Extreme and Exposed Points of Spaces of Integral Polynomials

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Abstract

We show that if E is a real Banach space such that E' has the approximation property and such that $\ell_1 \not\hookrightarrow \widehat{\bigotimes}_{n,s,\epsilon} E$ then the set of extreme points of the unit ball of $\mathcal{P}_I(^nE)$ is equal to $\{\pm\phi^n \colon \phi \in E', \|\phi\| = 1\}$. Under the additional assumption that E' has a countable norming set we see that the set of exposed points of the unit ball of $\mathcal{P}_I(^nE)$ is also equal to $\{\pm\phi^n \colon \phi \in E', \|\phi\| = 1\}$.

1 Introduction

The isometric study of Banach spaces tells us that there are certain subsets on the unit sphere of a Banach space which are fundamental in our understanding of the geometry of Banach spaces. Such sets include the set of all extreme points, the set of all exposed and weak*-exposed points, the set of all strongly exposed and weak*-strongly exposed points and the set of all denting points. We denote these set by $\mathrm{Ext}(E)$, $\mathrm{Exp}(E)$, $w^*\mathrm{Exp}(E)$, $\mathrm{sExp}(E)$, $w^*\mathrm{sExp}(E)$ and $\mathrm{Dent}(E)$ respectively. These sets are invariant under isometries and are used in the definition of such concepts as smoothness, strict convexity, Fréchet differentiablility and Gâteaux differentiability. In the 1980's Ruess and Stegall published a series of papers where the geometric structure of the spaces of projective tensor products and linear operators is investigated. They show in [12] that the extreme points of the unit ball of the projective tensor product of the Banach spaces E and F, $E\widehat{\otimes}_\pi F$, is the set $\{x \otimes y : x \in \mathrm{Ext}(E'), y \in \mathrm{Ext}(F')\}$. In [13] they prove that the analogous result holds for $\mathrm{Ext}(E)$ replaced with $\mathrm{Exp}(E)$, $w^*\mathrm{Exp}(E)$, $\mathrm{sExp}(E)$, $w^*\mathrm{sExp}(E)$ and $\mathrm{Dent}(E)$.

Perhaps some definitions are in order. A point x is said to be an extreme point of the (closed) unit ball of a Banach space E, $\overline{B}_E$ if x cannot be written as the midpoint of a line segment which is entirely contained in $\overline{B}_E$. We recall that a unit vector x in a Banach space E is exposed if there is a unit vector $\phi \in E'$ so that $\phi(x) = 1$ and $\phi(y) < 1$ for $y \in B_E \setminus \{x\}$. We will say that ϕ *exposes* B_E at x. If $E = F'$ is a dual space and the vector

2000 Mathematics Subject Classification: Primary 46G25, 46B04.

ϕ which exposes x is in F we shall say that x is weak*-exposed and that ϕ weak*-exposes the unit ball of E at x. Note that ϕ in F weak*-exposes the unit ball of E at x if and only if whenever (x_k) is a sequence in E so that $\phi(x_k)$ converges to 1 then (x_k) converges weak* to x. A point x in the unit ball of E is said to be a strongly exposed by ϕ in $S_{E'}$ if whenever (x_k) is a sequence in E so that $\phi(x_k)$ converges to 1 then (x_k) converges to x in norm. In this case we will say that ϕ *strongly exposes* B_E at x.

Given a Banach space E a point x in the closed unit ball of E is said to be a denting point if for every $\epsilon > 0$, x does not belong to $\overline{\Gamma(B_E \setminus B(x,\epsilon))}$, the closure of the absolutely convex hull of $B_E \setminus B(x,\epsilon)$. This is equivalent to the condition that x is contained in slices of the unit ball of E which have arbitrary small diameter.

The theory of tensor products is also important in the study of spaces of homogeneous polynomials on Banach spaces. Here however, it is the spaces of symmetric tensors which we need to investigate. The geometric structure of these spaces is very different to that of the 'ordinary' tensor products. In [14] Ryan and Turett show that if E is a real finite dimensional Banach space and $n > 1$ is a positive integer then the extreme points of the unit ball of $\widehat{\bigotimes}_{n,s,\pi} E$ is $\{\pm x \otimes \cdots \otimes x : x \in E, \|x\| = 1\}$. In [3] an upper bound for the set of extreme points of the unit ball of $\mathcal{P}_I({}^nE)$, the space of n-homogeneous integral polynomials on E, is given by $\{\pm\phi^n \colon \phi \in E', \|\phi\| = 1\}$. When E' has the approximation property and $\widehat{\bigotimes}_{n,s,\epsilon} E$ does not contain a copy of ℓ_1 it is shown in [3] that $\mathcal{P}_I({}^nE)$ and $\widehat{\bigotimes}_{n,s,\pi} E'$ are isometrically isomorphic (see also [5, Theorem 1.5]). Therefore, under these conditions $\{\phi^n \colon \phi \in E', \|\phi\| = 1\}$ is also an upper bound for the set of extreme points on the unit ball of $\widehat{\bigotimes}_{n,s,\pi} E'$. In [3] a lower bound for the set of extreme points of the unit ball of $\mathcal{P}_I({}^nE)$ is also obtained. Specifically it is shown that the set of extreme points of the unit ball $\mathcal{P}_I({}^nE)$ contains $\{\pm\phi^n \colon \phi \in E', \|\phi\| = 1 \text{ and } \phi \text{ attains its norm}\}$. Hence if E is a real reflexive Banach space and n is an integer which is greater than or equal to 2 then the set of extreme points of the unit ball of $P_I({}^nE)$ is precisely the set $\{\pm\phi^n \colon \phi \in E', \|\phi\| = 1\}$.

In this note we show that given E a real Banach space and fixed $n > 1$ any positive integer such that $\widehat{\bigotimes}_{n,s,\epsilon} E$ does not contain a copy of ℓ_1 then the set of extreme points of the unit ball of $\mathcal{P}_I({}^nE)$ is $\{\pm\phi^n \colon \phi \in E', \|\phi\| = 1\}$.

The extreme points of the unit ball of the space of n-homogeneous integral polynomials on complex Banach spaces are studied by Dineen in [9]. For further reading on polynomials on infinite dimensional Banach spaces we refer the reader to [8].

2 Extreme and exposed points

We begin by introducing some topologies on the space of symmetric tensor products. Given a tensor u in $\bigotimes_{n,s} E$ we define the symmetric projective norm of u indicated with π to be

$$\|u\|_\pi = \inf\{\sum_{i=1}^{k} \|x_i\|^n \colon u = \sum_{i=1}^{k} \lambda_i x_i \otimes \ldots \otimes x_i, \lambda_i = \pm 1, x_i \in E\}.$$

We denote the completion of $\bigotimes_{n,s} E$ with respect to the projective symmetric norm by $\widehat{\bigotimes}_{n,s,\pi} E$. The dual of $\widehat{\bigotimes}_{n,s,\pi} E$ is the space $\mathcal{P}(^n E)$ of all continuous n-homogeneous polynomials on E endowed with the norm $\|P\| = \sup_{\|x\|\leq 1} |P(x)|$.

An n-homogeneous polynomial P on E is said to be nuclear if there is a bounded sequence $(\phi_j)_{j=1}^\infty \subset E'$ and a sequence $(\lambda_j)_{j=1}^\infty$ in ℓ_1 such that

$$P(x) = \sum_{j=1}^{\infty} \lambda_j \phi_j(x)^n$$

for every x in E. The space of all nuclear n-homogeneous polynomials on E is denoted by $\mathcal{P}_N(^n E)$ and becomes a Banach space when the norm of P is given as the infimum of $\sum_{j=1}^\infty |\lambda_j| \, \|\phi_j\|^n$ taken over all representations of P of the form described above. This norm is called the nuclear norm of P and is denoted by $\|P\|_N$. Given ϕ in E' we denote by ϕ^n the n-homogeneous polynomial which takes x to $\phi(x)^n$. When E' has the approximation property $\mathcal{P}_N(^n E)$ is isometrically isomorphic to $\widehat{\bigotimes}_{n,s,\pi} E'$ under the map induced by $\phi^n \to \phi \otimes \cdots \otimes \phi$.

A polynomial P on E is said to be integral if there is a regular Borel measure μ on $(B_{E'}, \sigma(E', E))$ such that

$$P(x) = \int_{B_{E'}} \phi(x)^n \, d\mu(\phi) \tag{1}$$

for every x in E. We write $\mathcal{P}_I(^n E)$ for the space of all n-homogeneous integral polynomials on E. We define the integral norm of an integral polynomial P, $\|P\|_I$, as the infimum of $\|\mu\|$ taken over all regular Borel measures which satisfy (1). Given an n-fold symmetric tensor $u = \sum_{i=1}^k \lambda_i x_i \otimes \cdots \otimes x_i$ on E we define its symmetric injective norm indicated with ϵ as

$$\|u\|_\epsilon = \sup_{\phi \in B_{E'}} \left| \sum_{i=1}^{k} \lambda_i \phi(x_i)^n \right|.$$

We denote the completion of $\bigotimes_{n,s} E$ with respect to this norm by $\widehat{\bigotimes}_{n,s,\epsilon} E$. It follows from [7] that the dual of $\widehat{\bigotimes}_{n,s,\epsilon} E$ is isometrically isomorphic to $(\mathcal{P}_I(^n E), \| \, . \, \|_I)$.

The projective tensor product does not respect subspaces. A consequence of this fact is that there is no Hahn-Banach Theorem for n-homogeneous polynomials when n is at least 2. In [11] Kirwan and Ryan introduce the space of all extendible n-homogeneous polynomials. This is the subspace of all P in $\mathcal{P}(^nE)$ such that given any superspace G of E there is an extension of P to G as a continuous n-homogeneous polynomial. The space of all extendible n-homogeneous polynomials on E is denoted by $\mathcal{P}_e(^nE)$. Moreover, it is shown in [11, Proposition 1] that given P in $\mathcal{P}_e(^nE)$ there is $C > 0$ such that for every superspace G of E there is an extension Q of P to $\mathcal{P}(^nG)$ with $\|Q\| \leq C$. If we define $\|P\|_e$ to be the infimum of all C with the above property then $\|\cdot\|_e$ is a norm on $\mathcal{P}_e(^nE)$. Furthermore, by [11, Proposition 2] $(\mathcal{P}_e(^nE), \|\cdot\|_e)$ is a Banach space. In fact it is a dual space. To construct the predual of $(\mathcal{P}_e(^nE), \|\cdot\|_e)$ we first note that Carando [4] shows that for P in $\mathcal{P}(^nE)$ to be extendible it is sufficient that it can be extended to either $C(B_{E'}, w^*)$ or $\ell_\infty(B_{E'})$ where E is isometrically embedded into $C(B_{E'}, w^*)$ by the mapping $I_E(x)(\phi) = \phi(x)$ for ϕ in E' and E is isometrically embedded into $\ell_\infty(B_{E'})$ by the mapping $J_E(x) = (\phi(x))_{\phi\in B_{E'}}$. Given a Banach space E the space of n-fold symmetric tensors $\bigotimes_{n,s}E$ may be considered as a subspace of $\widehat{\bigotimes}_{n,s,\pi}\ell_\infty(B_{E'})$. We let $\bigotimes_{n,s,\eta}E$ denote $\widehat{\bigotimes}_{n,s,\pi}E$ with the topology induced from $\widehat{\bigotimes}_{n,s,\pi}\ell_\infty(B_{E'})$. The completion of $\bigotimes_{n,s,\eta}E$ is denoted by $\widehat{\bigotimes}_{n,s,\eta}E$. The space $\ell_\infty(B_{E'})$ may be replaced by $C(B_{E'}, w^*)$ in the construction of $\widehat{\bigotimes}_{n,s,\eta}E$. Kirwan and Ryan [11, Proposition 4] show that $\mathcal{P}_e(^nE)$ is isometrically isomorphic to the dual of $\widehat{\bigotimes}_{n,s,\eta}E$.

Although every integral polynomial P is extendible with $\|P\|_e \leq \|P\|_I$, there are extendible non-integral polynomials, for example c_0 contains non-integral extendible polynomials (see [6]). However, we shall see that the study of the geometry of $\widehat{\bigotimes}_{n,s,\eta}E$ reveals information about the geometry of the spaces of integral polynomials and n-fold symmetric tensors.

Lemma 1 *Let E be a real Banach space and $n > 1$ be a positive integer. Then* $\{x^n : \|x\| = 1\} \subseteq \mathrm{Ext}\left(\widehat{\bigotimes}_{s,n,\eta}E\right)$

Proof: From the inclusion $\widehat{\bigotimes}_{s,n,\eta}E \hookrightarrow \widehat{\bigotimes}_{s,n,\pi}\ell_\infty(B_{E'})$ and the fact that $\mathrm{Ext}(B)\cap A \subset \mathrm{Ext}(A)$ whenever $A \subset B$ we have that

$$\mathrm{Ext}\Big(\widehat{\bigotimes}_{s,n,\pi}\ell_\infty(B_{E'})\Big)\bigcap\widehat{\bigotimes}_{s,n,\eta}\,E \subseteq \mathrm{Ext}\Big(\widehat{\bigotimes}_{s,n,\eta}E\Big).$$

On the other hand, $\ell_\infty(B_{E'})$ has the approximation property then, $\widehat{\bigotimes}_{s,n,\pi}\ell_\infty(B_{E'}) = \mathcal{P}_N\big(^n\ell_1(B_{E'})\big)$. By [3, Proposition 1], the extreme points of the unit ball of $\mathcal{P}_I\big(^n\ell_1(B_{E'})\big)$ are all nuclear polynomials, therefore $\mathrm{Ext}\Big(\mathcal{P}_I\big(^n\ell_1(B_{E'})\big)\Big) \subseteq \mathrm{Ext}\Big(\mathcal{P}_N\big(^n\ell_1(B_{E'})\big)\Big)$. Applying [3, Proposition 5], we have that

$$\begin{aligned}\{\pm\phi^n\colon\ \phi\in\ \ell_\infty(B_{E'}); \|\phi\| = 1; \phi \text{ is norm attaining}\} &\subseteq \mathrm{Ext}\Big(\mathcal{P}_I\big(^n\ell_1(B_{E'})\big)\Big)\\ &\subseteq \mathrm{Ext}\Big(\widehat{\bigotimes}_{s,n,\pi}\ell_\infty(B_{E'})\Big).\end{aligned}$$

Use $\hat{x}$ to denote the canonical image of $x \in E$ in $\ell_\infty(B_{E'})$. Since $\hat{x}$ is norm attaining then, $\{\hat{x}^n\colon \ \|x\| = 1\} \subseteq \mathrm{Ext}\Big(\widehat{\bigotimes}_{s,n,\pi}\ell_\infty(B_{E'})\Big)$.

Finally, $\{x^n\colon \|x\| = 1\} \subseteq \mathrm{Ext}\Big(\widehat{\bigotimes}_{s,n,\pi}\ell_\infty(B_{E'})\Big) \cap \widehat{\bigotimes}_{s,n,\eta} E$, and therefore $\{x^n\colon \ \|x\| = 1\} \subseteq \mathrm{Ext}\Big(\widehat{\bigotimes}_{s,n,\eta} E\Big)$. ■

This result is the key to classify the extreme points of the space of n-homogeneous integral polynomials when ℓ_1 is not a subspace of $\widehat{\bigotimes}_{n,s,\epsilon}E$.

Theorem 2 *Let E be a real Banach space so that E' has the approximation property. Let $n > 1$ be a positive integer and suppose that $\widehat{\bigotimes}_{n,s,\epsilon}E$ does not contain a subspace isomorphic to ℓ_1. Then, the set of extreme points of the unit ball of $\mathcal{P}_I(^nE)$ is equal to $\{\pm\phi^n\colon \phi \in E', \|\phi\| = 1\}$.*

Proof: Since $\ell_1 \not\hookrightarrow \widehat{\bigotimes}_{n,s,\epsilon}E$ and E' has the approximation property it follow from [3, Theorem 2] (or [5, Theorem 1.5]) that $\mathcal{P}_I(^nE)$ is isometrically isomorphic to $\widehat{\bigotimes}_{n,s,\pi}E'$.

By [10, 18.3.7], the mapping $\widehat{\bigotimes}_{s,n,\pi} E' \hookrightarrow \widehat{\bigotimes}_{s,n,\epsilon} E'$ is injective, whenever E' has the approximation property. Hence, the canonical mapping $j : \widehat{\bigotimes}_{s,n,\pi} E' \hookrightarrow \widehat{\bigotimes}_{s,n,\eta} E'$ is also injective.

Let ϕ^n be in $\widehat{\bigotimes}_{s,n,\pi} E'$ with $\|\phi\| = 1$. Since, $\|\phi^n\|_\pi = \|\phi^n\|_\eta = \|\phi\|^n$, then, by Proposition 1, $\phi^n \in \mathrm{Ext}\Big(\widehat{\bigotimes}_{s,n,\eta} E'\Big)$.

In order to prove that ϕ^n is an extreme point of the unit ball of $\widehat{\bigotimes}_{s,n,\pi} E'$ suppose that is not the case. Then, we can find $u \in \widehat{\bigotimes}_{s,n,\pi} E'$, $u \neq 0$ so that $\|\phi^n \pm u\|_\pi \leq 1$. Thus, $\|j(\phi^n \pm u)\|_\eta \leq 1$ and $j(u) = 0$ in $\widehat{\bigotimes}_{s,n,\eta} E'$; which leads us into a contradiction since j is injective.

The reverse inclusion is a consequence of the fact that $\mathcal{P}_I(^nE) = \widehat{\bigotimes}_{s,n,\pi} E'$ together with [3, Proposition 1]. ■

Since the injective tensor product of Asplund spaces is Asplund and Asplund spaces cannot contain a copy of ℓ_1 we obtain the following corollary:

Corollary 3 *Let E be a real Banach space. Suppose that E is Asplund and that E' has the approximation property. Then the set of extreme points of the unit ball of $\mathcal{P}_I(^nE)$ is equal to $\{\pm\phi^n\colon \phi \in E', \|\phi\| = 1\}$, for any $n > 1$ a positive integer.*

In the course of proving Theorem 2 we have also proved the following result.

Corollary 4 *Let E be a real Banach space. Then, for any $n > 1$ a positive integer, the set $\{\pm x^n\colon x \in E, \|x\| = 1\}$ is contained in the set of extreme points of the unit ball of $\widehat{\bigotimes}_{s,n,\pi}E$.*

Let us now investigate the exposed points of the space of integral polynomials.

Lemma 5 *Let E be a real Banach space such that E' has a countable norming set. Let $n > 1$ be a positive integer. Then* $\{x^n : \|x\| = 1\} \subseteq \operatorname{Exp}\left(\widehat{\bigotimes}_{s,n,\eta} E\right)$.

Proof: Let x be a point of E with $\|x\| = 1$. Choose ϕ in E' so that $\phi(x) = 1$. Let K be a countable norming subset of E' and define K_x to be $K \cup \{\phi\}$. Applying an analogous argument to that given at the beginning of Lemma 1 we see that

$$\operatorname{Exp}\left(\widehat{\bigotimes}_{s,n,\pi} \ell_\infty(K_x)\right) \bigcap \widehat{\bigotimes}_{s,n,\eta} E \subseteq \operatorname{Exp}\left(\widehat{\bigotimes}_{s,n,\eta} E\right).$$

Since every exposed point of the unit ball of $\mathcal{P}_I(^n\ell_1(K_x))$ is also an exposed point of $\mathcal{P}_N(^n\ell_1(K_x))$ we get that

$$\operatorname{Exp}\left(\mathcal{P}_I\left(^n\ell_1(K_x)\right)\right) \subseteq \operatorname{Exp}\left(\mathcal{P}_N\left(^n\ell_1(K_x)\right)\right) = \operatorname{Exp}\left(\widehat{\bigotimes}_{s,n,\pi} \ell_\infty(K_x)\right).$$

Using [3, Theorem 8] and the fact that weak*-exposed points are exposed points we have

$$\begin{aligned} \{\pm\phi^n \colon \phi \in \ell_\infty(K_x), \|\phi\| = 1, \phi \text{ is norm attaining}\} &\subseteq \operatorname{Exp}\left(\mathcal{P}_I\left(^n\ell_1(K_x)\right)\right) \\ &\subseteq \operatorname{Exp}\left(\widehat{\bigotimes}_{s,n,\pi} \ell_\infty(K_x)\right). \end{aligned}$$

Consider $\hat{x}$, the image of x in $\ell_\infty(K_x)$. Then $\hat{x}$ attains its norm at the point δ_ϕ where

$$\delta_\phi(\psi) = \begin{cases} 1 & \text{if } \phi = \psi, \\ 0 & \text{if } \phi \neq \psi. \end{cases}$$

Therefore we see that $\{\hat{x}^n \colon \|x\| = 1\} \subseteq \operatorname{Exp}\left(\widehat{\bigotimes}_{s,n,\pi} \ell_\infty(K_x)\right)$. The remainder of the proof follows as in Lemma 1. ■

Theorem 6 *Let E be a real Banach space such that E' has the approximation property and a countable norming set. Let $n > 1$ be a positive integer and suppose that $\widehat{\bigotimes}_{n,s,\epsilon} E$ does not contain a subspace isomorphic to ℓ_1. Then the set of exposed points of the unit ball of $\mathcal{P}_I(^nE)$ is equal to $\{\pm\phi^n \colon \phi \in E', \|\phi\| = 1\}$.*

Proof: Since ℓ_1 is not a subspace of $\widehat{\bigotimes}_{n,s,\epsilon} E$ it follows from [3, Theorem 2] or [5, Theorem 1.5] that $\mathcal{P}_I(^nE)$ is isometically isomorphic to $\widehat{\bigotimes}_{s,n,\pi} E'$. Since $\|\cdot\|_\eta \leq \|\cdot\|_\pi$ each exposed point of the unit ball of $\widehat{\bigotimes}_{s,n,\eta} E'$ is also an exposed point of the unit ball of $\widehat{\bigotimes}_{s,n,\pi} E'$. (Simply restrict the exposing polynomial to $\widehat{\bigotimes}_{s,n,\pi} E'$.) Thus the exposed points of the unit ball of $\mathcal{P}_I(^nE)$ contains $\{\pm\phi^n \colon \phi \in E', \|\phi\| = 1\}$. The reverse inclusion is immediate. ■

The above result applies to all real Asplund Banach spaces E such that E' has the approximation property and a countable norming set. In particular, it holds for all real separable Asplund Banach spaces E such that E' has the approximation property.

The methods used in this paper to give that the extreme and exposed points of the space of integral polynomials are, up to multplication by ± 1, a power of a linear functional, will not work for denting or strongly exposed points. To see this we note that both methods are based on embedding the space of symmetric n-tensors of E in the space of symmetric n-tensors of $\ell_\infty(B_{E'})$ or $\ell_\infty(K_x)$ for some countable norming subset K_x of E'. However, it is show in [1, Proposition 2.2] (see also [2]) that if K is an infinite compact Hausdorff topological space then every slice of the unit ball of $\widehat{\bigotimes}_{n,s,\pi} C(K)$ has diameter 2. Hence the unit ball of $\widehat{\bigotimes}_{n,s,\pi} C(K)$ has no denting points and therefore no strongly exposed points. On the other hand [3, Section 4] gives examples of Banach spaces E where each x^n is a strongly exposed point of $\widehat{\bigotimes}_{n,s,\pi} E$.

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