit sphere, is a peak point. We suppose that $n \geq 2$. If $\sigma = \{j_1, \dots, j_n\}$ then $\Pi_\sigma(z) = (z_{j_1}, \dots, z_{j_n})$. Since $z_{j_k} \neq 0$, $\theta_k = -\arg(z_{j_k})$ is defined for all $k = 1, \dots, n$. Now, by (Tn2), we may consider the polynomial $g\colon B_{m_\Psi^\sigma} \to \mathbb{C}$ associated to $\Pi_\sigma(z)$, defined, for $x = (x_1, x_2, \dots, x_n)$, by

$$g(x) = \sum_{k=1}^n \left[\left(1 + \frac{e^{i\theta_k} x_k}{\Psi(n) - |z_{j_k}|}\right)\left(1 + \frac{1}{|z_{j_k}|} \sum_{l \neq k} e^{i\theta_l} x_l\right)\right]. \tag{1}$$

It is clear that g belongs to $\mathcal{A}_u(B_{m_\Psi^\sigma})$ and $g(\Pi_\sigma(z)) > 0$. Following the proof of [16, Theorem 2.3] it is possible to show that $|g(x)| < g(\Pi_\sigma(z))$ for all $x \neq \Pi_\sigma(z)$. It follows that $\Pi_\sigma(z)$ is a peak point of $\mathcal{A}_u(B_{m_\Psi^\sigma})$. ∎

As a consequence of [11, Theorem 4], we also obtain that $\Pi_\sigma(z)$ is a complex extreme point of $B_{m_\Psi^\sigma}$. The next proposition states that having finite support is a necessary condition to be a complex extreme point in m_Ψ^0. In its proof we adapt some techniques from [15].

Proposition 3 *Let Ψ be a symbol and $z \in \mathrm{Ext}_{\mathbb{C}}(B_{m_\Psi^0})$. Then z has finite support.*

Proof: Suppose that $z \in S_{m_\Psi^0}$ does not have finite support. Let $m \in \mathbb{N}$ be the largest positive integer such that $1 = \|z\| = \frac{1}{\Psi(m)} \sum_{j=1}^m z_j^*$. We have that

$z^*_{m+1} \neq 0$, otherwise $z^*_k = 0$ for all $k \geq m+1$ implying that z^* and therefore z have finite support. Take $0 < r < z^*_{m+1}$. Since $\lim |z_j| = 0$, there exists $n > m$ so that $|z_j| < r$ for all $j > n$. Finally, note that $a = 1 - \max\left\{\frac{1}{\Psi(l)} \sum_{j=1}^{l} z^*_j : l > m\right\} > 0$.

Consider $b = \min\{z^*_{m+1} - r, a\}$ and let $y = be_{n+1} \in B_{m^0_\Psi}$. We claim that $\|z + \lambda y\| \leq 1$ for all $\lambda \in \overline{\Delta}$. Indeed, for $j > m$ we have $|z_j + \lambda y_j| \leq r + b \leq z^*_{m+1}$. Then, if $k \leq m$ we have

$$\sum_{j=1}^{k} (z + \lambda y)^*_j = \sum_{j=1}^{k} z^*_j \leq \Psi(k)$$

while if $k > m$ we have

$$\sum_{j=1}^{k} (z + \lambda y)^*_j \leq \sum_{j=1}^{k} z^*_j + |\lambda| b \leq (1-a)\Psi(k) + a\Psi(k) = \Psi(k).$$

Hence z is not a complex extreme point of $B_{m^0_\Psi}$. ■

Proposition 4 *Let Ψ be a symbol and $z \in \mathrm{Ext}_{\mathbb{C}}(B_{m^0_\Psi})$. Then there exists a positive integer n so that $\Pi_n(z^*) \in \mathrm{Ext}_{\mathbb{C}}(B_{m^n_\Psi})$ and $\Psi(n) = \Psi(n+1)$.*

Proof: By Proposition 3 we know that z has finite support. Let us suppose that z has length n. It follows that $z^*_n \neq 0$. Clearly we have that $\Pi_n(z^*) \in \mathrm{Ext}_{\mathbb{C}}(B_{m^n_\Psi})$. Suppose that $\Psi(n) < \Psi(n+1)$. Let $a = \min\{z^*_n, 1 - \frac{\Psi(n)}{\Psi(n+1)}\} > 0$ and consider $y = ae_N \in B_{m^0_\Psi}$ where $N - 1$ is the index of the last nonzero coordinate of z. We have that $\sum_{j=1}^{n} z^*_j \leq \Psi(n) \leq (1-a)\Psi(n+1)$. For $|\lambda| \leq 1$

$$\sum_{j=1}^{n+1} (z + \lambda y)^*_j \leq \sum_{j=1}^{n} z^*_j + a \leq (1-a)\Psi(n+1) + a\Psi(n+1) = \Psi(n+1).$$

Also, if $k \leq n$, $\sum_{j=1}^{k} (z + \lambda y)^*_j = \sum_{j=1}^{k} z^*_j \leq \Psi(k)$. Since $(\Psi(n))$ is increasing the same holds for $k > n+1$. Therefore $\|z + \lambda y\| \leq 1$ for all $\lambda \in \overline{\Delta}$ contradicting the fact that z is a complex extreme point of $B_{m^0_\Psi}$. ■

Given a symbol Ψ we denote by $\mathbb{N}_\Psi$ the set of all the positive integers $n \in \mathbb{N}$ such that $\Psi(n) = \Psi(n+1)$. In [15] Kamińska and Lee prove that the unit ball of m^0_Ψ has an extreme point if an only if there is n in $\mathbb{N}$ with $\Psi(n) = \Psi(n+1)$. Specifically, they show that for each n in $\mathbb{N}_\Psi$ the point $\frac{\Psi(n)}{n} \sum_{i=1}^{n} e_i$ is an extreme point of the unit ball of m^0_Ψ. The following theorem gives a complete characterisation of the extreme points of the unit ball of m^0_Ψ and proves that for the unit ball of m^0_Ψ the sets of complex extreme points and peak points coincide.

Theorem 5 *Let Ψ be a symbol. The following are equivalent:*

(a) *z is a peak point for $\mathcal{A}_u(B_{m_\Psi^0})$,*

(b) $z \in \mathrm{Ext}_{\mathbb{C}}(B_{m_\Psi^0})$,

(c) *there is n in $\mathbb{N}_\Psi$ such that $z \in \mathcal{T}_n$.*

Proof: An application of [11, Theorem 4] gives us that (a) implies (b).

To show that (b) implies (c) take $z \in \mathrm{Ext}_{\mathbb{C}}(B_{m_\Psi^0})$. By Proposition 3 z has finite support. Let us suppose that z has support of length n. Clearly $\sum_{j=1}^k z_j^* \leq \Psi(k)$ for all $k \in \mathbb{N}$. Suppose that $\sum_{j=1}^n z_j^* < \Psi(n)$. Since $z_n^* \neq 0$ we may consider $a = \min\{z_n^*,\ \Psi(n) - \sum_{j=1}^n z_j^*\} > 0$ and $y = ae_N$ where $N-1$ is the index of the last nonzero coordinate of z.

Let $w = z + \lambda y$ with $|\lambda| \leq 1$, then $w^* = (z_1^*, z_2^*, \ldots, z_n^*, |\lambda| a, 0, \ldots)$. For $k \leq n$ we have $\sum_{j=1}^k w_j^* = \sum_{j=1}^k z_j^* \leq \Psi(k)$. For $k \geq n+1$

$$\sum_{j=1}^k w_j^* = \sum_{j=1}^{n+1} w_j^* = \sum_{j=1}^n z_j^* + |\lambda| a \leq \sum_{j=1}^n z_j^* + \Psi(n) - \sum_{j=1}^n z_j^* = \Psi(n) \leq \Psi(n+1) \leq \Psi(k).$$

Hence $\|z + \lambda y\| \leq 1$ for all $\lambda \in \overline{\Delta}$ contradicting the fact that z is a complex extreme point. Therefore $z \in \mathcal{T}_n$. The proof of Proposition 4 shows that $\Psi(n) = \Psi(n+1)$ for $n = |\mathrm{supp}(z)$.

Since m_Ψ^0 is a rearrangement invariant space, to prove that (c) implies (a) it is enough to show that z^* is a peak point for $A_u(B_{m_\Psi^0})$.

Consider the linear functional $\varphi \in (m_\Psi^0)'$ associated to z^*, defined by

$$\varphi(x) = \frac{1}{\Psi(n)} \sum_{j=1}^n x_j + \frac{1}{\Psi(n)} \sum_{j=1}^\infty \frac{x_{n+j}}{3^j}. \tag{2}$$

Since $\Psi(n) = \Psi(n+1)$ we have that

$$\begin{aligned} |\varphi(x)| &\leq \frac{1}{\Psi(n)} \sum_{j=1}^n |x_j| + \frac{1}{\Psi(n)} \sum_{j=1}^\infty \frac{|x_{n+j}|}{3^j} \\ &\leq \frac{1}{\Psi(n)} \sum_{j=1}^n x_j^* + \frac{1}{\Psi(n)} \sum_{j=1}^\infty \frac{x_{n+j}^*}{3^j} \\ &\leq \frac{1}{\Psi(n)} \sum_{j=1}^n x_j^* + \frac{1}{2} \frac{1}{\Psi(n)} x_{n+1}^* \\ &\leq \frac{1}{\Psi(n+1)} \sum_{j=1}^{n+1} x_j^*. \end{aligned}$$

Thus $|\varphi(x)| \leq 1$ for all $\|x\| \leq 1$ and $\varphi(z^*) = 1$. Note that the last inequality in the above estimate of $|\varphi(x)|$ is strict whenever $|supp(z)| > n$ and hence $|\varphi(x)| < 1$ if $|\mathrm{supp}(x)| > n$.

Let $g\colon B_{m_\Psi^n} \to \mathbb{C}$ be the polynomial associated to $(z_1^*, \ldots, z_n^*)$ defined as in (1)

$$g(x) = \sum_{j=1}^n \left[\left(1 + \frac{x_j}{\Psi(n) - z_j^*}\right)\left(1 + \frac{1}{z_j^*} \sum_{l \neq j} x_l\right) \right].$$

Now consider the holomorphic function $f\colon B_{m_\Psi^0} \to \mathbb{C}$ given by $f = \frac{1}{2}(g+\varphi)$. By the proof of [16, Theorem 2.3] we have that $|g(x)| < g(z^*)$ for all x with $\Pi_n(x) \neq \Pi_n(z^*)$. Whenever $\Pi_n(x) = \Pi_n(z^*)$ but $x \neq z^*$ we have that $\mathrm{supp}(x) > n$ and $|\varphi(x)| < \varphi(z^*)$. Therefore $|f(x)| < f(z^*)$ for any $x \neq z^*$ and $f/f(z^*)$ peaks over $B_{m_\Psi^0}$ at z^*. ■

Corollary 6 *Let Ψ be a symbol. Let $z \in \mathrm{Ext}_{\mathbb{C}}(B_{m_\Psi^n})$ and suppose that $|supp(z)|-k$. Then either $k = n$ or $k < n$ and $k \in \mathbb{N}_\Psi$.*

Proof: Suppose that $k = |supp(z)| < n$ and $\Psi(k) < \Psi(k+1)$. Let $z^* = (z_1^*, \ldots, z_k^*, 0, \ldots)$ and $a = \min\{z_k^*, 1 - \frac{\Psi(k)}{\Psi(k+1)}\} > 0$. Set $y = ae_{k+1}$. As in Proposition 4 we can show that z^*, and therefore z, is not an extreme point. The result now follows. ■

Denote by $\mathcal{P}$ the set of all the peak points of $\mathcal{A}_u(B_{m_\Psi^0})$ and for n in $\mathbb{N}$, denote by $\mathcal{P}_n$ the set of all points z in m_Ψ^0 with support of length at most n such that $\Pi_n(z^*)$ is a peak point for $\mathcal{A}_u(B_{m_\Psi^n})$.

The equivalence between (a) and (c) of Theorem 5 gives the following result.

Corollary 7 *Let Ψ be a symbol. Then $\mathcal{P} = \bigcup_{n\in\mathbb{N}_\Psi} \mathcal{T}_n$.*

As consequence of the above, we obtain another characterisation of the set of all peak points in terms of the sets of peaks points of level n. To be more precise we have:

Corollary 8 *Let Ψ be a symbol. Then $\mathcal{P} = \bigcup_{n\in\mathbb{N}_\Psi} \mathcal{P}_n$.*

Proof: Suppose z belongs to $\mathcal{P}_n$ for some n in $\mathbb{N}_\Psi$, then there exists g in $\mathcal{A}_u(m_\Psi^n)$ such that g peaks at $\Pi_n(z^*)$. Let k be the length of the support of z^*. It follows from [11, Theorem 4] and Corollary 6 that $k \in \mathbb{N}_\Psi$. Let φ in $(m_\Psi^0)'$ be the linear function associated to z^* defined as in (2). It follows as in Theorem 5 that $f = \frac{1}{2}(g+\varphi)$ is in $A_u(B_{m_\Psi^0})$ and $f/f(z^*)$ peaks over $B_{m_\Psi^0}$ at z^*. Consequently, z^* and z belong to $\mathcal{P}$.

Conversely, if $z \in \mathcal{P}$ then $z^* \in \mathcal{P}$. By Theorem 5, there exists $n \in \mathbb{N}_\Psi$ such that $|\mathrm{supp}(z^*)| = n$. Restricting the function f which peaks $B_{m_\Psi^0}$ at z^* we get that $z \in \mathcal{P}_n$. ■

It is perhaps worth observing that in general $\mathrm{Ext}_{\mathbb{C}}(B_{m_\Psi^0})$ fails to be equal to $\bigcup_{n\in\mathbb{N}_\Psi}\{z : \Pi_n(z^*) \in \mathrm{Ext}_{\mathbb{C}}(B_{m_\Psi^n})\}$. To see this consider any weight Ψ with $\Psi(1) = 1$, $\Psi(2) = 2$, $\Psi(3) = 2.5$, $\Psi(4) = 3.25$, $\Psi(5) = \Psi(6) = 4$. In follows from [8, Proposition 2.5] that the point $(1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is a complex extreme point of $B_{m_\Psi^5}$. However, using Theorem 5, we can readily check that $(1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, \ldots)$ is not a complex extreme point of $B_{m_\Psi^0}$.

Theorem 9 *The set of all peak points, $\mathcal{P}$, is closed.*

Proof: Let z belong to $\overline{\mathcal{P}}$, the closure of $\mathcal{P}$, and $\mathbb{N}_\Psi = \{n_j\}_{j\geq 1}$. Choose $N \in \mathbb{N}$ so that for all $n \geq N$,

$$\frac{1}{\Psi(n)} \sum_{i=1}^{n} z_i^* < \frac{1}{2}.$$

We first show that z belongs to $\overline{\bigcup_{n_j<N} \mathcal{T}_{n_j}}$ and then we show that $\bigcup_{n_j<N} \mathcal{T}_{n_j}$ is closed. Suppose $n_j \in \mathbb{N}_\Psi$, with $n_j \geq N$, and take y in $\mathcal{T}_{n_j}$. We have

$$\|y-z\| \geq \frac{1}{\Psi(n_j)} \sum_{i=1}^{n_j} |y_i - z_i| \geq \frac{1}{\Psi(n_j)} \left(\sum_{i=1}^{n_j} |y_i| - \sum_{i=1}^{n_j} z_i^* \right) \geq 1 - \frac{1}{2} = \frac{1}{2}.$$

Hence $z \notin \overline{\bigcup_{n_j \geq N} \mathcal{T}_{n_j}}$ and therefore, by Corollary 7, $z \in \overline{\bigcup_{n_j<N} \mathcal{T}_{n_j}}$. Choose a sequence $(z^n)_{n=1}^{\infty}$ in $\bigcup_{n_j<N} \mathcal{T}_{n_j}$ which converges to z. As N is finite there is a largest positive integer $M \in \mathbb{N}_\Psi$, $M < N$, so that $\mathcal{T}_M$ contains infinite many $(z^n)_n$. Without loss of generality, we may assume that $(z^n)_{n=1}^{\infty}$ is contained in $\mathcal{T}_M$.

We observe that the length of the support of z cannot be greater than M. To see this, note that if $z_i \neq 0$ we can find n_i so that $|z_i - z_i^n| < |z_i|/2$ for all $n > n_i$ and hence $z_i^n \neq 0$ for all n sufficiently large. As the length of the support of each term in the sequence $(z^n)_{n=1}^{\infty}$ is M, z has support of length at most M.

Given $\epsilon > 0$ choose n_0 so that $\|z - z^{n_0}\| < \epsilon/\Psi(M)$. Let $(i_1, i_2, \ldots, i_M)$ be the support of z^{n_0}. Then we have that

$$\Big| \sum_{j=1}^{M} |z_{i_j}| - \Psi(M) \Big| = \Big| \sum_{j=1}^{M} |z_{i_j}| - \sum_{j=1}^{M} |z_{i_j}^{n_0}| \Big| \leq \sum_{j=1}^{M} |z_{i_j} - z_{i_j}^{n_0}| \leq \Psi(M)\|z - z^{n_0}\| < \epsilon.$$

Since ϵ is arbitrary we get that $\sum_{j=1}^{M} z_j^* = \Psi(M)$. As $z \in B_{m_\Psi^0}$ we have that $\sum_{j=1}^{k} z_j^* \leq \Psi(k)$ for all k. Thus $\sum_{j=1}^{M} z_j^* = \Psi(M)$. Note that the length of the support of z could be $l < M$ but since z belongs to the unit ball this will merely imply that $\Psi(l) = \Psi(M)$ and therefore $\sum_{j=1}^{l} z_j^* = \Psi(l)$. Hence z belongs to $\mathcal{T}_l$, with $l \in \mathbb{N}_\Psi$. ∎

The above Theorem shows that any finite union of $\mathcal{T}_n$ is closed.

The real extreme points of a finite dimensional Lorentz space are characterised in [10, Proposition 8]. These correspond to the Marcinkiewicz space m_Ψ^n where $\Psi(j+1) - \Psi(j)$, $1 \leq j \leq n-1$, is a decreasing function of j. For an arbitrary symbol Ψ we obtain the following characterisation of real extreme points.

Theorem 10 *Let Ψ be a symbol. Then $z \in \mathrm{Ext}_{\mathbb{R}}(B_{m_\Psi^0})$ if and only if there is n in $\mathbb{N}_\Psi$ such that z belongs to $\mathcal{T}_n$ and $\Pi_n(z^*) \in \mathrm{Ext}_{\mathbb{R}}(B_{m_\Psi^n})$.*

Proof: First suppose that $z \in \mathrm{Ext}_{\mathbb{R}}(B_{m_\Psi^0})$. Then $z \in \mathrm{Ext}_{\mathbb{C}}(B_{m_\Psi^0})$ and so it follows from Theorem 5 that there is n in $\mathbb{N}_\Psi$ such that z belongs to $\mathcal{T}_n$. Moreover, if $\Pi_n(z^*) \notin \mathrm{Ext}_{\mathbb{R}}(B_{m_\Psi^n})$ then we can find y in m_Ψ^n, $y \neq 0$, so that

$\Pi_n(z^*) \pm y \in B_{m_\Psi^n}$. Define $\tilde{y}$ in m_Ψ^0 by

$$\tilde{y}_j = \begin{cases} y_j & \text{if } z_j \neq 0, \\ 0 & \text{if } z_j = 0. \end{cases}$$

It follows that $z \pm \tilde{y} \in B_{m_\Psi^0}$ contradicting the fact that $z \in \mathrm{Ext}_{\mathbb{R}}(B_{m_\Psi^0})$.

Now suppose that z belongs to $\mathcal{T}_n$ and $\Pi_n(z^*) \in \mathrm{Ext}_{\mathbb{R}}(B_{m_\Psi^n})$. Let $\sigma = \{j_1, \ldots, j_n\}$ be the support of z, with $n \in \mathbb{N}_\Psi$, and suppose y is a point of m_Ψ^0 with the property that $\|z \pm y\|_{m_\Psi^0} \leq 1$. Since $z \in \mathrm{Ext}_{\mathbb{R}}(B_{m_\Psi^n})$ and $\|z \pm y\|_{m_\Psi^\sigma} \leq \|z \pm y\|_{m_\Psi^0} \leq 1$ we have that $y_{j_1} = \cdots = y_{j_n} = 0$. For $k \notin \sigma$

$$\frac{1}{\Psi(n+1)} \sum_{l=1}^{n} |z_{j_l}| + \frac{|y_k|}{\Psi(n+1)} \leq \sum_{i=1}^{n+1} \frac{(z \pm y)_i^*}{\Psi(n+1)} \leq \|z \pm y\|_{m_\Psi^0} \leq 1.$$

We also have that

$$\frac{1}{\Psi(n+1)} \left(\sum_{l=1}^{n} |z_{j_l}| + |y_k| \right) = \frac{1}{\Psi(n)} \left(\Psi(n) + |y_k| \right) = 1 + \frac{|y_k|}{\Psi(n)}.$$

Hence

$$1 + \frac{|y_k|}{\Psi(n)} \leq 1$$

and $y_k = 0$ for all k proving that $z \in \mathrm{Ext}_{\mathbb{R}}(B_{m_\Psi^0})$. ∎

We note that the real and complex extreme points of m_Ψ^0 are, in general, different. To see this consider any symbol Ψ with $\Psi(1) = 1$, $\Psi(2) = 2$, $\Psi(3) = \Psi(4) = 2.5$. Using Theorem 5 we see that the point $(1, \frac{3}{4}, \frac{3}{4}, 0, 0, \ldots)$ is a complex extreme point of $B_{m_\Psi^0}$. However, since $(1, \frac{3}{4}, \frac{3}{4}, 0, 0, \ldots) = \frac{1}{2}(1, 1, \frac{1}{2}, 0, 0, \ldots) + \frac{1}{2}(1, \frac{1}{2}, 1, 0, 0, \ldots)$, it is not a real extreme point of $B_{m_\Psi^0}$.

We are also in a position to characterise the exposed points of the unit ball of m_Ψ^0.

Theorem 11 *Let Ψ be a symbol. Then z is an exposed point of $B_{m_\Psi^0}$ if and only if there is n in $\mathbb{N}_\Psi$ such that z belongs to $\mathcal{T}_n$ and $\Pi_n(z^*)$ is an exposed point of $B_{m_\Psi^n}$.*

Proof: If z is an exposed point of $B_{m_\Psi^0}$ then z is a complex extreme point of $B_{m_\Psi^0}$. It follows from Theorem 5 that there is n in $\mathbb{N}_\Psi$ such that z belongs to $\mathcal{T}_n$. Moreover, restricting the functional which exposes z to m_Ψ^n we see that z is an exposed point of $B_{m_\Psi^n}$.

Conversely, suppose there exists n in $\mathbb{N}_\Psi$ such that z belongs to $\mathcal{T}_n$ and $\Pi_n(z^*)$ is an exposed point of $B_{m_\Psi^n}$. Then we can find γ in $(m_\Psi^n)'$ such that $\gamma(\Pi_n(z^*)) = 1$ and $\gamma(w) < 1$ for w in $B_{m_\Psi^n}$ with $w \neq \Pi_n(z^*)$. Let φ be as in (2), arguing as in Theorem 5 we see that $\frac{1}{2}(\gamma + \varphi)$ peaks over $B_{m_\Psi^0}$ at z and hence z is an exposed point of the unit ball of m_Ψ^0. ∎

We finish our discussion on the geometry of m_Ψ^0 with two examples.

Example 12 *Consider the symbol with* $\Psi(1) = \Psi(2) = \ldots = \Psi(n) = \Psi(n+1) = 1$ *and* $\Psi(j) = j$ *for* $j \geq n+2$. *Using [14, Theorem 3.2 (4)] we see that* m^0_Ψ *is isomorphic to* c_0. *Moreover,* m^n_Ψ *is isometrically isomorphic to* ℓ^n_1. *Using Theorem 5 we see that every point of the unit sphere of* m^n_Ψ *is a complex extreme point. It follows from Theorems 10 and 11 that*

$$\mathrm{Ext}_{\mathbf{R}}(m^0_\Psi) = \mathrm{Exp}(m^0_\Psi) = \{\lambda e_i : i \in \mathbb{N}, |\lambda| = 1\}.$$

Example 13 *Consider the symbol with* $\Psi(1) = 1, \Psi(2) = 2, \ldots, \Psi(n-1) = n-1$, $\Psi(n) = \Psi(n+1) = n$ *and* $\Psi(j) = j$ *for* $j \geq n+2$. *Using [14, Theorem 3.2 (4)] again we obtain another renorming of* c_0. *This time we see that* m^n_Ψ *is isometrically isomorphic to* ℓ^n_∞. *The set of complex extreme points, real extreme points and exposed points all coincide with*

$$\{\lambda_1 e_{i_1} + \cdots + \lambda_n e_{i_n} : i_1, \ldots, i_n \text{ are distinct, } |\lambda_j| = 1\}.$$

2 Boundaries of $\mathcal{A}_u(B_{m^0_\Psi})$

Following the notation introduced in Section 1, for a finite ordered set $\sigma = \{j_1, \ldots, j_n\}$ we denote by $\mathcal{P}_\sigma$ the set of all points z in m^0_Ψ with support contained in σ such that $\Pi_\sigma(z)$ is a peak point of $\mathcal{A}_u(B_{m^\sigma_\Psi})$. This last condition is equivalent to requiring that $\Pi_n(z^*)$ be a peak point of $\mathcal{A}_u(B_{m^n_\Psi})$. Note that our definition of $\mathcal{P}_n$, in the previous section, does not correspond to any $\mathcal{P}_\sigma$ since an element in $\mathcal{P}_\sigma$ may have support of length strictly less than n. We denote by $|\sigma|$ the length of σ and by $\mathcal{T}_\sigma$ the restricted torus defined by $\mathcal{T}_\sigma = \{z \in m^\sigma_\Psi : z \in \mathcal{T}_{|\sigma|}\}$.

Lemma 14 *Let* σ *be a finite ordered set. Then* $\mathcal{P}_\sigma = \mathcal{T}_\sigma \cup \bigcup_{\substack{\gamma \subset \sigma \\ |\gamma| \in \mathbb{N}_\Psi}} \mathcal{T}_\gamma$. *In particular,* P_σ *is closed.*

Proof: Let $\sigma = \{j_1, \ldots, j_n\}$. For each z in $\mathcal{T}_\sigma$ the polynomial g, defined as in (1), associated to $(z^*_{j_1}, \ldots, z^*_{j_n})$ will peak at z^*. This implies that $\mathcal{T}_\sigma \subset \mathcal{P}_\sigma$. On the other hand, given z in $\mathcal{T}_\gamma$ with $\gamma \subset \sigma$, $|\gamma| \in \mathbb{N}_\Psi$, Theorem 5 implies that z is a peak point of $\mathcal{A}_u(B_{m^0_\Psi})$. Restricting the function which peaks at z to m^σ_Ψ we see that $\mathcal{P}_\sigma$ contains $\mathcal{T}_\gamma$.

Suppose that $\mathcal{P}_\sigma$ contains a point z which is not in $\mathcal{T}_\sigma \cup \bigcup_{\substack{\gamma \subset \sigma \\ |\gamma| \in \mathbb{N}_\Psi}} \mathcal{T}_\gamma$. Let g be the function in $\mathcal{A}_u(B_{m^\sigma_\Psi})$ which peaks at z. Define a symbol $\tilde{\Psi}$ by $\tilde{\Psi}(k) = \Psi(k)$ for $k \leq n$ and $\tilde{\Psi}(k) = \Psi(k-1)$ for $k \geq n+1$. Consider f in $\mathcal{A}_u(B_{m^\sigma_{\tilde{\Psi}}})$ given by $f = \frac{1}{2}(g + \varphi)$ where φ is a linear functional defined similarly to (2). It follows that $f/|f(z)|$ peaks at z and thus z belongs to $\mathcal{P}(\tilde{\Psi})$. This however contradicts Theorem 5 for the symbol $\tilde{\Psi}$. Thus $\mathcal{P}_\sigma = \mathcal{T}_\sigma \cup \bigcup_{\substack{\gamma \subset \sigma \\ |\gamma| \in \mathbb{N}_\Psi}} \mathcal{T}_\gamma$ and, in particular, P_σ is closed. ∎

Lemma 15 *Let* $(n_j)_{j \in \mathbb{N}}$ *be an increasing sequence of positive integers and* σ_{n_j} *be a sequence of finite ordered sets with* $|\sigma_{n_j}| = n_j$. *Then* $\bigcup_{j \geq 1} \mathcal{P}_{\sigma_{n_j}}$ *is closed in* $B_{m^0_\Psi}$.

Proof: Take z in the closure of $\bigcup_{j\geq 1} P_{\sigma_{n_j}}$. Since $\dfrac{\sum_{k=1}^n z_k^*}{\Psi(n)}$ tends to zero there exists $N \in \mathbb{N}$ so that for any $n \geq N$ we have $\dfrac{\sum_{k=1}^n z_k^*}{\Psi(n)} < \dfrac{1}{2}$.

Let $x \in \mathcal{T}_\gamma$ with $l = |\gamma| \geq N$ and $\gamma = \{i_1, \dots, i_l\} \subseteq \sigma_{n_j}$ some j in $\mathbb{N}$. We have

$$\|x - z\| \geq \frac{\sum_{k=1}^l |x-z|_{i_k}}{\Psi(l)} \geq \frac{\sum_{k=1}^l |x_{i_k}|}{\Psi(l)} - \frac{\sum_{k=1}^l |z_{i_k}|}{\Psi(l)} \geq 1 - \frac{\sum_{k=1}^l z_k^*}{\Psi(l)} > \frac{1}{2}.$$

By Lemma 14 it follows that $z \notin \overline{\bigcup_{j\geq 1} \bigcup_{\substack{\gamma\subseteq\sigma_{n_j}\\ |\gamma|\geq N}} \mathcal{T}_\gamma}$ and hence $z \in \overline{\bigcup_{j\geq 1} \bigcup_{\substack{\gamma\subseteq\sigma_{n_j}\\ |\gamma|< N}} \mathcal{T}_\gamma}$. Therefore there exists a sequence $(z^k)_k$ converging to z with z^k in some $\mathcal{T}_{\gamma_k}$ and $|\gamma_k| < N$. The proof of Theorem 9 now shows that $\bigcup_{j\geq 1} \bigcup_{\substack{\gamma\subseteq\sigma_{n_j}\\ |\gamma|< N}} \mathcal{T}_\gamma$ is closed and hence $\bigcup_{j\geq 1} \mathcal{P}_{\sigma_{n_j}}$ is closed. ∎

The following Theorem may also be deduced from [9, Corollary 2.3].

Theorem 16 *Let S be a subset of $B_{m_\Psi^0}$ and let $(\Pi_{\sigma_n})_n$ a sequence of finite dimensional projections with $\{\sigma_n\}_n$ increasing and $\bigcup_{n\in\mathbb{N}} \sigma_n = \mathbb{N}$. Then S is a boundary for $A_u(B_{m_\Psi^0})$ if and only if for every n, $\mathcal{P}_{\sigma_n}$ is contained in the closure of $\Pi_{\sigma_n}(S)$.*

Proof: Assume that P_{σ_n} is contained in the closure of $\Pi_{\sigma_n}(S)$ for every n and suppose that S is not a boundary for $\mathcal{A}_u(B_{m_\Psi^0})$. Then we can find $\epsilon > 0$ and $f \in \mathcal{A}_u(B_{m_\Psi^0})$ with $\|f\| = 1$ and $|f(z)| < 1 - \epsilon$ for all $z \in S$. Since m_Ψ^0 has a Schauder basis, the vectors with finite support are dense in $B_{m_\Psi^0}$. Therefore, there is a sequence of vectors (w_k) with finite support in $B_{m_\Psi^0}$ such that $\lim_{k\to\infty} |f(w_k)| = 1$.

For each k we consider m_k so that $\Pi_{\sigma_{m_k}}(w_k) = w_k$. As $\mathcal{P}_{\sigma_{m_k}}$ is a boundary for $\mathcal{A}_u(m_\Psi^{\sigma_{m_k}})$ we can choose x_k in $\mathcal{P}_{\sigma_{m_k}}$ so that $|f(w_k)| \leq |f(x_k)| \leq 1$, obtaining a sequence $(x_k)_k$ with $\lim_{k\to\infty} |f(x_k)| = 1$.

As the closure of $\Pi_{\sigma_{m_k}}(S)$ contains $\mathcal{P}_{\sigma_{m_k}}$ there exists u_k in $\Pi_{\sigma_{m_k}}(S)$ such that $|f(u_k)|$ converges to 1. Choose z_k in S with $u_k = \Pi_{\sigma_{m_k}}(z_k)$ and set $v_k = z_k - u_k$.

Fix $0 < \delta < 1$ and let $\xi \in \overline{\Delta}$. Note that since u_k and v_k are of disjoint support

$$\|u_k + \xi(1-\delta^2)v_k\| \leq \|u_k + v_k\| = \|z_k\| \leq 1.$$

By [11, Lemma 1.4] there exits $C > 0$ so that for any choice of α, β in $\overline{\Delta}$,

$$|f(u_k + \alpha(1-\delta)v_k) - f(u_k + \beta(1-\delta)v_k)| \leq C(1 - |f(u_k + \alpha(1-\delta)v_k)|).$$

Taking $\alpha = 0$ and $\beta = 1$ we have

$$|f(u_k) - f(u_k + (1-\delta)v_k)| \leq C(1 - |f(u_k)|).$$

From our choice of u_k we conclude that $|f(u_k + (1-\delta)v_k)| = |f(z_k - \delta v_k)| \to 1$, for any $0 < \delta < 1$. The uniform continuity of f gives that $\lim_{k\to\infty} |f(z_k)| = 1$. Since $(z_k)_k \subset S$ this is a contradiction. Thus S is a boundary for $\mathcal{A}_u(B_{m_\Psi^0})$.

Conversely, suppose S is a boundary for $\mathcal{A}_u(B_{m^0_\Psi})$ and that there exists σ_n so that the closure of $\Pi_{\sigma_n}(S)$ does not contain $\mathcal{P}_{\sigma_n}$. Then there exists $z \in \mathcal{P}_{\sigma_n}$ and $\delta > 0$ such that for all $x \in \Pi_{\sigma_n}(S)$ we have $\|x - z\| \geq \delta$. Take g as in (1) which peaks over $B_{m^0_\Psi}$ at z. Set $f = g/g(z)$.

Let $K = \{x \in B_{m^{\sigma_n}_\Psi} : \|x - z\| \geq \delta\}$ and $h = f \circ \Pi_{\sigma_n}$. Then there is $\epsilon > 0$ such that $|f(x)| \leq 1 - \epsilon$ for all $x \in K$. Since $\Pi_{\sigma_n}(S) \subset K$, we have that $|h(y)| \leq 1 - \epsilon$ for all $y \in S$ contradicting the fact that S is a boundary for $\mathcal{A}_u(B_{m^0_\Psi})$. ■

Theorem 16 extends [8, Proposition 4.2], [12, Theorem 1.5] and [16, Theorem 3.5].

Proposition 17 *Let Ψ be a symbol. Then $\mathcal{P}$ is a boundary for $\mathcal{A}_u(B_{m^0_\Psi})$ if and only if $\mathbb{N}_\Psi$ is infinite.*

Proof: Suppose that $\mathbb{N}_\Psi$ is finite. By Corollary 8, $\mathcal{P} = \bigcup_{\substack{n \in \mathbb{N}_\Psi \\ 1 \leq n \leq r}} \mathcal{P}_n$, where $r = \max \mathbb{N}_\Psi$. Then any z in $\mathcal{P}$ has support of length at most r. Moreover, Lemma 14 tells us that $\mathcal{P}_n$ contains elements with support of length n. Thus, for any $k > r$, $\Pi_k(\mathcal{P}) = \overline{\Pi_k(\mathcal{P})}$ cannot contain $\mathcal{P}_k$. Since Π_k is a sequence of finite dimensional projections satifying the conditions of Theorem 16 it follows that $\mathcal{P}$ is not a boundary for $\mathcal{A}_u(B_{m^0_\Psi})$.

Conversely, suppose $\mathbb{N}_\Psi$ is infinite and that $(\sigma_n)_n$ is an increasing sequence of finite ordered sets with $|\sigma_n|$ in $\mathbb{N}_\Psi$. Then $(\Pi_{\sigma_n})_n$ with $\{\sigma_n\}_n$ increasing and $\bigcup_{n \in \mathbb{N}} \sigma_n = \mathbb{N}$. By Corollary 8, $\mathcal{P} = \bigcup_{n \in \mathbb{N}_\Psi} \mathcal{P}_{\sigma_n}$ and hence $\Pi_{\sigma_n}(\mathcal{P}) \supset \mathcal{P}_{\sigma_n}$. Another application of Theorem 16 gives the desired result. ■

It follows from Theorem 5 that $\mathcal{P}$ is a boundary for $\mathcal{A}_u(B_{m^0_\Psi})$ if and only if $\mathrm{Ext}_{\mathbb{C}}(B_{m^0_\Psi})$ is.

Corollary 18 *Let Ψ be a symbol such that $\mathbb{N}_\Psi$ is finite. Then $\mathcal{A}_u(B_{m^0_\Psi})$ does not have a Šilov boundary in the sense of Globevnik.*

Proof: For each $j \in \mathbb{N}$, let $S_j = \bigcup_{k \geq j} \mathcal{P}_k$. Since $(\Pi_k)_{k \geq j}$ which satisfies Theorem 16 and $\Pi_k(S_j) \supset \mathcal{P}_k$, applying both Lemma 15 and Theorem 16, we have that S_j is a closed boundary.

Let $r = \max \mathbb{N}_\Psi$. For each $k \geq r$, Proposition 2 and Lemma 14 imply that $\mathcal{P}_k = \mathcal{P} \cup \mathcal{T}_k$. Then $\bigcap_{j=1}^k S_j = \mathcal{P} \cup \bigcup_{l=k}^\infty \mathcal{T}_l$ and letting k tend to infinity we get that $\bigcap_{j=1}^\infty S_j = \mathcal{P}$. Applying Corollary 8 we have that $\mathcal{P} = \bigcup_{\substack{n \in \mathbb{N}_\Psi \\ 1 \leq n \leq r}} \mathcal{P}_n = \bigcap_{j=1}^\infty S_j$. For each $k > r$, $\overline{\Pi_k(\bigcap_{j=1}^\infty S_j)} = \Pi_k(\bigcap_{j=1}^\infty S_j)$ does not contain $\mathcal{P}_{r+1}$ and therefore Theorem 16 implies that $\mathcal{P} = \bigcap_{j=1}^\infty S_j$ is not a boundary for $\mathcal{A}_u(B_{m^0_\Psi})$. ■

Theorem 19 *Let Ψ be a symbol such that $\mathbb{N}_\Psi$ is infinite and for each $k \in \mathbb{N}_\Psi$ the sequence $(\Phi_k(n))_{n > k+1}$ given by $\Phi_k(n) = \dfrac{\Psi(n) - \Psi(k+1)}{n - (k+1)}$ is decreasing. Then $\mathcal{A}_u(B_{m^0_\Psi})$ does not admit a Šilov boundary in the sense of Globevnik.*

Proof: For the proof we assume that $\mathbb{N}_\Psi$ does not have consecutive elements. The proof for a general symbol is more technical and can be adapted from what follows.

Suppose a Šilov boundary in the sense of Globevnik, $\mathcal{S}$, for $\mathcal{A}_u(B_{m^0_\Psi})$ exists. By Proposition 17, $\mathcal{P}$ contains $\mathcal{S}$.

Take $x \in \mathcal{S}$. As $x \in \mathcal{P}$, by Corollary 7, there exists $q \in \mathbb{N}_\Psi$ so that $x \in \mathcal{T}_q$. We claim that,

$$\frac{\Psi(n) - \Psi(q+1)}{\Psi(n-(q+1))}$$

is bounded below. Indeed, choose N sufficiently large so that $\frac{\Psi(q+1)}{\Psi(n-(q+1))} < \frac{1}{2}$ for all $n \geq N$. Then $\frac{\Psi(n)-\Psi(q+1)}{\Psi(n-(q+1))} > \frac{\Psi(n)}{\Psi(n-(q+1))} - \frac{1}{2} \geq \frac{1}{2}$, for $n \geq N$.

Now take $0 < \delta < \min\{\frac{1}{2}, x^*_q, \frac{\Psi(n)-\Psi(q+1)}{\Psi(n-(q+1))} : n = q+2, \dots, N\}$ and let $U = B(x, \frac{\delta}{2})$ be the open ball of radius $\frac{\delta}{2}$ centred at x. We will show that $\mathcal{S} \setminus U$ is also a boundary for $\mathcal{A}_u(B_{m^0_\Psi})$ which is a contradiction.

Let $y \in \mathcal{S}$. Since $y \in \mathcal{P}$, by Corollary 7, $y \in \mathcal{T}_k$ for some $k \in \mathbb{N}_\Psi$. Let $\sigma = \{i_1, \dots, i_k\}$ be the support of y with $|y_{i_1}| \geq |y_{i_2}| \geq \dots \geq |y_{i_k}|$.

In what follows we split the proof in two parts. First suppose that $k \geq q$. Take $\epsilon > 0$ such that $\epsilon < y^*_k$. Since $(\Phi_k(n))_n$ is decreasing to zero, we can find $p \in \mathbb{N}_\Psi$ so that $p > N$, $p > i_{\max} = \max\{i_j : j = 1, \dots, k\} + 2$ and $\alpha = \Phi_k(p) < \min\{\epsilon, y^*_k - \epsilon\}$. Now define $z = z(\epsilon)$ by

$$z = \sum_{j=1}^{k-1} y_{i_j} e_{i_j} + (y_{i_k} - \lambda\epsilon) e_{i_k} + \epsilon e_{i_k+1} + \sum_{j=i_{\max}+2}^{p} \alpha e_{l_j}$$

where the l_j are chosen to have support disjoint from that of x and λ is a complex number of modulus 1 such that $|y_{i_k} - \lambda\epsilon| = |y_{i_k}| - \epsilon$. As $\sum_{j=1}^{p-(k+1)} (\Pi_\mu(x) - z)$ contains $\sum_{j=i_{\max}+2}^{p} \alpha e_{l_j}$ we have that

$$\|\Pi_\mu(x) - z\| \geq \frac{\sum_{j=1}^{p-(k+1)} (\Pi_\mu(x) - z)^*_j}{\Psi(p-(k+1))} \geq \frac{\alpha(p-(k+1))}{\Psi(p-(k+1))} = \frac{\Psi(p) - \Psi(k+1)}{\Psi(p-(k+1))} > \delta.$$

We next show that z is in $\mathcal{T}_p$. Since $z^* = \sum_{j=1}^{k-1} y^*_j e_j + (y^*_k - \epsilon) e_k + \epsilon e_{k+1} + \sum_{j=k+2}^{p} \alpha e_j$,

we consider the following three cases

$$
\begin{aligned}
1 \leq n \leq k: \quad & \textstyle\sum_{j=1}^n z^*_j \leq \sum_{j=1}^n y^*_j \leq \Psi(n). \\
k < n \leq p: \quad & \textstyle\sum_{j=1}^n z^*_j = \sum_{j=1}^k y^*_j + \alpha(n-(k+1)) \\
& = \Psi(k) + \Phi_k(p)(n-(k+1)) \\
& \leq \Psi(k) + \Phi_k(n)(n-(k+1)) = \Psi(n). \\
n \geq p: \quad & \textstyle\sum_{j=1}^n z^*_j = \sum_{j=1}^p z^*_j = \Psi(k) + \alpha(p-(k+1)) = \Psi(p) \leq \Psi(n).
\end{aligned}
$$

In the second case we used that Φ_k is decreasing. From the third case, taking $n = p$, it follows that $\sum_{j=1}^p z^*_j = \Psi(p)$ and $z \in \mathcal{T}_p$.

Note that $y - \Pi_\sigma(z) = \lambda\epsilon e_{i_k}$ and therefore $\|y - \Pi_\sigma(z)\| = \epsilon$.

Since $\mathcal{S}$ is a boundary for $\mathcal{A}_u(B_{m^0_\Psi})$ by Theorem 16 we can find $w \in \mathcal{S}$ so that $\|\Pi_\mu(w) - z\| < \min\{\epsilon, \frac{\delta}{2}\}$.

Then

$$\|x - w\| \geq \|\Pi_\mu(x - w)\| \geq \|\Pi_\mu(x) - z\| - \|z - \Pi_\mu(w)\| > \delta - \delta/2 = \delta/2.$$

We also have

$$\|y-\Pi_\sigma(w)\| \leq \|y-\Pi_\sigma(z)\|+\|\Pi_\sigma(z-w)\| \leq \epsilon+\|\Pi_\mu(z-w)\| = \epsilon+\|z-\Pi_\mu(w)\| \leq 2\epsilon.$$

Note that this can be done for any ϵ sufficiently small. Therefore w belongs to $\mathcal{S} \setminus U$ and y is in the closure of $\Pi_\sigma(\mathcal{S} \setminus U)$.

To conclude the proof we have to see what happens when $y \in \mathcal{T}_k$ with $k < q$. In this case $\|x - y\| \geq x^*_q > \delta$ which implies that $y \notin U$.

Now fix $\epsilon > 0$. Since $\mathcal{S}$ is a boundary and $\mathbb{N}_\Psi$ is infinite, by Theorem 16, we can find $w \in \mathcal{S}$ so that $\|\Pi_\mu(w) - y\| < \min\{\epsilon, \frac{\delta}{2}\}$, where μ is a support containing σ. We conclude as before that $w \notin U$ and that y belongs to the closure of $\Pi_\sigma(\mathcal{S} \setminus U)$.

In both cases we arrive at the same situation which allows us to conclude, applying Theorem 16, that $\mathcal{S} \setminus U$ is a boundary for $\mathcal{A}_u(B_{m^0_\Psi})$ contradicting the fact that $\mathcal{S}$ is the minimal closed boundary. ∎

Bishop [6, Theorem 1] shows that when $\mathcal{A}$ is a separating algebra of continuous functions on $C(K)$ with K a compact Hausdorff metrizable space then the Bishop boundary of $\mathcal{A}$ and the Šilov boundary of $\mathcal{A}$ coincide. Theorem 19 shows that the condition that K is compact cannot be dropped and it is possible for the Bishop boundary of $\mathcal{A}_u(B_{m^0_\Psi})$ (or $\mathcal{P}$) to be boundary for $\mathcal{A}_u(B_{m^0_\Psi})$ and yet for $\mathcal{A}_u(B_{m^0_\Psi})$ to fail to have a Šilov boundary in the sense of Globevnik.

Let us now observe that we cannot replace exposed point with strongly exposed point in Theorem 11. We will show that when Ψ satisfies the conditions of Theorem 19 then the unit ball of m^0_Ψ cannot contain any strongly exposed point. In order to prove this we will need the concept of a strong peak point. We recall that a point in the unit sphere of a Banach space E is a strong peak point for $\mathcal{A}_u(B_{m^0_\Psi})$ if there is f in $\mathcal{A}_u(B_{m^0_\Psi})$ such that $f(x) = 1$ and given any $\epsilon > 0$ there is $\delta > 0$ such that $|1 - f(y)| < \delta$ implies that $\|x - y\| < \epsilon$. This is equivalent to the condition that there is f in $\mathcal{A}_u(B_{m^0_\Psi})$ such that $f(x) = 1$ and given any $\epsilon > 0$ there is $\delta > 0$ such that $\|x - y\| \geq \epsilon$ implies that $|f(y)| < 1 - \delta$.

Theorem 20 *Let Ψ be a symbol such that $\mathbb{N}_\Psi$ is infinite and for each $k \in \mathbb{N}_\Psi$ the sequence $(\Phi_k(n))_{n>k+1}$ given by $\Phi_k(n) = \dfrac{\Psi(n) - \Psi(k+1)}{n - (k+1)}$ is decreasing. Then $B_{\mathcal{A}_u(B_{m^0_\Psi})}$ does not contain strongly exposed points.*

Proof: It follows as in [11, Proposition 1] (replacing $\mathcal{A}_b(B_{m^0_\Psi})$ in the proof with $\mathcal{A}_u(B_{m^0_\Psi})$) that any strongly exposed point x of $\mathcal{A}_u(B_{m^0_\Psi})$ would be a strong peak point and hence a peak point for $\mathcal{A}_u(B_{m^0_\Psi})$. Let f denote a function in $\mathcal{A}_u(B_{m^0_\Psi})$ which strongly peaks at x and denote by $\mathcal{P}$ the set of all peak points of $\mathcal{A}_u(B_{m_\Psi^0})$. Taking $y = x$ in the proof of Theorem 19 we see that it is possible to remove a neighbourhood U of x and for the set $\mathcal{P} \setminus U$ to remain a boundary for $\mathcal{A}_u(B_{m_\Psi^0})$. However, from the definition of strong peak point, we see that $\|f\|_{\mathcal{P}\setminus U} < 1$ contradicting the fact that $\mathcal{P}\setminus U$ is a boundary for $\mathcal{A}_u(B_{m_\Psi^0})$. ■

Since the exposed and strongly exposed points in a finite dimensional space coincide we see that we cannot replace exposed points with strongly exposed points in Theorem 11.

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