ATOMIC DECOMPOSITIONS FOR TENSOR PRODUCTS AND POLYNOMIAL SPACES

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ABSTRACT. We study the existence of atomic decompositions for tensor products of Banach spaces and spaces of homogeneous polynomials. If a Banach space X admits an atomic decomposition of a certain kind, we show that the symmetrized tensor product of the elements of the atomic decomposition provides an atomic decomposition for the symmetric tensor product $\bigotimes_{s,\mu}^n X$, for any symmetric tensor norm μ. In addition, the reciprocal statement is investigated and analogous consequences for the full tensor product are obtained. Finally we apply the previous results to establish the existence of monomial atomic decompositions for certain ideals of polynomials on X.

INTRODUCTION

Function theory on infinite dimensional spaces comprises, among many other topics, the study of multilinear functions, polynomials and holomorphic functions defined on a Banach space. The linear structure and properties of the underlying Banach space reflect into the structure and properties of different type of functions defined on it. For example, for Banach spaces X and Y with shrinking bases, the space of bilinear forms $\mathcal{B}(X \times Y)$ has a monomial basis if and only if every linear operator form X to Y' is compact, see [28].

Many authors have studied the existence of bases in tensors products of Banach spaces and in spaces of homogeneous polynomials [1, 6, 12, 14, 15, 23, 27]. For a Banach space X with a Schauder basis, a natural question is if the monomials associated to the basis form a basis for the space of polynomials on X. This would result in a good approximation of polynomials and analytic functions by combinations of coordinate functionals. Also, it was shown that the existence of such a basis is closely related to the reflexivity of some spaces of polynomials and analytic functions [1, 14].

In this article, we face the analogous question regarding atomic decompositions instead of bases. Atomic decompositions were introduced by Gröchening [22] as a possible extension of the concept of Hilbert frames to the Banach space framework. Atomic decompositions are present in any separable space with the bounded approximation property. Moreover, a complemented subspace of a Banach space

with basis has always a natural atomic decomposition, easily obtained in terms of the basis of the superspace. Note that even when this subspace may have a basis, there is not a systematic way to find it. This makes atomic decompositions a less restrictive structure than bases.

In this setting, one of our main questions is the following: if a Banach space X has an atomic decomposition and $\mathcal{Q}(X)$ is some space of polynomials on X, are the corresponding monomials an atomic decomposition for $\mathcal{Q}(X)$? More precisely, given an atomic decomposition $((x_i'), (x_i))$ of X and any continuous n-homogeneous polynomial P on X, the series expansion

$$P(x) = \check{P}(x, \cdots, x) = \sum_{\alpha_1} \cdots \sum_{\alpha_n} \check{P}(x_{\alpha_1}, \ldots, x_{\alpha_n})\ x_{\alpha_1}'(x) \cdots x_{\alpha_n}'(x)$$

is pointwise convergent (here, $\check{P}$ denotes the symmetric n-linear form associated to P). The question is then to find conditions under which the monomials $(x_{\alpha_1}' \cdots x_{\alpha_n}')$, together with the n-tuples $(x_{\alpha_1}, \ldots, x_{\alpha_n})$, form an atomic decomposition for different spaces of polynomials. For the particular case when the atomic decomposition is a Schauder basis, we recover some of the results in [1, 6, 12, 14, 15, 23, 27].

It is worthwhile to note that homogeneous polynomials of degree 1 on a Banach space X are exactly the linear forms on X, that is the dual space X'. Also, many classes of polynomials on X are related to different symmetric tensor norms on $\bigotimes_s^n X'$. Therefore, to study the existence of atomic decompositions for spaces of polynomials, a natural approach is to investigate such structures on $\bigotimes_{s,\mu}^n X'$ for symmetric tensor norms μ. This fact suggests that a good start is to study when an atomic decomposition for X ensures the existence of an atomic decomposition for $\bigotimes_{s,\mu}^n X$. This question is discussed in Section 2. Also, the results on tensor products are combined with the duality theory for atomic decompositions presented in [9] to obtain atomic decompositions for $\bigotimes_{s,\mu}^n X'$ built from those on X.

The correspondence between symmetric tensor norms on $\bigotimes_s^n X'$ and ideals of n-homogeneous polynomials on X allows us to tackle, in Section 3, our main question: in which cases do monomials provide an atomic decomposition for spaces of polynomials? Finally, as applications, we relate the Asplund property with the existence of monomial atomic decompositions for integral polynomials. We characterize the reflexivity of the space of polynomials in terms of the existence of monomial atomic decompositions.

For further information on atomic decompositions see, for example, [10, 11, 22] and the references therein. We refer to [24] for Banach space theory, [13, 19, 20, 27] for notation and properties of tensor products and [16, 25] for polynomials on Banach spaces.

1. Definitions and basic results on atomic decompositions and duality

The definitions and results given in this section are mainly taken from [9]. Since these results will be used throughout the present article we include them here for the reader's convenience.

By a Banach sequence space we understand a Banach space of scalar sequences for which the coordinate functionals are continuous. We say that the space is a Schauder sequence space if, in addition, the unit vectors $\{e_i\}$ given by $(e_i)_j = \delta_{i,j}$ form a basis for it. In this case, a sequence $a = (a_i)$ can be written as $a = \sum_i a_i e_i$.

Definition 1. *Let X be a Banach space and Z be a Banach sequence space. Let (x_i') and (x_i) be sequences in X' and X respectively. We say that $((x_i'), (x_i))$ is an atomic decomposition of X with respect to Z if for all $x \in X$:*

(a) $(\langle x_i', x \rangle) \in Z$,

(b) $A\|x\| \leq \|(\langle x_i', x \rangle)\|_Z \leq B\|x\|$, *with A and B positive constants,*

(c) $x = \sum_i \langle x_i', x \rangle x_i$.

We will often refer to property (c) in the above definition as the *reconstruction formula* associated to the atomic decomposition.

Pelczyński [26] showed that a separable Banach space admits an atomic decomposition if an only if it has the bounded approximation property. In this case, if $((x_i'), (x_i))$ is an atomic decomposition of X with respect to some Banach sequence space Z, it is always possible to find a Schauder sequence space X_d and an operator $S\colon X_d \to X$ such that $Se_i = x_i$ and $((x_i'), (x_i))$ is also an atomic decomposition of X with respect to X_d [26, 10]. Then, in the sequel we will consider atomic decompositions of the form $((x_i'), (Se_i))$ associated to a Schauder sequence space X_d.

If $((x_i'), (Se_i))$ is an atomic decomposition of X with respect to X_d, the natural inclusion $J\colon X \to X_d$ is given by

$$(1) \qquad J(x) = (\langle x_i', x \rangle) = \sum_i \langle x_i', x \rangle e_i.$$

If (e_i') is the dual basic sequence of (e_i) then, $x_i' = J'e_i'$. Since $SJ = I_X$, X is isomorphic to a complemented subspace of X_d. On the other hand, if there exits $J\colon X \to X_d$ and $S\colon X_d \to X$ continuous operators so that $SJ = I_X$ and (e_i') is the dual basic sequence of (e_i), then the pair $((J'e_i'), (Se_i))$ is an atomic decomposition for X with respect to X_d.

In order to obtain an atomic decomposition for X' in terms of a given atomic decomposition of X, the notion of shrinking and strongly shrinking atomic decompositions were introduced in [9].

In what follows X will be a Banach space, X_d a Schauder sequence space and $S\colon X_d \to X$ a continuous operator such that $((x_i'), (Se_i))$ is an atomic decomposition of X with respect to X_d. We shall denote by $(X_d)'$ the usual dual space of X_d. Since $(X_d)'$ is not necessarily a Schauder sequence space, we will also consider X_d' the closed subspace spanned by (e_i') in $(X_d)'$.

The definition of a shrinking atomic decomposition requires the following operators: $T_N\colon X \to X$, $N \in \mathbb{N}$, given by $T_N(x) = \sum_{i \geq N} \langle x_i', x \rangle Se_i$. It can be seen that (T_N) is a uniformly bounded sequence. Now we are in conditions to state the following:

Definition 2. *The atomic decomposition $((x_i'), (Se_i))$ is said to be shrinking if for all $x' \in X'$*

$$\|x' \circ T_N\| \longrightarrow 0.$$

Theorem 3. *The pair $((Se_i), (x_i'))$ is an atomic decomposition for X' with respect to $(X_d)'$ if and only if $((x_i'), (Se_i))$ is shrinking.*

Note that in the theorem above we obtain an atomic decomposition for X' with respect to $(X_d)'$, which might not be a Schauder sequence space. A subtle modification to the definition of shrinking atomic decomposition allows us to replace $(X_d)'$ by X_d'.

Fixed N, consider the mapping $S_N \colon X_d \to X$ given by $S_N(a) = \sum_{i \geq N} a_i x_i$. Again, (S_N) is a uniformly bounded sequence. Now we have:

Definition 4. *The atomic decomposition $((x_i'), (Se_i))$ is said to be strongly shrinking if for all $x' \in X'$*

$$\|x' \circ S_N\| \longrightarrow 0.$$

Theorem 5. *The pair $((Se_i), (x_i'))$ is an atomic decomposition for X' with respect to X_d' if and only if $(x_i', (Se_i))$ is strongly shrinking.*

It is clear that any strongly shrinking atomic decomposition is shrinking. The converse is not true, as an example in [9] shows.

2. Atomic decomposition of symmetric tensor products

Given a Banach space X, we denote by $\bigotimes^n X$ the n-fold tensor product of X. The subspace of $\bigotimes^n X$ consisting of all tensors of the form $\sum_{j=1}^{l} \lambda_j x_j \otimes \cdots \otimes x_j$, where $x_j \in X$ and $\lambda_j = \pm 1$, is called the symmetric n-fold tensor product of X and is denoted by $\bigotimes_s^n X$. Fixed $x_1, \ldots, x_n$, we denote by $x_1 \otimes_s \cdots \otimes_s x_n$ the symmetrization of $x_1 \otimes \cdots \otimes x_n$, that is

$$x_1 \otimes_s \cdots \otimes_s x_n = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)},$$

where $\mathcal{S}_n$ is the symmetric group on $\{1, \ldots, n\}$. As a consequence of the polarization formula, $x_1 \otimes_s \cdots \otimes_s x_n$ is a symmetric tensor (see [19, Section 1.5]).

Given a Banach space Y and a continuous operator $T \colon X \to Y$, the symmetric n-tensor power of T is an operator from $\bigotimes_s^n X$ to $\bigotimes_s^n Y$ defined by

$$\left(\bigotimes_s^n T \right)(x \otimes_s \cdots \otimes_s x) = Tx \otimes_s \cdots \otimes_s Tx$$

on the elementary tensors and extended by linearity.

Given $x_1', \ldots, x_n' \in X'$, the so called trace duality between the full tensor products $\bigotimes^n X'$ and $\bigotimes^n X$ identifies the tensor $x_1' \otimes \cdots \otimes x_n'$ with the linear functional defined on $\bigotimes^n X$ by

$$\langle x_1' \otimes \cdots \otimes x_n', x_1 \otimes \cdots \otimes x_n \rangle = \langle x_1', x_1 \rangle \cdots \langle x_n', x_n \rangle.$$

for all $x_1, \ldots, x_n \in X$, and extended by linearity. For the symmetric tensor product, $x_1' \otimes_s \cdots \otimes_s x_n'$ corresponds to a linear functional on $\bigotimes_s^n X$ which applied on an elementary tensor $x \otimes_s \cdots \otimes_s x$ takes the value

$$\langle x_1' \otimes_s \cdots \otimes_s x_n', x \otimes_s \cdots \otimes_s x \rangle = \langle x_1', x \rangle \cdots \langle x_n', x \rangle.$$

A symmetric n-tensor norm μ assigns to each normed space X a norm on $\bigotimes_s^n X$ satisfying

(a) The tensor $\otimes^n 1 \in (\bigotimes_s^n \mathbb{K}, \mu)$ has unit norm, where $\mathbb{K}$ denotes the real or complex field.

(b) The metric mapping property: for all continuous linear mappings $T \colon E \to F$, we have:

$$\| \bigotimes_s^n T \colon (\bigotimes_s^n X, \mu) \to (\bigotimes_s^n Y, \mu) \| = \|T\|^n.$$

We denote the completion of $(\bigotimes_s^n X, \mu)$ with respect to this norm by $\bigotimes_{\mu,s}^n X$. Note that extending the definition of the n-tensor power of T from $(\bigotimes_s^n X, \mu)$ to $\bigotimes_{\mu,s}^n X$ by density we have $\bigotimes_s^n T \colon \bigotimes_{\mu,s}^n X \to \bigotimes_{\mu,s}^n Y$ a continuous linear operator of the same norm.

As well as for the full tensor product, for the symmetric n-tensor fold there is a least symmetric n-tensor norm, called the symmetric injective norm, noted by ε and a greatest symmetric n-tensor norm, called the symmetric projective norm, noted by π.

Given an n-fold symmetric tensor $z \in \bigotimes_s^n X$ the symmetric injective norm is defined by

$$\varepsilon(z) = \sup_{x' \in B_{X'}} \left| \sum_{i=1}^k \lambda_i \langle x', x_i \rangle^n \right|,$$

where $\sum_{i=1}^k \lambda_i x_i \otimes x_i \otimes \cdots \otimes x_i$ is any fixed representation of z.

On the other hand

$$\pi(z) = \inf \left\{ \sum_{i=1}^k \|x_i\|^n \right\}$$

is the symmetric projective norm, where the infimum is taken over all the representations of z of the form $\sum_{i=1}^k \lambda_i x_i \otimes x_i \otimes \ldots \otimes x_i$.

In his Ph D. Thesis [27], Ryan states without a proof that the n-fold symmetric tensor product of a Banach space X has a Schauder basis whenever X does. An implicit proof is given by Dimant and Dineen for complex Banach spaces with shrinking basis in [14] (see also [12]). Later, in [23], Grecu and Ryan provide a constructive proof for real or complex Banach spaces. To be more precise, if (e_i) is a Schauder basis for X and μ is a symmetric n-tensor norm, then the sequence $(e_\alpha)_{\alpha \in \mathcal{J}}$ is a Schauder basis for $\bigotimes_{\mu,s}^n X$, where $e_\alpha = e_{\alpha_1} \otimes_s \cdots \otimes_s e_{\alpha_n}$ and $\mathcal{J} = \{\alpha \in \mathbb{N}^n : \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n\}$ is the set of decreasing n-multi-indices with the square ordering in which the role of rows and columns is reversed. From now on we use this result without further mention.

In particular, for X_d a Schauder sequence space, the sequence $(e_\alpha)_{\alpha \in \mathcal{J}} = (e_{\alpha_1} \otimes_s \cdots \otimes_s e_{\alpha_n})_{\alpha \in \mathcal{J}}$ is a basis for the n-fold symmetric tensor product $\bigotimes_{\mu,s}^n X_d$. This means that $\bigotimes_{\mu,s}^n X_d$ can be considered as a sequence space, identifying the elements in $\bigotimes_{\mu,s}^n X_d$ with their coefficients in the basis $(e_\alpha)_{\alpha \in \mathcal{J}} = (e_{\alpha_1} \otimes_s \cdots \otimes_s e_{\alpha_n})_{\alpha \in \mathcal{J}}$. In order to describe $((e_\alpha)')_{\alpha \in \mathcal{J}}$ the dual basic sequence of the basis $(e_\alpha)_{\alpha \in \mathcal{J}}$ we need to introduce some notation.

For any n-multi-index α, we denote by $\mathrm{Inv}(\alpha)$ the number of permutations in $\mathcal{S}_n$ for which α is invariant, that is $\mathrm{Inv}(\alpha) = \sharp \{\sigma \in \mathcal{S}_n : \alpha_{\sigma(i)} = \alpha_i, \ \forall i = 1, \ldots, n\}$. Also, $\mathrm{Perm}(\alpha)$ denotes the number of the different multi-indexes obtained by permutations of α. Then, the relation $\mathrm{Perm}(\alpha)\mathrm{Inv}(\alpha) = n!$ holds.

Now, if $((e_\alpha)')_{\alpha \in \mathcal{J}}$ is the dual basic sequence of $(e_\alpha)_{\alpha \in \mathcal{J}}$, then $\langle e_\alpha', e_\beta \rangle = \delta_{\alpha,\beta}$, for any pair of decreasing n-multi-indices α and β. Note that for full tensors, $\langle e_{\xi_1}' \otimes \cdots \otimes e_{\xi_n}', e_{\chi_1} \otimes \cdots \otimes e_{\chi_n} \rangle = \langle e_{\xi_1}', e_{\chi_1} \rangle \ldots \langle e_{\xi_n}', e_{\chi_n} \rangle$. For decreasing α and β,

we then have $\langle e'_{\alpha_1} \otimes_s \cdots \otimes_s e'_{\alpha_n}, e_{\beta_1} \otimes_s \cdots \otimes_s e_{\beta_n} \rangle = 0$ whenever $\beta \neq \alpha$. Otherwise,

$$
\begin{aligned}
\langle e'_{\alpha_1} \otimes_s \cdots \otimes_s e'_{\alpha_n}, e_{\alpha_1} \otimes_s \cdots \otimes_s e_{\alpha_n} \rangle &= \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \langle e'_{\alpha_{\sigma(1)}}, e_{\alpha_1} \rangle \ldots \langle e'_{\alpha_{\sigma(n)}}, e_{\alpha_n} \rangle \\
&= \frac{\mathrm{Inv}(\alpha)}{n!}.
\end{aligned}
$$

Therefore, for any $\alpha \in \mathcal{J}$ we have

$$
(e_\alpha)' = \mathrm{Perm}(\alpha) e'_{\alpha_1} \otimes_s \cdots \otimes_s e'_{\alpha_n}.
$$

Let X be a Banach space, X_d be a Schauder sequence space and μ be a symmetric n-tensor norm. Suppose there exists a continuous linear operator $S \colon X_d \to X$ and a sequence $(x'_i) \subset X'$ such that $((x'_i), (Se_i))$ is an atomic decomposition for X with respect to X_d. If $J \colon X \to X_d$ is the natural inclusion defined in equation (1), both n-tensor power operators $\bigotimes_s^n J \colon \bigotimes_{\mu,s}^n X \to \bigotimes_{\mu,s}^n X_d$ and $\bigotimes_s^n S \colon \bigotimes_{\mu,s}^n X_d \to \bigotimes_{\mu,s}^n X$ are continuous with norms $\|J\|^n$ and $\|S\|^n$ respectively. Moreover, we have $\left(\bigotimes_s^n S \right) \circ \left(\bigotimes_s^n J \right) = \bigotimes_s^n SJ = \bigotimes_s^n I_X = I_{\bigotimes_{\mu,s}^n X}$ and since $\bigotimes_{\mu,s}^n X_d$ can be thought of as a sequence space, we have that $\left(\left((\bigotimes_s^n J)'(e_\alpha)' \right)_{\alpha \in \mathcal{J}}, \left((\bigotimes_s^n S)(e_\alpha) \right)_{\alpha \in \mathcal{J}} \right)$ is an atomic decomposition for $\bigotimes_{\mu,s}^n X$ with respect to $\bigotimes_{\mu,s}^n X_d$, see Section 1. Furthermore, since $(\bigotimes_s^n J)' = \bigotimes_s^n J'$ and $J'(e'_i) = x'_i$ the atomic decomposition has the form

$$
\left(\mathrm{Perm}(\alpha)(x'_{\alpha_1} \otimes_s \cdots \otimes_s x'_{\alpha_n})_{\alpha \in \mathcal{J}}, (Se_{\alpha_1} \otimes_s \cdots \otimes_s Se_{\alpha_n})_{\alpha \in \mathcal{J}} \right).
$$

We have shown one of the implications of the following:

Theorem 6. *Let X be a Banach space, X_d be a Schauder sequence space and let μ be a symmetric n-tensor norm. Take $S \colon X_d \to X$ a continuous operator and $(x'_i) \subset X'$ a sequence.*

If $((x'_i), (Se_i))$ is an atomic decomposition for X with respect to X_d then

$$
(2) \qquad \left(\mathrm{Perm}(\alpha)(x'_{\alpha_1} \otimes_s \cdots \otimes_s x'_{\alpha_n})_{\alpha \in \mathcal{J}}, (Se_{\alpha_1} \otimes_s \cdots \otimes_s Se_{\alpha_n})_{\alpha \in \mathcal{J}} \right)
$$

is an atomic decomposition for $\bigotimes_{\mu,s}^n X$ with respect to $\bigotimes_{\mu,s}^n X_d$.

Conversely, if $\bigotimes_{\mu,s}^n X$ admits an atomic decomposition with respect to $\bigotimes_{\mu,s}^n X_d$ as in (2) then, for some n-th root of unity θ, $((\theta x'_i), (Se_i))$ is an atomic decomposition for X with respect to X_d.

Proof. We have to prove the second statement. The first step will be to show that the operator $J \colon X \to X_d$ given by $(\langle x'_i, x \rangle)_i$ is well defined and continuous.

Since μ is a symmetric n-tensor norm, we can consider the continuous natural mapping

$$
\bigotimes_{\mu,s}^n X_d \hookrightarrow \bigotimes_{\varepsilon,s}^n X_d \hookrightarrow \mathcal{L}\left(\bigotimes_{\pi,s}^{n-1} (X_d)', X_d \right).
$$

For each element of the basis e_α we write $\tilde{e}_\alpha$ for its identification as an operator in $\mathcal{L}\left(\bigotimes_{\pi,s}^{n-1} (X_d)', X_d \right)$. The application $\tilde{e}_\alpha$ is given by

$$
\tilde{e}_\alpha \big(\underbrace{a' \otimes_s \cdots \otimes_s a'}_{n-1} \big) = \frac{1}{n} \sum_{j=1}^{n} \Big(\prod_{i \neq j} \langle a', e_{\alpha_i} \rangle \Big) e_{\alpha_j}.
$$

In particular, fixed $k \in \mathbb{N}$, $\tilde{e}_\alpha(e'_k \otimes_s \cdots \otimes_s e'_k) = \frac{1}{n} \sum_{j=1}^n \left(\prod_{i \neq j} \delta_{\alpha_i,k} \right) e_{\alpha_j} = 0$ unless α is a permutation of $(k, k, \ldots, k, l)$, for some $l \in \mathbb{N}$. In this case, $\tilde{e}_\alpha(e'_k) = \frac{1}{n} e_l$.

Now, fix $x \in X$ and take $\tilde{J} \colon \bigotimes_{\mu,s}^n X \to \bigotimes_{\mu,s}^n X_d$ the canonical inclusion associated to the atomic decomposition,

$$
\begin{aligned}
\tilde{J}(x \otimes_s \cdots \otimes_s x) &= \sum_{\alpha \in \mathcal{J}} \mathrm{Perm}(\alpha) \langle x'_{\alpha_1} \otimes_s \cdots \otimes_s x'_{\alpha_n}, x \otimes_s \cdots \otimes_s x \rangle e_\alpha \\
&= \sum_{\alpha \in \mathcal{J}} \mathrm{Perm}(\alpha) \langle x'_{\alpha_1}, x \rangle \ldots \langle x'_{\alpha_n}, x \rangle e_\alpha.
\end{aligned}
$$

This series is norm convergent. Since the order in $\mathcal{J}$ fills blocks, taking a subsequence of the partial sums we can write

$$
\tilde{J}(x \otimes_s \cdots \otimes_s x) = \lim_{N \to \infty} \sum_{\substack{\alpha \in \mathcal{J} \\ \alpha_1 \leq N}} \mathrm{Perm}(\alpha) \langle x'_{\alpha_1}, x \rangle \ldots \langle x'_{\alpha_n}, x \rangle e_\alpha,
$$

with strong convergence in $\bigotimes_{\mu,s}^n X_d$. Using the identification described above we have,

$$
\begin{aligned}
\left(\sum_{\substack{\alpha \in \mathcal{J} \\ \alpha_1 \leq N}} \mathrm{Perm}(\alpha) \quad \langle x'_{\alpha_1}, x \rangle \ldots \langle x'_{\alpha_n} x \rangle \tilde{e}_\alpha \right)(e'_k) & \\
&= \sum_{l=1}^N \mathrm{Perm}(k, k, \ldots, k, l) \langle x'_k, x \rangle \ldots \langle x'_k, x \rangle \langle x'_l, x \rangle \frac{1}{n} e_l \\
&= \langle x'_k, x \rangle^{n-1} \sum_{l=1}^N \langle x'_l, x \rangle e_l,
\end{aligned}
$$

and this series is norm convergent in X_d. Since $\tilde{J}$ is injective, there exits $k \in \mathbb{N}$ such that $x'_k(x) \neq 0$. Then, $\sum_{l=1}^\infty \langle x'_l, x \rangle e_l$ converges in X_d and $J(x) = \sum_{l=1}^\infty \langle x'_l, x \rangle e_l$ is well defined. An application of the Banach-Steinhaus theorem shows that $J \colon X \to X_d$ is a continuous operator.

Straightforward calculations show that $\tilde{J} = \bigotimes_s^n J$. Since $\tilde{S}$ is given by $\bigotimes_s^n S$, we obtain $\bigotimes_s^n SJ = I_{\bigotimes_{\mu,s}^n X}$, that is, $SJx \otimes_s \cdots \otimes_s SJx = x \otimes_s \cdots \otimes_s x$. Therefore, it is easy to see that $SJx = \theta(x)x$ for some n-th root $\theta(x)$ of 1. We claim that $\theta(x)$ is independent of x. Indeed, let θ_0 be a primitive n-th root of the unit and define $A_j = \{x \in X \colon \|x\| = 1, SJx = \theta_0^j x\}$. The (path) connected set $\{x \in X \colon \|x\| = 1\}$ is the union of the closed sets A_j, $j = 1, \ldots, n$, so A_j is empty for all but one j, say j_0. Thus, setting $\theta = \theta_0^{j_0}$, we have $SJ(x) = \theta x$ for all x on the unit sphere of X and the claim is proved.

Changing J by $\theta^{-1} J$ if necessary we have $SJ = I_X$ and the result follows. $\square$ $\square$

The n-th root of the unit θ is unavoidable in the previous theorem (unless, of course, we deal with real Banach spaces and n is odd). Indeed, suppose $((x'_i), (Se_i))$ is an atomic decomposition and $\theta \neq 1$ is an n-th root of 1. If $y'_i = \theta x'_i$, $((y'_i), (Se_i))$ is not an atomic decomposition for X (the pair does not satisfy the reconstruction formula). However,

$$
\left(\mathrm{Perm}(\alpha)(y'_{\alpha_1} \otimes_s \cdots \otimes_s y'_{\alpha_n})_{\alpha \in \mathcal{J}}, (Se_{\alpha_1} \otimes_s \cdots \otimes_s Se_{\alpha_n})_{\alpha \in \mathcal{J}} \right)
$$

is an atomic decomposition for $\bigotimes_{\mu,s}^n X$ with respect to $\bigotimes_{\mu,s}^n X_d$.

The proof of the previous theorem can be adapted to show the converse of the result by Grecu - Ryan [23] and Dimant - Dineen [14]. We have not found this converse in literature, so we state the following:

Theorem 7. *Let X be a Banach space and (x_i) be a sequence in X. Then, the following statements are equivalent*

(i) *(x_i) is a basis for X.*
(ii) *$(x_{\alpha_1} \otimes_s \cdots \otimes_s x_{\alpha_n})_{\alpha \in \mathcal{J}}$ is a basis for $\bigotimes_{\mu,s}^n X$.*

If the conditions hold and (x_i') is the dual basic sequence of the basis (x_i), then dual basic sequence of $(x_{\alpha_1} \otimes_s \cdots \otimes_s x_{\alpha_n})_{\alpha \in \mathcal{J}}$ is $\mathrm{Perm}(\alpha)(x_{\alpha_1}' \otimes_s \cdots \otimes_s x_{\alpha_n}')_{\alpha \in \mathcal{J}}$.

The existence of a basis for the full tensor product of a Banach space is due to Gelbaum and Gil de Lamadrid in [21], and is previous to the result for symmetric tensor products. Arguing as in the proof of Theorem 6, we obtain the following result for atomic decompositions and full tensor products. In this case the set $\mathbb{N}^n$ may be considered either with the square ordering or with the order given in [23].

Theorem 8. *Let $X_1, \ldots, X_n$ be Banach spaces, $X_{1,d}, \ldots, X_{n,d}$ be sequence spaces and μ be a n-tensor norm. For each $j = 1, \ldots, n$, take $S_j \colon X_{j,d} \to X_j$ a continuous operator and a sequence $(x_{j,i}') \subset X_j'$. Then, $((x_{j,i}'), (Se_{j,i}))$ is an atomic decomposition for X_j with respect to $X_{j,d}$ for each $j = 1, \ldots, n$ if and only if $\left((x_{1,\alpha_1}' \otimes \cdots \otimes x_{n,\alpha_n}')_{\alpha \in \mathbb{N}^n}, (Se_{1,\alpha_1} \otimes \cdots \otimes Se_{n,\alpha_n})_{\alpha \in \mathbb{N}^n} \right)$ is an atomic decomposition for $\bigotimes_{\mu,j=1}^n X_j$ with respect to $\bigotimes_{\mu,j=1}^n X_{j,d}$.*

As in the case for symmetric tensor products, a simple modification of the above gives the converse of Gelbaum and Gil de Lamadrid's result for the Schauder basis case [21].

Now we combine the previous results with those of Section 1 to investigate the existence of atomic decompositions on tensor products of dual Banach spaces. This will be used in next section, in the setting of spaces of polynomials.

Corollary 9. *Let X be a Banach space and X_d be a sequence space. Let $S \colon X_d \to X$ be a continuous operator and $(x_i') \subset X'$ be a sequence such that $((x_i'), (Se_i))$ is an atomic decomposition for X with respect to X_d. Then, for any symmetric n-tensor norm μ, the following are equivalent:*

(i) *the atomic decomposition $(x_i', (Se_i))$ is strongly shrinking,*
(ii) *the pair $((Se_i), (x_i'))$ is an atomic decomposition for X' with respect to X_d',*
(iii) *the pair $\left(\left(Se_{\alpha_1} \otimes_s \cdots \otimes_s Se_{\alpha_n} \right)_{\alpha \in \mathcal{J}}, \mathrm{Perm}(\alpha)(x_{\alpha_1}' \otimes_s \cdots \otimes_s x_{\alpha_n}')_{\alpha \in \mathcal{J}} \right)$ is an atomic decomposition for $\bigotimes_{\mu,s}^n X'$ with respect to $\bigotimes_{\mu,s}^n X_d'$.*

Proof. (i) $\Leftrightarrow$ (ii) is Theorem 5. (ii) $\Rightarrow$ (iii) follows from Theorem 6. Now, if (iii) holds, by Theorem 6 we know that $((\theta Se_i), (x_i'))$ is an atomic for X' with respect to X_d', with θ some n-th root of the unit. Since $((x_i'), (Se_i))$ is an atomic decomposition, we must have $\theta = 1$. $\qquad\square \qquad\qquad\qquad \square$

The analogous equivalence remains true if in statement (i) we have that $(x_i', (Se_i))$ is a shrinking atomic decomposition and X_d' is replaced by $(X_d)'$ in (ii) and (iii).

However, since $\bigotimes_{\mu,s}^n (X_d)'$ is not necessarily a Schauder sequence space, the situation here is more complicated and we cannot combine previous results as in the corollary above to obtain these new equivalences. Indeed, for one of the implications we had to adapt the ideas from [23, Section 3] and we follow their notation. Also, we need the next lemma, the proof of which was kindly provided to us by Santiago Muro:

Lemma 10. *Let $(z_k)_k \subset X$, $z \in X$ and let μ be a tensor norm. If $\otimes^n z_k$ converges to $\otimes^n z$ in $\bigotimes_{\mu,s}^n X$, then z_k accumulates on $A_z := \{\theta^j z : j = 0, \ldots, n-1\}$, where θ is any primitive n-th root of 1.*

If we also have that (z_k) converges to z in some (Hausdorff) locally convex topology, then (z_k) converges to z in norm.

Proof. Note that for $z = 0$ the result is immediate. If $z \neq 0$, the sequence $(z_k)_k$ is bounded and bounded below. Suppose the result does not hold. We may assume that $d(z_k, A_z) > \epsilon$ for some $\epsilon > 0$. Again, passing to a subsequence if necessary, we obtain $d(z_k, [z]) > \delta$ for some positive δ. Indeed, if there exist scalars λ_k such that $\|z_k - \lambda_k z\| \to 0$, for any $x' \in X$ we have $|\langle x', z_k\rangle^n - \lambda_k^n \langle x', z\rangle^n| \to 0$. Since $\otimes^n z_k$ converges to $\otimes^n z$ in $\bigotimes_{\mu,s}^n X$, we also have $|\langle x', z_k\rangle^n - \langle x', z\rangle^n| \to 0$. Then, $|\lambda_k| \to 1$ and the sequence accumulates in A_z. Clearly, we may also assume that $d(z, [z_k]) > \delta$ for all k.

For each k, let γ_k be the linear functional on $[z_k, z]$ verifying $\gamma_k(z_k) = \gamma_k(z - z_k) = \|z - z_k\|$. From the lower bound of the distances above, it is easy to check that the norms of the γ_k's are uniformly bounded. By the Hahn-Banach extension theorem, we can consider γ_k defined on X.

Since $\otimes^n z_k$ converges to $\otimes^n z$ in $\bigotimes_{\mu,s}^n X$, we have

$$\sup_{x' \in B_{X'}} |\langle x', z_k\rangle^n - \langle x', z\rangle^n| \to 0.$$

Therefore, $\lim_k |\gamma_k(z_k)^n - \gamma_k(z)^n| = 0$. But, on the other hand, it is not hard to see that $|\gamma_k(z_k)^n - \gamma_k(z)^n| \geq \|z_k - z\|^n > \delta^n$, obtaining a contradiction. $\square$ $\square$

Theorem 11. *Let X be a Banach space and X_d be a sequence space. Let $S \colon X_d \to X$ be a continuous operator and $(x_i') \subset X'$ be a sequence such that $((x_i'), (Se_i))$ is an atomic decomposition for X with respect to X_d. Then, for any symmetric n-tensor norm μ, the following are equivalent:*

(i) *the atomic decomposition $(x_i', (Se_i))$ is shrinking,*

(ii) *the pair $((Se_i), (x_i'))$ is an atomic decomposition for X' with respect to $(X_d)'$,*

(iii) *the pair $\left(\left(Se_{\alpha_1} \otimes_s \cdots \otimes_s Se_{\alpha_n} \right)_{\alpha \in \mathcal{J}}, \mathrm{Perm}(\alpha)(x_{\alpha_1}' \otimes_s \cdots \otimes_s x_{\alpha_n}')_{\alpha \in \mathcal{J}} \right)$ is an atomic decomposition for $\bigotimes_{\mu,s}^n X'$ with respect to $\bigotimes_{\mu,s}^n (X_d)'$.*

Proof. The equivalence between (i) and (ii) is theorem 3. Suppose (ii) holds. Note that, to establish (iii), the reconstruction formula is the non direct part of the proof of the statement (properties (a) and (b) of the definition follow as in the comments before Theorem 6).

First, we show the reconstruction formula for the full tensor product. Following [23, Section 3], for $\beta \in \mathcal{J}$ we consider the finite rank operator $P_\beta^n \colon \bigotimes_\mu^n X' \to$

$\bigotimes_\mu^n X'$ given on the elementary tensors by

$$(3) \quad P_\beta^n(y_1' \otimes \cdots \otimes y_n') = \sum_{\substack{\alpha \in \mathcal{J} \\ \alpha \leq \beta}} \langle y_1' \otimes \cdots \otimes y_n', Se_{\alpha_1} \otimes \cdots \otimes Se_{\alpha_n} \rangle \, x_{\alpha_1}' \otimes \cdots \otimes x_{\alpha_n}',$$

and extended by linearity and density. We must show that $P_\beta^n(z)$ converges to z for all $z \in \bigotimes_\mu^n X'$.

For $j \in \mathbb{N}$ we denote by $x_j' \underline{\otimes} Se_j \colon X' \to X'$ the operator defined as $x_j' \underline{\otimes} Se_j(x') = \langle x', Se_j \rangle x_j'$. Note that $P_m^1 = \sum_{j=1}^m x_j' \underline{\otimes} Se_j$ and since $((x_i'), (Se_i))$ is shrinking, by Theorem 3, P_m^1 is pointwise convergent to the identity. Thus, by the Banach-Steinhaus theorem, (P_m^1) is uniformly bounded: say $\|P_m^1\| \leq K_1$, for all $m \in \mathbb{N}$. Moreover, fixed $n \in \mathbb{N}$, $(P_\beta^n)_{\beta \in \mathcal{J}}$ is uniformly bounded. To see this, consider μ' the symmetric tensor norm associated to the inclusion $\bigotimes_{\mu'}^n X \hookrightarrow \left(\bigotimes_\mu^n X' \right)'$, that is $\mu'(z) = \sup_{\|w\|_\mu \leq 1} \langle w, z \rangle$ where the supremum in taken over $w \in \bigotimes_\mu^n X'$. In an analogous way to $(P_\beta^n)_{\beta \in \mathcal{J}}$ we may define $Q_\beta^n \colon \bigotimes_{\mu'}^n X \to \bigotimes_{\mu'}^n X$. By Theorem 8, $\left((x_{\alpha_1}' \otimes \cdots \otimes x_{\alpha_n}')_{\alpha \in \mathcal{J}}, (Se_{\alpha_1} \otimes \cdots \otimes Se_{\alpha_n})_{\alpha \in \mathcal{J}} \right)$ is an atomic decomposition for $\bigotimes_\mu^n X$ with respect to $\bigotimes_\mu^n X_d$ which implies that Q_β^n converges to the identity on each $z \in \bigotimes_{\mu'}^n X$. Therefore, Q_β^n is uniformly bounded: say $\|Q_\beta^n\| \leq K_n$.

Then, we have

$$|\langle P_\beta^n w, z \rangle| = |\langle w, Q_\beta^n z \rangle| \leq K_n \|w\|_\mu \|z\|_{\mu'}$$

and $(\|P_\beta^n\|)_{\beta \in \mathcal{J}}$ is also bounded by K_n.

To see that $P_\beta^n(z)$ converges to z for all $z \in \bigotimes_\mu^n X'$, it is enough to consider elementary tensors $y_1' \otimes \cdots \otimes y_n'$, by the uniform bound on the norms of the projections P_β. Now, equality (3) together with another application of Theorem 8, imply that $P_\beta^n(y_1' \otimes \cdots \otimes y_n')$ converges weakly to $y_1' \otimes \cdots \otimes y_n'$. So we must show that $P_\beta^n(y_1' \otimes \cdots \otimes y_n')$ is a Cauchy sequence to obtain the desired result.

We will present the case $n = 2$. The general case follows by induction. The induction step in [23] can be adapted to our setting in the same way as the case $n = 2$, so we omit it. It is worth mentioning that the symmetry of the tensor norm μ plays its role in this induction step.

Take an elementary tensor $x' \otimes y' \in X' \otimes_\mu X'$ and $\beta = (\beta_1, \beta_2) \in \mathcal{J}$.

If $\beta = (m, m)$ then,

$$P_{(m,m)}^2(x' \otimes y') = \sum_{1 \leq i,j \leq m} \langle x', Se_i \rangle \langle y', Se_j \rangle \, x_i' \otimes x_j' = P_m^1(x') \otimes P_m^1(y').$$

Then, by Theorem 3, $P_{(m,m)}^2$ converges pointwise to the identity. Hence, $P_{(m,m)}^2(x' \otimes y')$ is a Cauchy sequence.

Now, take $m \in \mathbb{N}$ so that $m + 1 = \max\{\beta_1, \beta_2\}$. Then, $(m, m) < \beta$ in the order given in [23]. In order to compare $P_\beta^2(x' \otimes y')$ with $P_{(m,m)}^2(x' \otimes y')$, suppose first that $\beta = (k, m + 1)$. In this case,

$$P_\beta^2 - P_{(m,m)}^2 = P_k^1 \otimes (x_{m+1}' \underline{\otimes} Se_{m+1}) + (x_{m+1}' \underline{\otimes} Se_{m+1}) \otimes P_{k-1}^1.$$

If $\beta = (m + 1, k)$ we have

$$P_\beta^2 - P_{(m,m)}^2 = P_k^1 \otimes (x_{m+1}' \underline{\otimes} Se_{m+1}) + (x_{m+1}' \underline{\otimes} Se_{m+1}) \otimes P_k^1.$$

Note that as β increases, m goes to infinity. Then, in both cases we obtain

$$\|(P_\beta^2 - P_{(m,m)}^2)(x' \otimes y')\|$$

$$\leq C\|x'\| |\langle y', Se_{m+1} \rangle| \|x'_{m+1}\| + C\|y'\| |\langle x', Se_{m+1} \rangle| \|x'_{m+1}\|,$$

which converges to zero. Finally, we need to estimate the difference between $P_\beta^2(x' \otimes y')$ and $P_{(m+1,m+1)}^2(x' \otimes y')$. We write $P_{(m+1,m+1)}^2 - P_\beta^2 = (P_{(m+1,m+1)}^2 - P_{(m,m)}^2) + (P_{(m,m)}^2 - P_\beta^2)$ which, using estimations as above, goes to zero on each $x' \otimes y'$ when m goes to infinity.

Now, let $\alpha, \beta \in \mathcal{J}$ and suppose $\alpha < \beta$. To show that $(P_\beta^2 - P_\alpha^2)(x' \otimes y')$ converges to zero, take k and m in $\mathbb{N}$ so that $m + 1 = \max\{\beta_1, \beta_2\}$ and $k + 1 = \max\{\alpha_1, \alpha_2\}$. It is enough to consider the case $k < m$. Then, writing $P_\beta^2 - P_\alpha^2 = (P_\beta^2 - P_{(m,m)}^2) + (P_{(m,m)}^2 - P_{(k+1,k+1)}^2) + (P_{(k+1,k+1)}^2 - P_\alpha^2)$, we get the desired result.

For the symmetric tensor product, the operators $\Pi_\beta \colon \bigotimes_{\mu,s}^n X' \to \bigotimes_{\mu,s}^n X'$ should be considered instead of P_β, where $\Pi_\beta(y'_1 \otimes_s \cdots \otimes_s y'_n)$ is computed as

$$\sum_{\substack{\alpha \in \mathcal{J} \\ \alpha \leq \beta}} \mathrm{Perm}(\alpha) \langle y'_1 \otimes_s \cdots \otimes_s y'_n, Se_{\alpha_1} \otimes_s \cdots \otimes_s Se_{\alpha_n} \rangle \, x'_{\alpha_1} \otimes_s \cdots \otimes_s x'_{\alpha_n}.$$

Now, (iii) follows from the fact that P_β coincides with Π_β on symmetric tensors.

Finally, we have to show the implication (iii)$\Rightarrow$(ii). Consider the canonical operators $J \colon X \to X_d$ and $S \colon X_d \to X$ associated to the atomic decomposition of X with respect to X_d. Their adjoints $S' \colon X' \to (X_d)'$ and $J' \colon (X_d)' \to X'$ satisfy $J' \circ S' = I_{X'}$. Since $(X_d)'$ is not necessarily a Schauder sequence space, we cannot conclude (ii) straightforwardly. We need to show that the reconstruction formula holds, i.e., that for any $x' \in X'$ we have

$$x' = \sum_{k=1}^\infty \langle x', Se_k \rangle x'_k.$$

Since $((x'_i), (Se_i))$ is an atomic decomposition for X, it is easy to show that the equality holds pointwise, that is, $\sum_{k=1}^N \langle x', Se_k \rangle x'_k$ converges to x' in the weak-star topology. On the other hand, from the atomic decomposition for the tensor product, we have that the sequence

$$\sum_{k=1}^N \langle x', Se_k \rangle x'_k \otimes \cdots \otimes \sum_{k=1}^N \langle x', Se_k \rangle x'_k$$

converges to $x' \otimes \cdots \otimes x'$ in $\bigotimes_{\mu,s}^n X'$. The reconstruction formula is then a consequence of Lemma 10. $\qquad\square$ $\qquad\qquad\square$

Note that in the proof we have also shown the analogous result for full tensor products.

3. Atomic decompositions and spaces of polynomials

Tensor products are closely related with multilinear forms and symmetric tensor products with homogeneous polynomials. When endowed with different topologies, the spaces of symmetric tensors correspond with different classes of homogeneous polynomials. Before going on, we recall some notation and definitions.

Let X be Banach spaces and denote $\mathbb{K}$ the real or complex scalar field. A function $P\colon X \to \mathbb{K}$ is said to be a (continuous) n-homogeneous scalar-valued polynomial if there exists a (continuous) n-linear map $\check{P}\colon \underbrace{X \times \cdots \times X}_{n-\text{times}} \to \mathbb{K}$ such that $P(x) = \check{P}(x,\ldots,x)$ for all $x \in X$. Continuous n-homogeneous polynomials are bounded on the unit ball. We denote by $\mathcal{P}(^nX)$ the Banach space of all continuous n-homogeneous polynomials on X endowed with the supremum norm $\|P\| := \sup_{\|x\|\leq 1} |P(x)|$. Since we only consider continuous scalar-valued polynomials, we will usually omit the adjectives continuous and scalar-valued.

Definition 12. *A pair $(\mathcal{Q}, \|.\|_{\mathcal{Q}})$ is a Banach ideal of n-homogeneous polynomials if for any Banach spaces X and Y we have*

(a) $\mathcal{Q}(X) = \mathcal{Q} \cap \mathcal{P}(^nX)$ is a linear subspace of $\mathcal{P}(^nX)$ and $\|\cdot\|_{\mathcal{Q}(X)}$ is a norm on $\mathcal{Q}(X)$ that makes it a Banach space.

(b) If $T \in \mathcal{L}(X;Y)$ and $P \in \mathcal{Q}(Y)$; then $P \circ T \in \mathcal{Q}(X)$ and $\|P \circ T\|_{\mathcal{Q}} \leq \|P\|_{\mathcal{Q}}\|T\|^n$.

(c) $\otimes^n 1 = [\mathbb{K} \ni z \rightsquigarrow z^n \in \mathbb{K}] \in \mathcal{Q}$ and $\|\otimes^n 1\colon \mathbb{K} \to \mathbb{K}\|_{\mathcal{Q}} = 1$.

We present some of the usual ideals of polynomials. An n-homogeneous polynomial $P \in \mathcal{P}(^nX)$ is said to be of finite type if there are $x'_1,\ldots,x'_k$ in X' and scalars $\lambda_1,\ldots,\lambda_k$ such that $P(x) = \sum_{j=1}^{k} \lambda_j \langle x'_j, x\rangle^n$ for all x in X. Polynomials in the closure of the finite type n-homogeneous polynomials are called approximable. We use $\mathcal{P}_f(^nX)$ to denote the space of finite type n-homogeneous polynomials and $\mathcal{P}_A(^nX)$ to denote the space of all n-homogeneous approximable polynomials.

A polynomial $P \in \mathcal{P}(^nX)$ is said to be nuclear if it can be written as $P(x) = \sum_{j=1}^{\infty} \lambda_j \langle x'_j, x\rangle^n$, where (λ_j) is a bounded sequence of scalars and $(x'_j) \subset X'$ verifies $\sum_{j=1}^{\infty} \|x'_j\|^n < \infty$. The space of nuclear n-homogeneous polynomials on X will be denoted by $\mathcal{P}_N(^nX)$. It is a Banach space when considered with the norm

$$\|P\|_N = \inf\left\{\sum_{j=1}^{\infty} |\lambda_j|\|x'_j\|^n\right\}$$

where the infimum is taken over all representations of P as above.

A polynomial P on X is said to be integral if there is a regular Borel measure Γ on $(B_{X'}, \sigma(X', X))$ such that

$$(4) \qquad\qquad P(x) = \int_{B_{X'}} \langle x', x\rangle^n \, d\Gamma(x')$$

for every x in X. We write $\mathcal{P}_I(^nX)$ for the space of all n-homogeneous integral polynomials on X. The integral norm of an integral polynomial P, $\|P\|_I$, is defined as the infimum of $\|\Gamma\|$ taken over all regular Borel measures which satisfy (4). It is shown in [17] that the dual of $\bigotimes_{\epsilon,s}^n X$ is isometrically isomorphic to $(\mathcal{P}_I(^nX), \|.\|_I)$.

Given a Banach ideal of n-homogeneous polynomials $\mathcal{Q}$, the minimal ideal $\mathcal{Q}^{\min}$ is defined as the composition ideal

$$\mathcal{Q}^{\min} = \mathcal{Q} \circ \overline{\mathcal{F}},$$

where $\overline{\mathcal{F}}$ is the ideal of all operators which are approximable by finite rank operators. In other words, P belongs to $\mathcal{Q}^{\min}$ if there exists a factorization $P = QT$ with $T \in \overline{\mathcal{F}}$

and $Q \in \mathcal{Q}$. Also,

$$\|P\|_{\mathcal{Q}^{\min}} = \inf\{\|Q\|_{\mathcal{Q}}\|T\|^n\},$$

where the infimum is taken over all such factorizations $P = QT$.

A Banach ideal $\mathcal{Q}$ is said to be minimal if $\mathcal{Q} = \mathcal{Q}^{\min}$.

Let $\mathcal{Q}$ be a Banach ideal of n-homogeneous polynomials and M a finite dimensional space. We define in $\bigotimes_s^n M$ the symmetric tensor norm $\mu_\mathcal{Q}$ associated to $\mathcal{Q}$ by

$$\bigotimes_{s,\mu_\mathcal{Q}}^n M \stackrel{1}{=} \left(\mathcal{Q}(M'), \|\cdot\|_\mathcal{Q}\right).$$

For an arbitrary normed space X, $FIN(X)$ denotes the class of all finite dimensional subspaces of X. Then we define

$$\mu_\mathcal{Q}(z, \bigotimes_s^n X) := \inf\left\{\mu_\mathcal{Q}(z, \bigotimes_s^n M) \ / \ z \in \bigotimes_s^n M, \ M \in FIN(X)\right\}.$$

Let X be a Banach space and X_d a Schauder sequence space. Suppose there exists a continuous operator $S \colon X_d \to X$ and a sequence $(x_i') \subset X'$ such that $((x_i'), (Se_i))$ is an atomic decomposition for X with respect to X_d.

If $((x_i'), (Se_i))$ is shrinking, Theorem 11 states that

$$\left((Se_{\alpha_1} \otimes_s \cdots \otimes_s Se_{\alpha_n})_{\alpha \in \mathcal{J}}, (\text{Perm}(\alpha) \ x'_{\alpha_1} \otimes_s \cdots \otimes_s x'_{\alpha_n})_{\alpha \in \mathcal{J}}\right)$$

is an atomic decomposition for $\bigotimes_{\mu_\mathcal{Q},s}^n X'$ with respect to $\bigotimes_{\mu_\mathcal{Q},s}^n (X_d)'$. On the other hand, since X' admits an atomic decomposition, it has the bounded approximation property. We have the following isometric isomorphism [20, Corollary 5.2]:

$$(5) \qquad \mathcal{Q}^{\min}(X) \stackrel{1}{=} \bigotimes_{s,\mu_\mathcal{Q}}^n X'.$$

Therefore, we have shown that

$$(6) \qquad \left((Se_{\alpha_1} \otimes_s \cdots \otimes_s Se_{\alpha_n})_{\alpha \in \mathcal{J}}, (\text{Perm}(\alpha) \ x'_{\alpha_1} \otimes_s \cdots \otimes_s x'_{\alpha_n})_{\alpha \in \mathcal{J}}\right)$$

is an atomic decomposition for $\mathcal{Q}^{\min}(X)$ with respect to the Banach sequence space $\bigotimes_{\mu_\mathcal{Q},s}^n (X_d)'$.

Observe that when $((x_i'), (Se_i))$ is strongly shrinking, by Corollary 9, the atomic decomposition (6) is in fact associated to the Schauder sequence space $\bigotimes_{\mu_\mathcal{Q},s}^n X_d'$.

To simplify the statement of the results just obtained, we will introduce the concept of monomial atomic decomposition, which generalizes the concept of monomial basis.

Whenever $((x_i'), (x_i))$ is an atomic decomposition of X, each $x \in X$ can be written as $x = \sum_i \langle x_i', x \rangle x_i$. Therefore, if $P \in \mathcal{P}(^n X)$ we always have the pointwise series expansion

$$\begin{aligned}
P(x) &= \check{P}(x, \ldots, x) \\
&= \sum_{\alpha_1} \cdots \sum_{\alpha_n} \check{P}(x_{\alpha_1}, \ldots, x_{\alpha_n}) \ \langle x'_{\alpha_1}, x \rangle \cdots \langle x'_{\alpha_n}, x \rangle \\
&= \sum_{\alpha \in \mathcal{J}} \text{Perm}(\alpha) \ \check{P}(x_{\alpha_1}, \ldots, x_{\alpha_n}) \ \langle x'_{\alpha_1}, x \rangle \cdots \langle x'_{\alpha_n}, x \rangle.
\end{aligned}$$

Now, take $P \in \mathcal{Q}^{\min}(X)$ and let $\check{P} \colon X \times \cdots \times X \to \mathbb{K}$ be the symmetric n-linear form associated to P. If we set $x_i = S(e_i)$, we have shown that P can be written as:

$$P = \sum_{\alpha \in \mathcal{J}} \mathrm{Perm}(\alpha) \, \check{P}(x_{\alpha_1}, \ldots, x_{\alpha_n}) \, x'_{\alpha_1} \cdots x'_{\alpha_n},$$

and this series expansion converges in $\| \cdot \|_{\mathcal{Q}^{\min}}$.

Therefore, the atomic decomposition for $\mathcal{Q}^{\min}(X)$ gives a monomial expansion for polynomials with respect to the atomic decomposition of X. This motivates the following definition:

Definition 13. *Let $((x'_i),(x_i))$ be any atomic decomposition of X with respect to X_d. We say that $\mathcal{Q}(X)$ has a monomial atomic decomposition with respect to $((x'_i),(x_i))$ if*

$$\left((x_{\alpha_1} \otimes_s \cdots \otimes_s x_{\alpha_n})_{\alpha \in \mathcal{J}}, (\mathrm{Perm}(\alpha) \, x'_{\alpha_1} \otimes_s \cdots \otimes_s x'_{\alpha_n})_{\alpha \in \mathcal{J}} \right)$$

is an atomic decomposition for $\mathcal{Q}(X)$ with respect to the Banach sequence space $\bigotimes_{\mu_{\mathcal{Q}},s}^n (X_d)'$.

Whenever the sequence space $\bigotimes_{\mu_{\mathcal{Q}},s}^n (X_d)'$ can be replaced by $\bigotimes_{\mu_{\mathcal{Q}},s}^n X'_d$, we say that the monomial atomic decomposition is sharp.

Now we are ready to state the following:

Proposition 14. *Suppose there exists a continuous operator $S \colon X_d \to X$ and a sequence $(x'_i) \subset X'$ such that $((x'_i),(Se_i))$ is an atomic decomposition for X with respect to X_d. Then, $((x'_i),(Se_i))$ is (strongly) shrinking if and only if $\mathcal{Q}^{\min}(X)$ has a (sharp) monomial decomposition with respect to $((x'_i),(Se_i))$.*

Proof. By the discussion preceding Definition 13 it only remains to prove one implication. Suppose $\mathcal{Q}^{\min}(X)$ admits a (sharp) monomial decomposition of the form (6). In particular, it has the bounded approximation property. As we will see in Lemma 15, X' is isomorphic to a complemented subspace of any ideal of polynomials on X and, in particular, of $\mathcal{Q}^{\min}(X)$. Then, X' inherits the bounded approximation property and therefore (5) holds. This means that (6) is an atomic decomposition for $\bigotimes_{s,\mu_{\mathcal{Q}}}^n X'$ with respect to $\bigotimes_{\mu_{\mathcal{Q}},s}^n X'_d$. By Corollary Theorem 11(Theorem 9), the atomic decomposition $((x'_i),(Se_i))$ is (strongly) shrinking. □   □

The following lemma should be compared to [4, Proposition 5.3] and [5]:

Lemma 15. *Let $(\mathcal{Q}, \|.\|_{\mathcal{Q}})$ be a Banach ideal of n-homogeneous polynomials. Then, if X is Banach space, X' is isomorphic to a complemented subspace of $\mathcal{Q}(X)$.*

Proof. Fix $x_0 \in X$ a unit vector and $x'_0 \in X'$ a norm one functional so that $\langle x'_0, x_0 \rangle = 1$. Now, consider the mapping $\iota \colon X' \to \mathcal{Q}(X)$, defined by $(\iota x')(x) = \langle x'_0, x \rangle^{n-1} \langle x', x \rangle$, for all $x \in X$ and every $x' \in X'$. By [8, Corollary 8.a.], we have $\|\iota(x')\|_{\mathcal{Q}} \le e\|x'\|$. On the other hand, take $q \colon \mathcal{Q}(X) \to X'$ the operator given by $q(P)(x) = n\check{P}(x, x_0, \ldots, x_0) - (n-1)P(x_0)\langle x'_0, x \rangle$, for all $x \in X$. By [8, Corollary 8.b.], $\|q(P)\| \le ne\|P\| + (n-1)\|P\| \le (ne + n - 1)\|P\|_{\mathcal{Q}}$. As in [8, Lemma 4], we get for every $x' \in X'$ and all $x \in X$

$$q \circ \iota(x')(x) = q(\langle x'_0, \cdot \rangle^{n-1} \langle x', \cdot \rangle)(x) = \langle x', x \rangle.$$

Then, ι is an isomorphism onto its image and $\iota \circ q$ is a projection onto $\iota(X')$. □   □

Remark 16. Note that Proposition 14 shows the existence of monomial decompositions for $\mathcal{Q}^{\min}(X)$. A natural question is whether it is possible to obtain monomial decompositions for arbitrary ideals of polynomials. To answer this question suppose that $\mathcal{Q}(X)$ admits a monomial decomposition. Let us show that in this case $\mathcal{Q}(X)$ must coincide with $\mathcal{Q}^{\min}(X)$.

First note that $\mathcal{Q}(X)$ has the bounded approximation property and so does X', since it is complemented in $\mathcal{Q}(X)$ by Lemma 15. As a consequence of the Factorization Lemma [20, Section 3.5], it can be seen that for X' with the bounded approximation property, the norms $\| \cdot \|_{\mathcal{Q}^{\min}}$ and $\| \cdot \|_{\mathcal{Q}}$ are equivalent on $Q^{\min}(X)$. Now, all polynomials in the monomial decomposition for $Q(X)$ are of finite type, so they all belong to $\mathcal{Q}^{\min}(X)$. By the equivalence of norms, the closure of the span of the monomials must be $\mathcal{Q}(X)$ and, at the same time, be contained in $\mathcal{Q}^{\min}(X)$. Therefore, $\mathcal{Q}(X) = \mathcal{Q}^{\min}(X)$.

As a consequence of Proposition 14 we have:

Corollary 17. Let $((x_i'), (Se_i))$ be an atomic decomposition for X with respect to X_d. The following are equivalent:

(a) $\mathcal{Q}(X) = \mathcal{Q}^{\min}(X)$ and $((x_i'), (Se_i))$ is (strongly) shrinking;
(b) $\mathcal{Q}(X)$ has a (sharp) monomial decomposition with respect to $((x_i'), (Se_i))$.

Proof. It is clear that (a) implies (b). Conversely, if $\mathcal{Q}(X)$ admits a sharp monomial decomposition with respect to $((x_i'), (Se_i))$, arguing as in the proofs of Theorem 6 and Corollary 9 we can see that $((x_i'), (Se_i))$ is strongly shrinking. By Remark 16, $\mathcal{Q}(X) = \mathcal{Q}^{\min}(X)$. □ □

It is clear that Proposition 14 and Corollary 17 have their analogous for multilinear forms. The existence of monomial bases for spaces of polynomials and multilinear forms was studied in Dimant-Dineen [14] and Dimant-Zalduendo [15], where similar results to the "only if" part of Proposition 14 and Corollary 17 are obtained. The converse for Schauder basis can be proved as above.

Now we apply the previous results to the polynomial ideals presented above. To this end, suppose we have a continuous operator $S \colon X_d \to X$ and a sequence $(x_i') \subset X'$ such that $((x_i'), (Se_i))$ is an atomic decomposition for X with respect to X_d. All the examples below have their analogous in terms of strongly shrinking atomic decompositions and sharp monomial decompositions. We chose to state them in their simpler form.

Recall that the Banach ideal of approximable polynomials $\mathcal{P}_A(^nX)$ is minimal and is associated to the symmetric injective n-tensor norm ε. Therefore, the polynomial ideal $\mathcal{P}_A(^nX)$ has a monomial atomic decomposition with respect to $((x_i'), (Se_i))$ if and only if $((x_i'), (Se_i))$ is shrinking.

Moreover, if $((x_i'), (Se_i))$ is shrinking, X' has the bounded approximation property, so every polynomial that is weakly continuous on bounded sets must be approximable [3]. Also, X' is separable and then X does not contain an isomorphic copy of ℓ_1. This means that weakly sequentially continuous polynomials are weakly continuous on bounded sets [2]. Therefore, the space $\mathcal{P}_{wsc}(^nX)$ of weakly sequentially continuous polynomials coincide with $\mathcal{P}_A(^nX)$. Therefore, we have that $\mathcal{P}_{wsc}(^nX)$ has a monomial atomic decomposition with respect to $((x_i'), (Se_i))$ (see [14] for a similar result in the Schauder basis setting).

Regarding nuclear polynomials, $\mathcal{P}_N(^nX)$ is a minimal polynomial ideal associated to the symmetric projective n-tensor norm π. Then, $((x_i'),(Se_i))$ gives an monomial atomic decomposition for $\mathcal{P}_N(^nX)$ if and only if $((x_i'),(Se_i))$ is shrinking.

If $((x_i'),(Se_i))$ is shrinking, the Banach space X has a separable dual and, in consequence, X is an Asplund space. In this case, the spaces of nuclear and integral polynomials on X coincide isometrically (see [1, 6, 7]), whence $\mathcal{P}_I(^nX)$ has a monomial atomic decomposition. On the other hand, if $\mathcal{P}_I(^nX)$ admits a monomial decomposition with respect to $((x_i'),(Se_i))$, by Corollary 17 $((x_i'),(Se_i))$ is shrinking and $\mathcal{P}_I(^nX) = \mathcal{P}_N(^nX)$. Also, X is Asplund.

We resume the previous discussions in the following statement:

Remark 18. *Let* $((x_i'),(Se_i))$ *be an atomic decomposition for* X *with respect to* X_d. *The following are equivalent:*

(a) $((x_i'),(Se_i))$ *is (strongly) shrinking;*
(b) $\mathcal{P}_{wsc}(^nX)$ *has a (sharp) monomial decomposition with respect to* $((x_i'),(Se_i))$.
(c) $\mathcal{P}_I(^nX)$ *has a (sharp) monomial decomposition with respect to* $((x_i'),(Se_i))$.

In addition, if the conditions hold, X *is an Asplund space.*

Now we turn our attention to the reflexivity of the space of polynomials. This should be compared with the results in [1] and [14]. For a reflexive space X with the approximation property, the reflexivity of $\mathcal{P}(^nX)$ is equivalent to every polynomial $P \in \mathcal{P}(^nX)$ being approximable, that is, to $\mathcal{P}(^nX) = \mathcal{P}_A(^nX)$ (see [1, 27]). The characterization of reflexivity in terms of monomial bases relies in a result analogous to Corollary 17 and the following fact: a basis for a reflexive Banach space is always shrinking (which for Schauder basis, is equivalent to being strongly shrinking). In [9], an example of an atomic decomposition for a reflexive Banach space that is not strongly shrinking is presented. We do not know if an atomic decomposition of a reflexive Banach space is always shrinking. However, this is the case if the canonical basis (e_i) of X_d is unconditional, see [9]. Whenever (e_i) is an unconditional basis, we say that $((x_i'),(Se_i))$ is an unconditional atomic decomposition. Note that we always have $\mathcal{P}^{\min}(^nX) = \mathcal{P}_A(^nX)$, then Corollary 17 gives:

Theorem 19. *Let* X *be a Banach space with an unconditional atomic decomposition* $((x_i'),(Se_i))$. *The following statements are equivalent:*

(a) $\mathcal{P}(^nX)$ *is reflexive*
(b) X *is reflexive and* $\mathcal{P}(^nX)$ *admits a monomial decomposition with respect to* $((x_i'),(Se_i))$.

If we drop off the unconditionality assumption, we obtain a similar characterization for the reflexivity of $\mathcal{P}(^nX)$ imposing the atomic decomposition to be shrinking or strongly shrinking.

Theorem 20. *Let* X *be a Banach space with an atomic decomposition* $((x_i'),(Se_i))$. *The following statements are equivalent:*

(a) $\mathcal{P}(^nX)$ *is reflexive and* $((x_i'),(Se_i))$ *is (strongly) shrinking*
(b) X *is reflexive and* $\mathcal{P}(^nX)$ *admits a (sharp) monomial decomposition with respect to* $((x_i'),(Se_i))$.

Acknowledgements

References

[1] Alencar, Raymundo. *On reflexivity and basis for $P(^m E)$*, Proc. Roy. Irish Acad. Sect. A 85 (1985), no. 2, 131–138.

[2] Aron, Richard M.; Hervés, Carlos; Valdivia, Manuel. Weakly continuous mappings on Banach spaces. J. Funct. Anal. 52 (1983), no. 2, 189–204.

[3] Aron, Richard M.; Prolla, João B. Polynomial approximation of differentiable functions on Banach spaces. J. Reine Angew. Math. 313 (1980), 195–216.

[4] Aron, Richard M.; Schottenloher, Martin. Compact holomorphic mappings on Banach spaces and the approximation property. J. Funct. Anal. 21 (1976) 7-30.

[5] Blasco, Fernando. Complementation in spaces of symmetric tensor products and polynomials. Studia Math. 123 (1997), no. 2, 165–173.

[6] Boyd, Christopher; Ryan, Raymond. *Geometric theory of spaces of integral polynomials and symmetric tensor products*, J. Funct. Anal. 179 (2001), no. 1, 18–42.

[7] Carando, Daniel; Dimant, Verónica. *Duality in spaces of nuclear and integral polynomials*, J. Math. Anal. Appl. 241 (2000), no. 1, 107–121.

[8] Carando, Daniel; Dimant, Verónica; Santiago Muro. *Coherent sequences of polynomial ideals on Banach spaces*, J. Math. Anal. Appl. 336 (2007), no. 2, 1324–1340.

[9] Carando, Daniel; Lassalle, Silvia. *Duality, reflexivity and atomic decompositions for Banach spaces*, preprint.

[10] Casazza, Peter; Han, Deguang; Larson, David. *Frames for Banach spaces*, The functional and harmonic analysis of wavelets and frames (San Antonio, TX, 1999), Contemp. Math., 247, 149–182, Amer. Math. Soc., Providence, RI, (1999).

[11] Christensen, Ole; Heil, Christopher. *Perturbations of Banach frames and atomic decompositions*, Math. Nachr. 185 (1997), 33–47.

[12] Defant, Andreas; Daz, Juan Carlos; Garca, Domingo; Maestre, Manuel. *Unconditional basis and Gordon-Lewis constants for spaces of polynomials.* J. Funct. Anal. 181 (2001), no. 1, 119–145.

[13] Defant, Andreas; Floret, Klaus. Tensor norms and operator ideals. North Holland, Amsterdam, 1993.

[14] Dimant, Verónica; Dineen, Seán. *Banach subspaces of spaces of holomorphic functions and related topics*, Math. Scand., 83 (1998), no. 1, 142–160.

[15] Dimant, Verónica; Zalduendo, Ignacio. Bases in spaces of multilinear forms over Banach spaces. J. Math. Anal. Appl. 200 (1996), no. 3, 548–566.

[16] Dineen, Sean. Complex analysis on infinite dimensional spaces, Monographs in Mathematics, Springer-Verlag, (1999).

[17] Dineen, Sean. *Holomorphy types on Banach space*, Studia Math. 39, (1971), 241–288.

[18] Duffin, Richard J. and Schaeffer, A. C. *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc., 72, (1952), 341–366.

[19] Floret, Klaus. *Natural norms on symmetric tensor products of normed spaces.* Note Mat. 17 (1999) 153–188.

[20] Floret, Klaus. *Minimal ideals of n-homogeneous polynomials on Banach spaces*, Results Math. 39 (2001), no. 3-4, 201–217.

[21] Gelbaum, Bernard R.; Gil de Lamadrid, Jesús. *Bases of tensor products of Banach spaces*, Pacific J. Math. 11 (1961), 1281–1286.

[22] Gröchenig, Karlheinz. *Describing functions: Atomic decompositions versus frames*, Monatsh. Math., 112 (1) (1991), 1–42.

[23] Grecu, Bogdan; Ryan, Raymond. *Schauder bases for symmetric tensor products*, Publ. Res. Inst. Math. Sci. 41 (2005), no. 2, 459–469.

[24] Lindenstrauss, Joram; Tzafriri, Lior. *Classical Banach spaces I and II*, Springer, (1977).

[25] Mujica, Jorge. Complex Analysis in Banach Spaces, Math. Studies 120, North-Holland, Amsterdam (1986).

[26] Pelczyński, Aleksander. *Any separable Banach space with the bounded approximation property is a complemented subspace of a Banach space with a basis.* Studia Math. 40 (1971), 239–243.

[27] Ryan, Raymond. *Applications of topological tensor products to infinite dimensional holomorphy*, Ph.D. Thesis, University College, Dublin (1980).

[28] Ryan, Raymond *The Dunford-Pettis property and projective tensor products*, Bull. Polish Acad. Sci. Math. 35 (1987), no. 11-12, 785–792.