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# Adapted meshes for numerical approximation of singularly perturbed problems

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# Mallas adaptadas para la aproximación numérica de problemas singularmente perturbados


**Resumen**

En esta tesis se analiza la aproximación numérica de problemas singularmente perturbados de reacción–difusión, con y sin convección. En primer lugar, consideramos la aproximación por elementos finitos bilineales utilizando mallas adaptadas a priori para problemas modelos. En este caso, obtenemos resultados de superconvergencia, es decir, la diferencia entre la solución dada por el método de elementos finitos y la interpolada de Lagrange de la solución exacta es de mayor orden que el error numérico, en una norma apropiada. Las estimaciones obtenidas son casi óptimas respecto al orden en función del número de nodos $N$, y con constantes que dependen débilmente del parámetro de perturbación $\varepsilon$. Es decir, salvo factores logarítmicos, las constantes son independientes de $\varepsilon$ y el orden es el mismo que se obtiene utilizando mallas uniformes en problemas con soluciones suaves. Como consecuencia de estos resultados, obtenemos estimaciones casi óptimas del error en norma $L^2$, mejorando de esta manera resultados conocidos anteriormente.

Para problemas más generales, para obtener mallas adaptadas adecuadamente es necesario usar estimadores a posteriori. En la última parte de esta tesis, construímos y analizamos este tipo de estimador de error para un problema de convección-reacción-difusión y presentamos un método de refinamiento anisotrópico basado en este estimador de error a posteriori.



**Palabras Claves:** Elementos finitos; mallas graduadas; convección-difusión; reacci'on-difusión; superconvergencia; supercercanía; estimadores de error a posteriori.

# Adapted meshes for numerical approximation of singularly perturbed problems


**Abstract**

In this thesis we analyze the numerical approximation of singularly perturbed problems of reaction-diffusion, with and without convection term. First, we consider the standard bilinear finite element approximation with a priori adapted meshes for model problems. In this case, we obtain superconvergence results, that is, that the difference between the finite element solution and the Lagrange interpolation of the exact solution, is of higher order than the error itself in an appropriate norm. The obtained estimates are almost optimal respect to the order in terms of the number of nodes $N$ and with constants which depend weakly on the singular perturbation parameter $\varepsilon$. That is, up to logarithmic factors the constants are independent of $\varepsilon$ and the order is the same as that for obtained for problems with smooths solutions using uniform meshes. As a consequence of these results, we obtain almost optimal error estimates in the $L^2$-norm improving in this ways previously known results.

For more general problems, to obtain appropriate adapted meshes it is necessary to use a posteriori error estimators. In the last part of this thesis, we construct and analyze this kind of error estimators for a convection-reaction–diffusion problem. Also we present an anisotropic adaptive refinement method based on a posteriori error estimator.



**Key words:** Finite elements; graded meshes; convection–diffusion; reaction–diffusion; superconvergence; supercloseness; a posteriori error estimates.

# Contents

# Introducción

En esta tesis se analiza la aproximación numérica de problemas singularmente perturbados de reacción–difusión, con y sin convección.
Se sabe que los métodos de elementos finitos estándares para problemas singularmente perturbados no producen buenos resultados cuando se utilizan mallas uniformes o cuasi-uniformes, salvo que sean suficientemente refinadas. Por ello, esta clase de mallas no son útiles en las aplicaciones prácticas, y por lo tanto, otras alternativas han sido estudiadas en numerosos trabajos para tratar este tipo de problemas. Una opción es usar algún tipo de up-wind o difusión artificial, y otra opción es usar métodos estándares con mallas adaptadas. En este trabajo, consideramos el segundo enfoque.
En general, las mallas adaptadas deben obtenerse por algún tipo de control a posteriori. Sin embargo, en algunos casos particulares donde se tiene información sobre el comportamiento de la solución, es posible diseñar mallas adaptadas a priori para aproximar las capas límites.
Esta tesis tiene dos partes. En primer lugar, analizamos la aproximación por elementos finitos bilineales de dos problemas modelo en los cuales es posible construir mallas adaptadas a priori. Nuestro objetivo es probar superconvergencia de la aproximación por elementos finitos, es decir, que la diferencia entre la interpolada de Lagrange de la solución exacta y la solución discreta es de mayor orden que el error de aproximación. En la segunda parte de la tesis, construímos y analizamos estimadores de error a posteriori para el problema de reacción-difusión.
En la primer parte de la tesis, consideramos la aproximación numérica de los problemas modelo:

$$\begin{aligned} -\varepsilon^2 \Delta u + u = f \qquad & \text{en } \Omega \\ u = 0 \qquad & \text{en } \partial\Omega \end{aligned} \tag{1}$$

y

$$\begin{aligned} -\varepsilon \Delta u + b \cdot \nabla u + cu = f \qquad & \text{en } \Omega \\ u = 0 \qquad & \text{en } \partial\Omega \end{aligned} \tag{2}$$

En ambos casos, $\Omega = (0,1)^2$ y $\varepsilon$ es un parámetro positivo y pequeño. Asumimos que $b = (b_1, b_2)$, $c$ y $f$ son suaves en $\Omega$ y que

$$b_i < -\gamma, \quad \text{con } \gamma > 0 \quad \text{para } i = 1, 2.$$

El primero, es el problema de reacción-difusión, mientras que en el segundo se introduce un término de convección. Observemos que se usa una notación diferente en el término de difusión ($\varepsilon^2$ en el primer problema y $\varepsilon$ en el segundo), como es usual en la bibliografía de este tema, por conveniencia de notación en el análisis.

El análisis de estos problemas simples ayudan a entender el comportamiento de los métodos estudiados y numerosos trabajos se han dedicado a obtener estimaciones de error para diferentes tipos de mallas adaptadas para problemas con capas límites (ver por ejemplo [RST08, Ro12] y sus referencias). Las mallas más conocidas para este tipo de problemas son las de Shishkin, cf. Linß [Ln00], Roos y Linß [RL01], Stynes y Tobiska [ST03], Roos, Stynes y Tobiska [RST08]. En particular, se prueba orden óptimo de convergencia cuando se utilizan mallas de Shishkin en combinación con elementos finitos estándares o algún método de difusión artificial con streamline, cf. Apel [Ap99] y [RST08, ST03]. Otro tipo de mallas conocidas son las de Bakhvalov, cf. Linß [Ln10], Linß y Madden [LM09] y [RST08].
Más recientemente, en el trabajo de Durán y Lombardi [DL05, DL06], se analizó el uso de mallas graduadas para los problemas (1) y (2) y se obtuvieron estimaciones de error casi óptimos. En el mismo trabajo, se probó que si se utilizan mallas graduadas adecuadamente, entonces para el problema (1) vale

$$\|u - u_h\|_\varepsilon \leq C\frac{\log N}{\sqrt{N}}.$$

donde $u_h$ es la solución bilineal a trozos de elementos finitos, $h$ es un parámetro positivo relacionado con la definición de la malla, $N$ es el número de nodos (relacionado con $h$), y $\|.\|_\varepsilon$ es la norma pesada $\varepsilon$ de $\mathrm{H}^1$ asociada con la ecuación diferencial, es decir:

$$\|v\|_\varepsilon^2 = \varepsilon^2\|\nabla v\|_{\mathrm{L}^2(\Omega)}^2 + \|v\|_{\mathrm{L}^2(\Omega)}^2$$

Acá y en el resto de la tesis $C$ denota una constante genérica independiente de $\varepsilon$, $h$ y $N$. Observar que, salvo factores logarítmicos, el orden en términos de $N$ es el mismo que se obtiene con mallas uniformes para un problema con solución suave.
Para el problema (2) que involucra también un término de convección, se probó en el trabajo [DL06] una estimación similar:

$$\|u - u_h\|_\varepsilon \leq C\frac{\log^2(1/\varepsilon)}{\sqrt{N}},$$

pero con un tipo diferente de mallas graduadas. En este caso, la norma $\varepsilon$ pesada de $\mathrm{H}^1$ es

$$\|v\|_\varepsilon^2 = \varepsilon\|\nabla v\|_{\mathrm{L}^2(\Omega)}^2 + \|v\|_{\mathrm{L}^2(\Omega)}^2.$$

La mayor diferencia entre los dos tipos de mallas, es que las consideradas en [DL05] son independientes de $\varepsilon$ mientras que las del trabajo [DL06] dependen de $\varepsilon$.
Nuestro objetivo es obtener resultados de superconvergencia cuando se utilizan mallas graduadas. La propiedad de superconvergencia para problemas elípticos con soluciones suaves fue estudiada en numerosas publicaciones desde el trabajo de Zlamal [Zl78] (ver por ejemplo el libro de Wahlbin [Wa95]). Para problemas singularmente perturbados como los que estamos considerando, resultados de superconvergencia para la aproximación numérica usando mallas de Shishkin fueron probados en [Ln00, ST03] y Zhang [Zh02, Zh03]. En esta tesis, probamos que resultados similares a los obtenidos para las mallas de Shishkin son válidos para las mallas graduadas.

Para el problema con convección, nuestros resultados son ligeramente más débiles a los obtenidos en los trabajos citados anteriormente, debido a los factores logarítmicos dependientes de $\varepsilon$ que aparecen en las estimaciones. Sin embargo, las mallas graduadas tienen algunas propiedades que las mallas de Shishkin no verifican. En efecto, cuando uno aproxima un problema singularmente perturbado con mallas diseñadas a priori, es natural esperar que una malla diseñada para algún valor del parámetro de perturbación $\varepsilon$ funcione bien para valores más grandes del mismo. En el trabajo [DL06] se mostró que esto se verifica para las mallas graduadas estudiadas en ese trabajo, pero no para las mallas de Shishkin. Vamos a presentar algunos experimentos numéricos mostrando que lo mismo vale cuando analizamos superconvergencia. Por otro lado, en el trabajo [DL05] se mostró que para el problema más simple (1) es posible construir mallas graduadas independientes del parámetro de perturbación $\varepsilon$. Este hecho puede ser importante en problemas donde el parámetro de difusión no es constante, o incluso, para estudiar sistemas de ecuaciones en los cuales diferentes ecuaciones tienen perturbaciones singulares de diferentes órdenes. Para este tipo de sistemas, en [LM09] se utilizaron mallas de Shishkin (ver también Valarathi y Miller [VM10] donde un método similar se usa para problema de valores iniciales).
En [LM09], los autores modificaron las mallas de Shishkin clásicas, que constan de dos partes uniformes, dividiendo el dominio en varios subdominios y partiendo uniformemente cada uno de estos subdominios. Puede verse que de esta manera, se obtiene algo intermedio entre las mallas de Shishkin usuales y las mallas graduadas.
Para establecer nuestros resultados, recordemos que superconvergencia (también conocida como supercercanía) significa que $\|u_h - \Pi u\|_\varepsilon$ es de mayor orden que $\|u - u_h\|_\varepsilon$, donde $\Pi u$ es la interpolada de Lagrange de la solución exacta $u$.
Para el problema (2) probamos que

$$\|u_h - \Pi u\|_\varepsilon \leq C \frac{\log^5(1/\varepsilon)}{N}$$

si usamos las mallas graduadas introducidas en [DL06]. Usando las mismas mallas, un resultado análogo puede obtenerse para el problema de reacción-difusión (1). Sin embargo, en este caso podemos usar un tipo diferente de mallas graduadas, independientes del parámetro $\varepsilon$. Para este problema, usando las mallas graduadas introducidas en [DL05] probamos que

$$\|u_h - \Pi u\|_\varepsilon \leq C \log(1/\varepsilon)^{1/2} \frac{(\log N)^2}{N}$$

Una consecuencia importante de estos resultados es que puede obtenerse orden óptimo en la norma $L^2$, combinando los resultados de superconvergencia con los errores de interpolación probados en [DL05, DL06]. Tanto la superconvergencia en la norma $\|.\|_\varepsilon$ como la convergencia en la norma $L^2$ son casi óptimas en el sentido que las constantes dependen del parámetro de perturbación solo a través de factores logarítmicos.
Como aplicación de nuestro resultado de superconvergencia, mostramos cómo obtener una aproximación de mayor orden vía un postproceso local de la solución numérica. Haremos esto para el problema (2). No es difícil ver que lo mismo puede hacerse para el problema (1). Métodos de postproceso para la ecuación de convección-difusión usando mallas de Shishkin fueron analizados en [RL01, ST03]. Si $u_h^*$ es la función postprocesada

que se obtiene de $u_h$, tenemos que existe una constante $C$ tal que,

$$\|u - u_h^*\|_\varepsilon \leq C\frac{\log(1/\varepsilon)^5}{N}.$$

En la segunda parte de esta tesis (capítulo 4), analizamos un estimador de error a posteriori para el problema (2).
Recientemente, diversos métodos para obtener estimadores de error garantizados han sido desarrollados. Algunos de estos métodos están basados en la reconstrucción de flujo, y utilizan una malla dual a la triangulación original sobre la que se calcula la soluci'on aproximada (ver por ejemplo, [CFPV09, LW04, Vo11]). Nosotros aplicamos algunas de estas ideas al problema (2). Finalmente, trabajamos en un refinamiento adaptivo anisotrópico. Usamos el estimador de error para construir una aproximación de la matriz Hessiana de la solución, que puede ser utilizada para definir un procedimiento adaptivo.
Presentamos algunos experimentos numéricos realizados en el paquete FreeFem++ [HPHO07].

## Publicaciones incluídas

La mayor parte de los resultados de esta tesis fueron publicados como artículos de investigación en diferentes revistas.
Los trabajos incluídos en esta tesis son:

- Durán R., Lombardi, A., Prieto, M "Superconvergence for finite element approximation of a convection-diffusion equation using graded meshes".
  IMA J. Numer. Anal. (2012) 32 (2): 511-533.

- Durán R., Lombardi, A., Prieto, M "Superconvergence on graded meshes for $\mathcal{Q}_1$ finite element approximation of a reaction-diffusion equation". Enviado a Journal of Computational and Applied Mathematics.

# Introduction

In this thesis we analyze the numerical approximation of singularly perturbed problems of reaction-diffusion, with and without convection term. It is well known that standard finite element methods for singularly perturbed problems produce very poor results when uniform or quasi-uniform meshes are used unless they are sufficiently refined. Consequently, this kind of meshes are not useful in practical applications, and therefore, several alternatives to deal with these problems have been considered in many papers. One possibility is to use some kind of up-wind or artificial diffusion and other possibility is to use standard methods with adapted meshes. In this work we consider the second approach . In general, adapted meshes should be obtained by some a posteriori error control. However, in some particular cases where some information on the behavior of the solution is known, it is possible to design a priori well adapted meshes to approximate well the boundary layer.
This thesis has two parts. First, we analyze the approximation by standard bilinear elements of two model problems in which it is possible to construct a priori adapted meshes. Our goal is to prove superconvergence of the finite element approximation, that is, the difference between the Lagrange interpolation of the exact solution and the discrete solution is of higher order than the approximation error. In the second part of the thesis, we construct and analyze a posteriori error estimates for a convection-reaction-diffusion problem and present some adaptive anisotropic refinement method.
In the first part of the thesis, we consider the numerical approximation of the model problems:

$$\begin{aligned} -\varepsilon^2 \Delta u + u &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{1}$$

and

$$\begin{aligned} -\varepsilon \Delta u + b \cdot \nabla u + cu &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{2}$$

In both cases, $\Omega = (0,1)^2$ and $\varepsilon$ is a small positive parameter. We assume that $b = (b_1, b_2)$, $c$ and $f$ are smooth on $\Omega$ and that

$$b_i < -\gamma, \quad \text{with } \gamma > 0 \quad \text{for } i = 1, 2.$$

The first one is a reaction-diffusion problem while in the second one we introduce a convection term. Note that we are using a different notation for the coefficient of the diffusion term ($\varepsilon^2$ in the first problem and $\varepsilon$ in the second one), as is usual in the literature of this subject for notational convenience in the analysis.

The analysis of these simple problems helps to understand the behavior of the methods and many papers have been dedicated to obtain error estimates for different types of adapted meshes for problems with boundary layers (see [RST08, Ro12] and their references). The best known meshes for this type of problems are the Shishkin ones, cf. Linß [Ln00], Roos and Linß [RL01], Stynes and Tobiska [ST03], Roos, Stynes and Tobiska [RST08]. In particular, optimal order of convergence has been proved when Shishkin meshes are used in combination with standard finite elements or some streamline artificial diffusion methods, cf. Apel [Ap99] and [RST08, ST03]. Other well known meshes are the Bakhvalov ones, cf. Linß [Ln10], Linß and Madden [LM09] and [RST08].
More recently, in Durán and Lombardi [DL05, DL06], the use of graded meshes for problems (1) and (2) was analyzed and almost optimal error estimates were obtained. It was proved in the first paper that if adequate graded meshes are used, then for problem (1) it holds

$$\|u - u_h\|_\varepsilon \leq C\frac{\log N}{\sqrt{N}}.$$

where $u_h$ is the bilinear piece-wise finite element solution, $h$ is a positive parameter related with the definition of the meshes, $N$ is the number of nodes (related with $h$), and $\|.\|_\varepsilon$ is the $\varepsilon$-weighted $\mathrm{H}^1$-norm associated with the differential equation, that is:

$$\|v\|_\varepsilon^2 = \varepsilon^2\|\nabla v\|^2_{\mathrm{L}^2(\Omega)} + \|v\|^2_{\mathrm{L}^2(\Omega)}$$

Here and in the rest of the thesis $C$ denotes a generic constant independent of $\varepsilon$, $h$ and $N$. Observe that, up to the logarithmic factor, the order in terms of $N$ is the same as that obtained with uniform meshes for a problem with a smooth solution.
When the problem involves also a convection term, it was proved in [DL06] a similar estimate for problem (2), namely,

$$\|u - u_h\|_\varepsilon \leq C\frac{\log^2(1/\varepsilon)}{\sqrt{N}},$$

but with a different kind of graded meshes. Here, the $\varepsilon$-weighted $\mathrm{H}^1$-norm is

$$\|v\|_\varepsilon^2 = \varepsilon\|\nabla v\|^2_{\mathrm{L}^2(\Omega)} + \|v\|^2_{\mathrm{L}^2(\Omega)}.$$

The main difference between the two type of meshes is that those considered in [DL05] are independent of $\varepsilon$ while those in [DL06] are $\varepsilon$-dependent.
Our main goal is to obtain superconvergence results when adapted meshes are used. Superconvergence for elliptic problems with smooth solutions has been developed in a lot of papers since the work of Zlamal [Zl78] (see for example the book Wahlbin [Wa95]). For singularly perturbed problems as those that we are considering, superconvergence results for approximations based on the use of Shishkin meshes have been proved in [Ln00, ST03] and Zhang [Zh02, Zh03]. In this thesis we prove that similar results than those obtained for Shishkin type meshes are valid for graded meshes.
For the problem with convection term our results are slightly weaker because of logarithmic factors of $\varepsilon$ involved in our estimates. However, the graded meshes have some desirable properties which the Shishkin meshes do not satisfy. Indeed, when one is approximating a singularly perturbed problem with an a priori adapted mesh, it is natural

to expect that a mesh designed for some value of the perturbation parameter $\varepsilon$ work well also for larger values of it. It was shown in [DL06] that this is the case for the meshes introduced in that paper but not for the Shishkin meshes. We will present some numerical results showing that the same is true for superconvergence. On the other hand, for the simpler problem (1) it is possible to construct graded meshes independent of the perturbation parameter $\varepsilon$ as it was shown in [DL05]. These facts can be important in problems where the diffusion parameter is not constant or, also, to treat systems of equations in which different equations have singular perturbations of different orders. Let us mention that, for this kind of systems, Shishkin type meshes have been used in [LM09] (see also Valarathi and Miller [VM10] where a similar method is used for initial value problems). In [LM09], the authors modify the classic two part Shishkin meshes dividing the domain in several parts and dividing uniformly each one of these parts. One can see that in this way one obtains something intermediate between the usual Shishkin meshes and the graded ones.

To state our results let us recall that superconvergence (also known as supercloseness) means that $\|u_h - \Pi u\|_\varepsilon$ is of higher order than $\|u - u_h\|_\varepsilon$, where $\Pi u$ is the Lagrange interpolation of the exact solution $u$.

For problem (2) we prove that

$$\|u_h - \Pi u\|_\varepsilon \leq C \frac{\log^5(1/\varepsilon)}{N}$$

if we use the graded meshes introduced in [DL06]. Using the same meshes, an analogous result can be obtained for the reaction-diffusion problem. However, in this case we can use a different kind of meshes, independent of the parameter $\varepsilon$. For this problem, using the meshes introduced in [DL05] we prove that

$$\|u_h - \Pi u\|_\varepsilon \leq C \log(1/\varepsilon)^{1/2} \frac{(\log N)^2}{N}$$

An important consequence of these results is that optimal order in $L^2-$norm can be obtained. This follows combining the superconvergence results with the interpolation error estimates obtained in [DL05, DL06]. Both superconvergence in the $\|.\|_\varepsilon$ norm as well as convergence in the $L^2-$norm are almost optimal, in the sense that the constants depend only on the logarithm of the singular perturbation parameter.

As an application of our superconvergence error estimate, we show how to obtain a higher order approximation by a simple local postprocessing of the computed solution. We will do this for problem (2). It is not difficult to see that the same can be done for problem (1). Let us mention that postprocessing procedures for convection-diffusion equations using Shishkin type meshes have been given in [RL01, ST03]. If $u_h^*$ is the postprocessed function obtained from $u_h$ we have that there exists a constant $C$ such that,

$$\|u - u_h^*\|_\varepsilon \leq C \frac{\log(1/\varepsilon)^5}{N}.$$

In the second part of this thesis, Chapter 4, we analyze a posteriori error estimates for the problem (2).
In recent years several methods to obtain so-called guaranteed error estimators have been developed. Some of these methods are based on flux reconstruction using a mesh which is dual to the original triangulation used to compute the approximate solution (see for example [CFPV09, LW04, Vo11]) . We apply some of these ideas to problem (2). Finally, we deal with adaptive anisotropic refinement. We use our error estimator to constructed an approximation of the Hessian matrix of the solution, which can be used to define an adaptive procedure.
We present some numerical experiments using the package FreeFem++ [HPHO07].

## Included publications

Most of the results of this thesis have been published as research articles in different journals.
The papers included in the thesis are:

- Durán R., Lombardi, A., Prieto, M "Superconvergence for finite element approximation of a convection-diffusion equation using graded meshes".
  IMA J. Numer. Anal. (2012) 32 (2): 511-533.

- Durán R., Lombardi, A., Prieto, M "Superconvergence on graded meshes for $\mathcal{Q}_1$ finite element approximation of a reaction-diffusion equation". Submitted to Journal of Computational and Applied Mathematics.

# Chapter 1

# Preliminaries

In this chapter we collect the notation and some preliminary results that will be used in the rest of the thesis.
For a domain $D$ we use the following notation for Sobolev space norms,

$$\|u\|_{H^m(D)} := \left\{ \sum_{\alpha \leq m} \|\mathcal{D}^\alpha u\|^2_{\mathrm{L}^2(D)} \right\}^{1/2} .$$

When $m = 0$, we write $\|u\|_{\mathrm{L}^2(D)}$.
Let $I = [a, b]$ be an interval of the real line. Then $\mathcal{P}_k(I)$ denotes the space of polynomials on $I$ of total degree less than or equal to $k$ over $I$. For a rectangle $R \subset \mathbb{R}^2$, $\mathcal{P}_k(R)$ and $\mathcal{Q}_k(R)$ denote the spaces of polynomials on $R$ of total degree less than or equal to $k$ and of degree less than or equal to $k$ in each variable, respectively. With $h_1$ and $h_2$ we denote the lengths of the edges of $R$ (see Figure 1 ). We denote with $S$ the reference element $[0, 1]^2$.

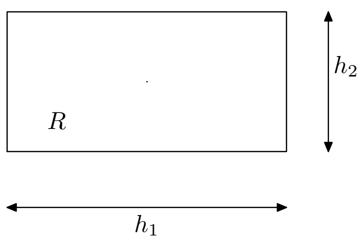



Figure 1.1: General Element

In the following sections we recall some results regarding trace and Poincaré inequalities, polynomial approximation and interpolation that will be useful in the proofs of our theorems.

## 1.1 Trace and Poincaré inequalities

Let $I = [a, b]$ be an interval of the real line. The standard trace theorem states that for $w \in \mathrm{H}^1(I)$,

$$|w(a)| \leq C \left\{ |I|^{-1/2} \|w\|_{\mathrm{L}^2(I)} + |I|^{1/2} \|w'\|_{\mathrm{L}^2(I)} \right\} \tag{1.1.1}$$

When $w \in \mathcal{P}_1(I)$, the second term on the right hand side can be bounded by the first one using an inverse inequality. Therefore, in that case we have,

$$|w(a)| \leq C|I|^{-1/2}\|w\|_{\mathrm{L}^2(I)}. \tag{1.1.2}$$

For the two dimensional case, it holds that for $w \in \mathrm{H}^1(R)$,

$$\begin{aligned}
\|w\|_{\mathrm{L}^2(l_v)} &\leq C\left\{h_1^{-1/2}\|w\|_{\mathrm{L}^2(R)} + h_1^{1/2}\left\|\frac{\partial w}{\partial x_1}\right\|_{\mathrm{L}^2(R)}\right\},\\
\|w\|_{\mathrm{L}^2(l_h)} &\leq C\left\{h_2^{-1/2}\|w\|_{\mathrm{L}^2(R)} + h_2^{1/2}\left\|\frac{\partial w}{\partial x_2}\right\|_{\mathrm{L}^2(R)}\right\},
\end{aligned} \tag{1.1.3}$$

with $l_v$ and $l_h$ any of the vertical or horizontal edges of $R$, respectively. When $w \in \mathcal{Q}_1(R)$, the second term on the right hand side can be bounded by the first one using an inverse inequality. Therefore, in that case we have,

$$\begin{aligned}
\|w\|_{\mathrm{L}^2(l_v)} &\leq C\left\{h_1^{-1/2}\|w\|_{\mathrm{L}^2(R)}\right\},\\
\|w\|_{\mathrm{L}^2(l_h)} &\leq C\left\{h_2^{-1/2}\|w\|_{\mathrm{L}^2(R)}\right\}.
\end{aligned} \tag{1.1.4}$$

We will need a weighted version of the trace theorem, which we prove in the following

**Lemma 1.1.1.** *Let $l_v$ be one of the vertical edges of the reference element $S = [0,1]$ and $l_h$ one of its horizontal edges. Given $w \in \mathrm{H}^1(S)$ we have, for $0 \leq \alpha < 1/2$,*

$$\begin{aligned}
\|w\|_{L^1(l_v)} &\leq C\left\{\|w\|_{\mathrm{L}^2(S)} + \frac{1}{(1-2\alpha)^{1/2}}\left\|x_2^\alpha\frac{\partial w}{\partial x_1}\right\|_{\mathrm{L}^2(S)}\right\},\\
\|w\|_{L^1(l_h)} &\leq C\left\{\|w\|_{\mathrm{L}^2(S)} + \frac{1}{(1-2\alpha)^{1/2}}\left\|x_1^\alpha\frac{\partial w}{\partial x_2}\right\|_{\mathrm{L}^2(S)}\right\}.
\end{aligned} \tag{1.1.5}$$

*Proof.* Assume that $l_h$ is the edge contained in $x_2 = 0$ (the other cases are, of course, analogous). We have

$$w(x_1,0) - w(x_1,x_2) = -\int_0^{x_2}\frac{\partial w}{\partial x_2}(x_1,t)dt,$$

By integrating on $[0,1]$ and using the Cauchy-Schwartz inequality (and multiplying and dividing by $x_1^\alpha$ with $\alpha < 1/2$), we obtain

$$\int_0^1 |w(x_1,0)|dx_1 \leq \int_0^1 |w(x_1,x_2)|dx_1 + \int_0^1\int_0^1 x_1^{-\alpha}x_1^\alpha\left|\frac{\partial w}{\partial x_2}(x_1,t)\right|dtdx_1.$$

Now, by integrating in the variable $x_2$ on $[0,1]$, we have

$$\int_0^1 |w(x_1,0)|dx_1 \leq \int_0^1\int_0^1 |w(x_1,x_2)|dx_1\,dx_2 + \int_0^1\int_0^1 x_1^{-\alpha}x_1^\alpha\left|\frac{\partial w}{\partial x_2}(x_1,t)\right|dtdx_1.$$

By means of the Cauchy-Schwarz inequality we obtain (1.1.5) □

We are going to use the following Poincaré inequality. Let $R$ be a rectangle with edges of length $h_1$ and $h_2$, respectively. If $v$ vanishes at one of the vertical edges of $R$, then it follows that

$$\|w\|_{\mathrm{L}^2(R)} \leq Ch_1 \left\| \frac{\partial w}{\partial x_1} \right\|_{\mathrm{L}^2(R)} . \tag{1.1.6}$$

Now we are going to prove a weighted Poincaré inequality that will be useful later in Chapter 3. The proof of this inequality, in the following lemma, uses an argument given in a more general context in [DD08]. Actually, this argument was generalized by the authors of [DD08], in an unpublished paper, to prove estimates of type (1.1.9).
We will make use of the Hardy-Littlewood maximal function defined, for $g \in L^1_{loc}(\mathbb{R}^2)$, as

$$Mg(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |g(y)|\, dy.$$

It is a classic result (see for example [St70]) that there exists a constant $C$ such that

$$\|Mg\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq C\|g\|_{\mathrm{L}^2(\mathbb{R}^2)}. \tag{1.1.7}$$

We will also need the following result which can be found, for example, in [Zi89]. There exists a constant $C$ such that, for any $\delta > 0$ and any $g \in L^1_{loc}(\mathbb{R}^2)$,

$$\int_{|x-y|\leq\delta} \frac{|g(y)|}{|x-y|}\, dy \leq C\delta Mg(x). \tag{1.1.8}$$

Given $w \in L^1(S)$ we will use the weighted average defined as $\bar{w} := \int_S w\varphi$, where $\varphi$ is a smooth function with integral equal to one and supported in a ball $B$ such that it expansion by two is contained in $S$.

**Lemma 1.1.2** (Weighted Poincaré inequality)**.** *For $w \in \mathrm{H}^1(S)$ and $\sigma \geq 0$ we have*

$$\|x_1^\sigma (w - \bar{w})\|_{\mathrm{L}^2(S)} \leq C \left\| x_1^{\sigma+1} \nabla w \right\|_{\mathrm{L}^2(S)} . \tag{1.1.9}$$

*Proof.* The argument is based on the following representation formula for $w(y) - \bar{w}$. Although this formula is known (see for example [DM01]) we reproduce its proof for the sake of completeness.
For all $y \in S$ we have

$$w(y) - \bar{w} = \int_S G(x,y) \cdot \nabla w(x)\, dx, \tag{1.1.10}$$

where

$$G(x,y) = \int_0^1 \frac{(y-x)}{t^3} \varphi\left(y + \frac{x-y}{t}\right) dt.$$

Indeed, for $y \in S$ and $z \in B$ we have,

$$w(y) - w(z) = \int_0^1 (y-z) \cdot \nabla w(y + t(z-y))\, dt,$$

therefore, multiplying by $\varphi(z)$ and integrating in $z$,

$$w(y) - \bar{w} = \int_S \int_0^1 (y - z) \cdot \nabla w(y + t(z - y))\, \varphi(z)\, dt\, dz.$$

Then, interchanging the order of integration and making the change of variable $x = y + t(z - y)$ we obtain (1.1.10).
We will use two properties of $G(x, y)$. The first one (see [BS94, DM01]) is that there exists a constant $C_1$ such that

$$|G(x, y)| \leq \frac{C_1}{|x - y|}. \tag{1.1.11}$$

Indeed, $G(x, y)$ vanishes unless $y + (x - y)/t \in B \subset S$. But, if $y + (x - y)/t \in S$ and $y \in S$, the diference between them is less than or equal the diameter of $S$, i. e.,

$$\frac{|x - y|}{t} \leq \sqrt{2}, \tag{1.1.12}$$

and then we have,

$$G(x, y) = \int_{|x-y|/\sqrt{2}}^1 \frac{(y - x)}{t^3}\, \varphi\left(y + \frac{x - y}{t}\right)\, dt.$$

Therefore, using again (1.1.12) we obtain,

$$|G(x, y)| \leq \sqrt{2}\|\varphi\|_\infty \int_{|x-y|/\sqrt{2}}^1 \frac{1}{t^2}\, dt,$$

and (1.1.11) follows immediately from this estimate.
The second important property of $G(x, y)$, which is the key point used in [DD08], is that $G(x, y)$ vanishes unless

$$|x - y| \leq C_2 d(x),$$

where $d(x)$ denotes the distance of $x$ to the boundary of $S$ and $C_2$ is a constant which depends only on the relation between the diameters of $S$ and $B$.
To prove this property recall that $x = tz + (1 - t)y$ with $z \in B$. Then, using that the ball obtained expanding $B$ by two is contained in $S$, an elementary geometric argument shows that $d(x) \geq c_3 t$, where $c_3$ is a positive constant which depends only on the relation between the diameters of $S$ and $B$. Consequently,

$$|x - y| = t|z - y| \leq \frac{\sqrt{2}}{c_3} d(x)$$

as we wanted to show.
In particular, since $d(x) \leq x_1$, we have

$$\operatorname{supp} G \subset \{(x, y) \in S \,:\, |x - y| \leq C_2 x_1\}. \tag{1.1.13}$$

Define now

$$g(x) = \begin{cases} x_1^\sigma (w(x) - \bar{w}) & \text{if} \quad x \in S, \\ 0 & \text{if} \quad x \notin S. \end{cases}$$

Then, using (1.1.10) we have

$$\|y_1^\sigma(w-\bar{w})\|^2_{L^2(S)} = \int_S y_1^\sigma |w(y)-\bar{w}||g(y)|\,dy$$

$$\leq \int_S\int_S y_1^\sigma |G(x,y)||\nabla w(x)||g(y)|\,dx\,dy.$$

Therefore, interchanging the order of integration and using (1.1.11) and (1.1.13), we obtain

$$\|y_1^\sigma(w-\bar{w})\|^2_{L^2(S)} \leq C_1 \int_S \left( \int_{|x-y|\leq C_2 x_1} \frac{y_1^\sigma |g(y)|}{|x-y|}\,dy \right) |\nabla w(x)|\,dx.$$

But, in the domain of integration of the interior integral we have,

$$y_1 \leq |y_1 - x_1| + x_1 \leq (C_2+1)x_1$$

and therefore,

$$\|y_1^\sigma(w-\bar{w})\|^2_{L^2(S)} \leq C \int_S \left( \int_{|x-y|\leq C_2 x_1} \frac{|g(y)|}{|x-y|}\,dy \right) x_1^\sigma |\nabla w(x)|\,dx.$$

with a constant $C$ depending on $C_1$, $C_2$ and $\sigma$. Now, (1.1.9) follows from this inequality using (1.1.8), the Schwarz inequality, and (1.1.7). □

## 1.2 Polynomial approximation

The following result concerning polynomial approximation is well-known: given $w \in H^3(I)$, there exists $p \in \mathcal{P}_2(I)$ such that

$$\left\|(w-p)''\right\|_{L^2(I)} \leq C|I| \left\|w'''\right\|_{L^2(I)}$$

Analogously, given $w \in H^3(R)$, there exists $p \in \mathcal{P}_2(R)$ such that

$$\left\| \frac{\partial^2(w-p)}{\partial x_1^2} \right\|_{L^2(R)} \leq C \left\{ h_1 \left\| \frac{\partial^3 w}{\partial x_1^3} \right\|_{L^2(R)} + h_2 \left\| \frac{\partial^3 w}{\partial x_1^2 \partial x_2} \right\|_{L^2(R)} \right\} \tag{1.2.1}$$

and

$$\left\| \frac{\partial^2(w-p)}{\partial x_1 \partial x_2} \right\|_{L^2(R)} \leq C \left\{ h_1 \left\| \frac{\partial^3 w}{\partial x_1^2 \partial x_2} \right\|_{L^2(R)} + h_2 \left\| \frac{\partial^3 w}{\partial x_1 \partial x_2^2} \right\|_{L^2(R)} \right\}. \tag{1.2.2}$$

Indeed, we can take $p$ as an averaged Taylor polynomial of $w$ (see for example [Ap99, BBDDFF08]).
Finally, another ingredient of our proofs is the following polynomial approximation result. The proof uses the well-known argument based on averaged Taylor polynomials and a weighted Poincaré inequality proved in the previous section.

**Lemma 1.2.1.** *Let $\alpha \geq 0$. For the reference element $S = [0,1]^2$ and $w \in H^3(S)$, there exists $p \in \mathcal{P}_2(S)$ such that*

$$\left\| x_1^\alpha \frac{\partial^2 (w-p)}{\partial x_1^2} \right\|_{L^2(S)} \leq C \left\{ \left\| x_1^{\alpha+1} \frac{\partial^3 w}{\partial x_1^3} \right\|_{L^2(S)} + \left\| x_1^{\alpha+1} \frac{\partial^3 w}{\partial x_1^2 \partial x_2} \right\| \right\}_{L^2(S)}, \tag{1.2.3}$$

$$\left\| x_1^\alpha \frac{\partial^2 (w-p)}{\partial x_1 \partial x_2} \right\|_{L^2(S)} \leq C \left\{ \left\| x_1^{\alpha+1} \frac{\partial^3 w}{\partial x_1^2 \partial x_2} \right\|_{L^2(S)} + \left\| x_1^{\alpha+1} \frac{\partial^3 w}{\partial x_1 \partial x_2^2} \right\| \right\}_{L^2(S)}. \tag{1.2.4}$$

*Consequently, for a general rectangle $R = [a,b] \times [c,d]$ where $h_1 = b-a$ and $h_2 = d-c$ we have*

$$\left\| (x_1-a)^\alpha \frac{\partial^2 (w-p)}{\partial x_1^2} \right\|_{L^2(R)} \leq C \left\{ \left\| (x_1-a)^{\alpha+1} \frac{\partial^3 w}{\partial x_1^3} \right\|_{L^2(R)} + h_1^{-1} h_2 \left\| (x_1-a)^{\alpha+1} \frac{\partial^3 w}{\partial x_1^2 \partial x_2} \right\|_{L^2(R)} \right\}, \tag{1.2.5}$$

$$\left\| (x_1-a)^\alpha \frac{\partial^2 (w-p)}{\partial x_1 \partial x_2} \right\|_{L^2(R)} \leq C \left\{ \left\| (x_1-a)^{\alpha+1} \frac{\partial^3 w}{\partial x_1^2 \partial x_2} \right\|_{L^2(R)} + h_1^{-1} h_2 \left\| (x_1-a)^{\alpha+1} \frac{\partial^3 w}{\partial x_1 \partial x_2^2} \right\|_{L^2(R)} \right\}. \tag{1.2.6}$$

*Proof.* Let $p \in \mathcal{P}_2(S)$ be the averaged Taylor polynomial of $w$ over $S$ with respect to the same weight function $\varphi$ used in the previous lemma (see for example [BS94] for the precise definition). Then, it is known that (recall that $\bar{w} := \int_S w\varphi$),

$$\frac{\partial^2 p}{\partial x_1^2} = \overline{\frac{\partial^2 w}{\partial x_1^2}} \qquad \text{and} \qquad \frac{\partial^2 p}{\partial x_1 \partial x_2} = \overline{\frac{\partial^2 w}{\partial x \partial x_2}}$$

and therefore, it follows from Lemma 1.1.2 that

$$\begin{aligned} \left\| x_1^\alpha \frac{\partial^2 (w-p)}{\partial x_1^2} \right\|_{L^2(S)} &\leq C \left\| x_1^{\alpha+1} \nabla \frac{\partial^2 w}{\partial x_1^2} \right\|_{L^2(S)} \\ &\leq C \left\{ \left\| x_1^{\alpha+1} \frac{\partial^3 w}{\partial x_1^3} \right\|_{L^2(S)} + \left\| x_1^{\alpha+1} \frac{\partial^3 w}{\partial x_1^2 \partial x_2} \right\|_{L^2(S)} \right\}. \end{aligned}$$

Changing of variables, we obtain inequality (1.2.5) for a general rectangle. Inequalities (1.2.4) and (1.2.6) follow in an analogous way.

□

## 1.3 Interpolation error

Let $R = [a,b] \times [c,d]$ be a rectangle in $\mathbb{R}^2$. For each continuous function $w$ on the rectangle $R$, we define the interpolant $\Pi_R w \in \mathcal{Q}_1(R)$ as the unique function in $\mathcal{Q}_1$ which has the same value of $w$ on the four vertices of $R$. Given partitions $a = x_0 < x_1 < \cdots < x_N = b$ and $c = y_0 < y_1 < \cdots < y_M = d$ of $[a,b]$ and $[c,d]$, respectively, we call $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$. Then, we define the Lagrange interpolant $\Pi w$ of $w$ over $R$ as the unique piecewise $\mathcal{Q}_1$ function which satisfies

$$(\Pi w)|_{R_{ij}} = \Pi_{R_{ij}} w.$$

We will use the following well-known results. For $w \in H^2(R)$, the $\mathcal{Q}_1$ linear interpolation satisfies the error estimate (see for example [Sh94])

$$\left\| \frac{\partial(w - \Pi_R w)}{\partial x_1} \right\|_{\mathrm{L}^2(R)} \leq C \left\{ h_1 \left\| \frac{\partial^2 w}{\partial x_1^2} \right\|_{\mathrm{L}^2(R)} + h_2 \left\| \frac{\partial^2 w}{\partial x_1 \partial x_2} \right\|_{\mathrm{L}^2(R)} \right\}. \tag{1.3.1}$$

In the following Lemmas we prove some weighted interpolation inequalities.

**Lemma 1.3.1.** *Let $l$ be one of the vertical edges of the reference element $S = [0,1]$ and $u \in \mathrm{H}^2(S)$. Then, for $0 \leq \alpha < 1/2$, it holds*

$$\|u - \Pi_S u\|_{\mathrm{L}^2(S)} \leq \frac{C}{(1-2\alpha)^{1/2}} \left\{ \left\| x_1^\alpha \frac{\partial u}{\partial x_1} \right\|_{\mathrm{L}^2(S)} + \left\| x_1^\alpha \frac{\partial (\Pi_S u)}{\partial x_1} \right\|_{\mathrm{L}^2(S)} + \|u - \Pi_S u\|_{\mathrm{L}^2(l)} \right\}. \tag{1.3.2}$$

*In particular, if $u$ vanishes on $l$, it holds*

$$\|u - \Pi_S u\|_{\mathrm{L}^2(S)} \leq \frac{C}{(1-2\alpha)^{1/2}} \left\{ \left\| x_1^\alpha \frac{\partial u}{\partial x_1} \right\|_{\mathrm{L}^2(S)} + \left\| x_1^\alpha \frac{\partial (\Pi_S u)}{\partial x_1} \right\|_{\mathrm{L}^2(S)} \right\}. \tag{1.3.3}$$

*For a general rectangle $R = [a,b] \times [c,d]$, if $u$ vanishes on one of its vertical edges, it holds*

$$\|u - \Pi_R u\|_{\mathrm{L}^2(R)} \leq \frac{C h_1^{1-\alpha}}{(1-2\alpha)^{1/2}} \left\{ \left\| (x_1 - a)^\alpha \frac{\partial u}{\partial x_1} \right\|_{\mathrm{L}^2(R)} + \left\| (x_1 - a)^\alpha \frac{\partial (\Pi_R u)}{\partial x_1} \right\|_{\mathrm{L}^2(R)} \right\}. \tag{1.3.4}$$

*Proof.* Suppose that $l = \{(0, x_2) \in \mathbb{R}^2 : 0 \leq x_2 \leq 1\}$ (clearly, the other case can be treated analogously). We have

$$\begin{aligned}
\|u - \Pi_S u\|^2_{\mathrm{L}^2(S)} &= \int_0^1 \int_0^1 (u - \Pi_S u)^2(x_1, x_2)\, dx_2\, dx_1 \\
&\leq \int_0^1 \int_0^1 \left[ \int_0^1 \left| \frac{\partial}{\partial x_1}(u - \Pi_S u)(t, x_2) \right| dt + (u - \Pi_S u)(0, x_2) \right]^2 dx_2\, dx_1 \\
&\leq 2 \left\{ \int_0^1 \int_0^1 \left[ \int_0^1 \left| \frac{\partial}{\partial x_1}(u - \Pi_S u)(t, x_2) \right| dt \right]^2 + (u - \Pi_S u)^2(0, x_2)\, dx_2\, dx_1 \right\} \\
&\leq 2 \int_0^1 \int_0^1 \left[ \int_0^1 \left| t^{-\alpha} t^\alpha \frac{\partial}{\partial x_1}(u - \Pi_S u)(t, x_2) \right| dt \right]^2 dx_2\, dx_1 + 2\, \|u - \Pi_S u\|^2_{\mathrm{L}^2(l)} \\
&\leq 2 \int_0^1 \int_0^1 \left( \int_0^1 t^{-2\alpha} dt \right) \left( \int_0^1 t^{2\alpha} \left| \frac{\partial}{\partial x_1}(u - \Pi_S u)(t, x_2) \right|^2 dt \right) dx_2\, dx_1 + \\
&\qquad + 2\, \|u - \Pi_S u\|^2_{\mathrm{L}^2(l)} \\
&\leq \frac{C}{1 - 2\alpha} \left\| x_1^\alpha \frac{\partial}{\partial x_1}(u - \Pi_S u) \right\|^2_{\mathrm{L}^2(S)} + 2\, \|u - \Pi_S u\|^2_{\mathrm{L}^2(l)},
\end{aligned}$$

obtaining (1.3.2). If $u$ vanishes on $l$, then $\Pi_S u$ vanishes on $l$ too, and hence $\|u - \Pi_S u\|^2_{\mathrm{L}^2(l)}$. So we have inequality (1.3.3) in this case. Inequality (1.3.4) follows by scaling arguments. □

**Lemma 1.3.2.** *For a general rectangle $R = [a,b] \times [c,d]$, $u \in \mathrm{H}^2(R)$ and $0 \leq \alpha < 1/2$ we have*

$$\left\| \frac{\partial}{\partial x_1}(u - \Pi_R u) \right\|_{\mathrm{L}^2(R)} \leq$$

$$\leq \frac{C}{(1-2\alpha)^{1/2}} \left\{ h_1^{1-\alpha} \left\| (x_1 - a)^\alpha \frac{\partial^2 u}{\partial x_1^2} \right\|_{\mathrm{L}^2(R)} + h_1^{-\alpha} h_2 \left\| (x_1 - a)^\alpha \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{\mathrm{L}^2(R)} \right\},$$

$$\left\| \frac{\partial}{\partial x_2}(u - \Pi_R u) \right\|_{\mathrm{L}^2(R)} \leq$$

$$\leq \frac{C}{(1-2\alpha)^{1/2}} \left\{ h_1 h_2^{-\alpha} \left\| (x_2 - c)^\alpha \frac{\partial^2 u}{\partial x_1^2} \right\|_{\mathrm{L}^2(R)} + h_2^{1-\alpha} \left\| (x_2 - c)^\alpha \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{\mathrm{L}^2(R)} \right\}.$$

*Proof.* First, we consider the reference element $S = [0,1]^2$. Let $p \in \mathcal{P}_1$ be the averaged Taylor polynomial of $u$ with respect to the weight function $\varphi$ introduced in Section 2. We have

$$\left\| \frac{\partial}{\partial x_1}(u - \Pi_S u) \right\|_{\mathrm{L}^2(S)} \leq \left\| \frac{\partial}{\partial x_1}(u - p) \right\|_{\mathrm{L}^2(S)} + \left\| \frac{\partial}{\partial x_1}(p - \Pi_S u) \right\|_{\mathrm{L}^2(S)}. \tag{1.3.5}$$

The first term in (1.3.5) can be bounded using Lemma 1.1.2. Indeed, we know that $\frac{\partial p}{\partial x_1} = \overline{\frac{\partial u}{\partial x_1}}$, and therefore, it follows from that Lemma that

$$\left\| \frac{\partial}{\partial x_1}(u - p) \right\|_{\mathrm{L}^2(S)} \leq C \left\| x_1^\alpha \nabla\Big(\frac{\partial u}{\partial x_1}\Big) \right\|_{\mathrm{L}^2(S)}. \tag{1.3.6}$$

To estimate the second term of (1.3.5), we define $v = p - \Pi_S u$. Since $v \in \mathcal{Q}_1$ we have

$$\left\| \frac{\partial v}{\partial x_1} \right\|_{\mathrm{L}^2(S)}^2 \leq C \left\{ |v(1,0) - v(0,0)|^2 + |v(1,1) - v(0,1)|^2 \right\}.$$

Now,

$$\begin{aligned} v(1,0) - v(0,0) &= (p - \Pi_S u)(1,0) - (p - \Pi_S u)(0,0) \\ &= (p - u)(1,0) - (p - u)(0,0) \\ &= \int_0^1 \frac{\partial (p-u)}{\partial x_1}(t,0)\, dt, \end{aligned}$$

and then,

$$|v(1,0) - v(0,0)| \leq \left\| \frac{\partial (p-u)}{\partial x_1} \right\|_{L^1(l)}$$

where $l = \{(x_1, 0) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1\}$. Now, we apply Lemmas 1.1.1 and 1.1.2 (the second one with $\alpha = 0$) to obtain

$$\begin{aligned} |v(1,0) - v(0,0)| &\leq C \left\{ \left\| \frac{\partial (u-p)}{\partial x_1} \right\|_{\mathrm{L}^2(S)} + \frac{1}{(1-2\alpha)^{1/2}} \left\| x_1^\alpha \frac{\partial}{\partial x_2} \frac{\partial (u-p)}{\partial x_1} \right\|_{\mathrm{L}^2(S)} \right\} \\ &\leq \frac{C}{(1-2\alpha)^{1/2}} \left\{ \left\| x_1 \frac{\partial^2 u}{\partial x_1^2} \right\|_{\mathrm{L}^2(S)} + \left\| x_1^\alpha \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{\mathrm{L}^2(S)} \right\} \end{aligned}$$

An analogous estimate holds for $|v(1,1) - v(0,1)|$, and so we have for the second term of (1.3.5)

$$\left\| \frac{\partial}{\partial x_1}(p - \Pi_S u) \right\|_{L^2(S)} \leq \frac{C}{(1-2\alpha)^{1/2}} \left\{ \left\| x_1 \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2(S)} + \left\| x_1^\alpha \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(S)} \right\}. \qquad (1.3.7)$$

Collecting inequalities (1.3.6) and (1.3.7) we have

$$\left\| \frac{\partial}{\partial x_1}(u - \Pi_S u) \right\|_{L^2(S)} \leq \frac{C}{(1-2\alpha)^{1/2}} \left\{ \left\| x_1^\alpha \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2(S)} + \left\| x_1^\alpha \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(S)} \right\}.$$

Then, the first inequality in the statement of the Lemma follows by scaling arguments. The second one can be proved analogously. □

# Chapter 2

# Convection-Diffusion-Reaction problem

## 2.1 Introduction

In this chapter we discuss the numerical approximation of the following singularly perturbed model problem:

$$\begin{aligned} -\varepsilon\Delta u + b\cdot\nabla u + cu = f \qquad & \text{in } \Omega \\ u = 0 \qquad & \text{on } \partial\Omega \end{aligned} \tag{2.1.1}$$

where $\varepsilon$ is a small positive parameter, and $\Omega = (0,1)^2$. We assume that $b = (b_1, b_2)$, $c$ and $f$ are smooth on $\Omega$ and that

$$b_i < -\gamma, \quad \text{with } \gamma > 0 \quad \text{for } i = 1, 2.$$

Then, the solution will have a boundary layer at the boundary $\{(x_1, x_2) \in \partial\Omega : x_1 = 0 \text{ or } x_2 = 0\}$ (see [RST08]). We will make the usual assumption in order to have coerciveness of the bilinear form associated with Problem (4.1.1), namely, there exists a constant $\mu$ such that

$$c - \frac{1}{2}\text{div}(b) \geq \mu > 0. \tag{2.1.2}$$

It is known that if $f \in \mathcal{C}^4(\Omega)$ and satisfies the following compatibility conditions:

$$f(0,0) = f(1,0) = f(0,1) = f(1,1) = 0,$$

$$\frac{\partial^{i+j} f}{\partial x_1^i \partial x_2^j}(1,1) = 0 \text{ for } 0 \leq i+j \leq 3,$$

then Problem (4.1.1) has a classical solution $u \in \mathcal{C}^3(\Omega)$, and for all $(x_1, x_2) \in \Omega$ we have

$$\left| \frac{\partial^{i+j} u}{\partial x_1^i \partial x_2^j}(x_1, x_2) \right| \leq C\left(1 + \varepsilon^{-i} e^{-\gamma x_1/\varepsilon} + \varepsilon^{-j} e^{-\gamma x_2/\varepsilon} + \varepsilon^{-(i+j)} e^{-\gamma x_1/\varepsilon} e^{-\gamma x_2/\varepsilon}\right) \tag{2.1.3}$$

for $0 \leq i+j \leq 3$. We refer to [RL01, Section 2] and [LS01] for details.

In the following Lemma we give several weighted a priori estimates for the solution of Problem (4.1.1). Some of these estimates were proved in [DL06] and all of them are consequences of (2.1.3).

**Lemma 2.1.1.** *Let $u$ be the solution of Problem* (4.1.1). *Then, there exists a constant $C$ such that*

$$\varepsilon^{\alpha}\left\|x_2^{\beta}\frac{\partial^2 u}{\partial x_2^2}\right\|_{L^2(\Omega)}, \varepsilon^{\alpha}\left\|x_1^{\beta}\frac{\partial^2 u}{\partial x_1^2}\right\|_{L^2(\Omega)} \leq C \quad \textit{for} \quad \alpha+\beta\geq 3/2,\ \alpha\geq 0, \beta>-1/2$$

$$\varepsilon^{\alpha}\left\|x_2^{\beta}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(\Omega)}, \varepsilon^{\alpha}\left\|x_1^{\beta}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(\Omega)} \leq C \quad \textit{for} \quad \alpha+\beta\geq 1,\ \alpha\geq 1/2, \beta>-1/2$$

$$\varepsilon^{\alpha}\left\|x_2^{\beta}\frac{\partial^3 u}{\partial x_2^3}\right\|_{L^2(\Omega)}, \varepsilon^{\alpha}\left\|x_1^{\beta}\frac{\partial^3 u}{\partial x_1^3}\right\|_{L^2(\Omega)} \leq C \quad \textit{for} \quad \alpha+\beta\geq 5/2,\ \alpha\geq 0, \beta>-1/2$$

$$\varepsilon^{\alpha}\left\|x_2^{\beta}\frac{\partial^3 u}{\partial x_1\partial x_2^2}\right\|_{L^2(\Omega)}, \varepsilon^{\alpha}\left\|x_1^{\beta}\frac{\partial^3 u}{\partial x_1^2\partial x_2}\right\|_{L^2(\Omega)} \leq C \quad \textit{for} \quad \alpha+\beta\geq 2,\ \alpha\geq 1/2, \beta>-1/2$$

$$\varepsilon^{\alpha}\left\|x_2^{\beta}\frac{\partial^3 u}{\partial x_1^2\partial x_2}\right\|_{L^2(\Omega)}, \varepsilon^{\alpha}\left\|x_1^{\beta}\frac{\partial^3 u}{\partial x_1\partial x_2^2}\right\|_{L^2(\Omega)} \leq C \quad \textit{for} \quad \alpha+\beta\geq 2,\ \alpha\geq 3/2, \beta>-1/2$$

$$\varepsilon^{\alpha}\left\|x_1x_2\frac{\partial^3 u}{\partial x_1^2\partial x_2}\right\|_{L^2(\Omega)}, \varepsilon^{\alpha}\left\|x_1x_2\frac{\partial^3 u}{\partial x_1\partial x_2^2}\right\|_{L^2(\Omega)} \leq C \quad \textit{for} \quad \alpha\geq 1/2$$

*Proof.* Let us prove the second inequality in the third line given in Lemma 2.1.1. The other inequalities can be obtained in a similar way.
From (2.1.3) we have,

$$\int_{\Omega} x_1^{2\beta}\left|\frac{\partial^3 u}{\partial x_1^3}\right|^2 dx\,dx_2 \leq C\int_0^1\int_0^1 x_1^{2\beta}(1+\varepsilon^{-6}e^{-2\gamma x_1/\varepsilon})dx\,dx_2$$

and so, integrating first in the variable $x_2$ and making the change of variable $z = x_1/\varepsilon$, we obtain

$$\int_{\Omega} x_1^{2\beta}\left|\frac{\partial^3 u}{\partial x_1^3}\right|^2 dx\,dx_2 \leq C\left(1+\varepsilon^{2\beta-5}\int_0^{\infty} z^{2\beta}e^{-2\gamma z}\,dz\right)$$

and therefore, the inequality is proved. □

In [DL06], the weighted *a priori* estimates of the exact solution were used to design a priori adapted meshes which approximate well the boundary layer. We want to show that a result of superconvergence is valid for these kind of meshes.
The rest of the chapter is organized as follow. In section 2.2 we set the weak formulation of problem (4.1.1), recall the graded meshes introduced in [DL06] and the finite element approximation over these meshes and we recall results from [DL06] concerning interpolation and convergence error over the graded meshes. In section 2.3 we prove the superconvergence theorem which is the main result of this chapter. In section 2.4 we construct a higher order approximation $u_h^*$ of the solution of problem (4.1.1) by a local postprocessing. Finally, in section 2.5 we show some numerical experiments.

## 2.2 Weak formulation and finite element approximation

In this section we recall the graded meshes used in [DL06] and set the finite element approximation over these meshes.
The standard weak formulation of Problem (4.1.1) is given by: find $u \in \mathrm{H}_0^1(\Omega)$ such that

$$\mathcal{B}(u,\varphi) = \int_\Omega f\,\varphi\,dx \qquad \forall\varphi \in \mathrm{H}_0^1(\Omega),$$

where the bilinear form $\mathcal{B}$ is defined as

$$\mathcal{B}(u,\varphi) = \int_\Omega (\varepsilon\nabla u\cdot\nabla\varphi + b\cdot\nabla u\,\varphi + cu\varphi)\,dx.$$

We will work with the $\varepsilon$-weighted $\mathrm{H}^1$-norm defined by

$$\|\varphi\|_\varepsilon^2 := \varepsilon\,\|\nabla\varphi\|_{\mathrm{L}^2(\Omega)}^2 + \|\varphi\|_{\mathrm{L}^2(\Omega)}^2\,.$$

It is well-known that, under the hypothesis (2.1.2), the bilinear form $\mathcal{B}$ is coercive in the $\varepsilon$-norm, moreover, there exists $\beta > 0$, independent of $\varepsilon$, such that

$$\beta\,\|\varphi\|_\varepsilon \leq \mathcal{B}(\varphi,\varphi) \quad \forall\varphi \in \mathrm{H}_0^1(\Omega). \tag{2.2.1}$$

Now we construct the graded mesh. We start defining a graded mesh over the interval $[0,1]$: given $h > 0$, consider the partition $\{\xi_i\}_{i=0}^M$ of the interval $[0,1]$ given by

$$\begin{cases} \xi_0 &= 0 \\ \xi_1 &= h\varepsilon \\ \xi_{i+1} &= (1+h)\xi_i = h\varepsilon(1+h)^{i-1}, \qquad \text{for } 1 \leq i \leq M-2 \\ \xi_M &= 1 \end{cases}$$

where $M$ is such that $\xi_{M-1} < 1$ and $\xi_{M-1} + h\xi_{M-1} \geq 1$. It is assumed that the last interval $(\xi_{M-1}, 1)$ is not too small in comparison with the previous one $(\xi_{M-2}, \xi_{M-1})$.

**Remark 2.2.1.** *In practice it is natural to take $h_i := \xi_i - \xi_{i-1}$ to be monotonically increasing. Therefore, it is convenient to modify the partition by taking $h_i = h_1$ for $i$ such that $\xi_{i-1} < \varepsilon$ and starting with the graded mesh after that. It is not difficult to check that all our arguments can be extended to this case.*

We define $R_{ij} = [\xi_{i-1}, \xi_i] \times [\xi_{j-1}, \xi_j]$, $1 \leq i,j \leq M$, and the graded meshes $\mathcal{T}_h = \{R_{ij}\}_{i,j=1}^M$ on $\Omega$. Note that the total number of nodes is $N = M^2$.
For these meshes, it was proved in [DL06, Corollary 2.3] that

$$h \leq C\frac{\log(1/\varepsilon)}{\sqrt{N}}. \tag{2.2.2}$$

Associated with $\mathcal{T}_h$ we introduce the standard piecewise bilinear finite element space

$$V_h = \left\{\varphi \in \mathcal{C}(\Omega) : \varphi\,|_{R_{ij}} \in \mathcal{Q}_1(R_{ij}), 1 \leq i,j \leq M\right\},$$

and the finite element approximation $u_h \in V_h$ given by: find $u_h \in V_h$ such that

$$\mathcal{B}(u_h, \varphi) = \int_\Omega f\varphi \, dx \quad \forall \varphi \in V_h.$$

For the proof of our estimates, we will need to decompose $\overline{\Omega}$ as $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \overline{\Omega}_3$, where

$$\begin{aligned}
\overline{\Omega}_1 &= \bigcup \{R_{ij} : \xi_{i-1} < c_1\varepsilon \log(1/\varepsilon)\} \\
\overline{\Omega}_2 &= \bigcup \{R_{ij} : \xi_{i-1} \geq c_1\varepsilon \log(1/\varepsilon), \xi_{j-1} < c_1\varepsilon \log(1/\varepsilon)\} \\
\overline{\Omega}_3 &= \bigcup \{R_{ij} : \xi_{i-1} \geq c_1\varepsilon \log(1/\varepsilon), \xi_{j-1} \geq c_1\varepsilon \log(1/\varepsilon)\}
\end{aligned}$$

and the constant $c_1$ is such that

$$\left| \frac{\partial^{i+j} u}{\partial x_1^i \partial x_2^j} \right| \leq C \text{ for } 0 \leq i+j \leq 3, \text{ if } x_1, x_2 > c_1\varepsilon \log(1/\varepsilon). \tag{2.2.3}$$

Note that, in view of (2.1.3), it is enough to take $c_1 > 3/\gamma$.
The following results regarding interpolation and convergence error over graded meshes were proven in [DL06]. See Theorems 2.1 and 2.2 in [DL06], for the respective proofs. As in the previous Chapter, for a continuous function $u$, $\Pi u \in V_h$ denotes the standard piecewice $\mathcal{Q}_1$ Lagrange interpolation of $u$.

**Theorem 2.2.2.** *Let $u$ be the solution of Problem* (4.1.1) *and let $h > 0$ be fixed. We have the following estimates for the interpolation error*

$$\|u - \Pi u\|_{L^2(\Omega)} \leq Ch^2 \quad \text{and} \quad \varepsilon^{1/2} \|\nabla(u - \Pi u)\|_{L^2(\Omega)} \leq Ch.$$

*with a constant $C$ independent of $h$ and $\varepsilon$. In particular,*

$$\|u - \Pi u\|_\varepsilon \leq Ch$$

**Theorem 2.2.3.** *Let $u$ be the solution of Problem* (4.1.1) *and $u_h \in V_h$ its finite element approximation. Then,*

$$\|u - u_h\|_\varepsilon \leq Ch \log(1/\varepsilon)$$

*with a constant $C$ independent of $h$ and $\varepsilon$.*

## 2.3 Superconvergence for graded meshes

In this section we prove that the finite element approximation defined in the previous section is superconvergent in the $\varepsilon$-weighted $H^1$-norm, i.e., the difference between the computed solution and the Lagrange interpolation of the exact solution is of higher order than the error itself. In particular, it follows from this result and previously known interpolation error estimates, that the method is almost optimal convergent in the $L^2$-norm.
For each element $R_{ij} = (\xi_{i-1}, \xi_i) \times (\xi_{j-1}, \xi_j)$ we define $h_i = \xi_i - \xi_{i-1}, h_j = \xi_j - \xi_{j-1}$ and we denote with $(\overline{x}_i, \overline{y}_j)$ the barycenter of $R_{ij}$, and with $\ell_k^{i,j}$, for $k = 1, 2, 3, 4$, its edges, as indicated in Figure 2.1.

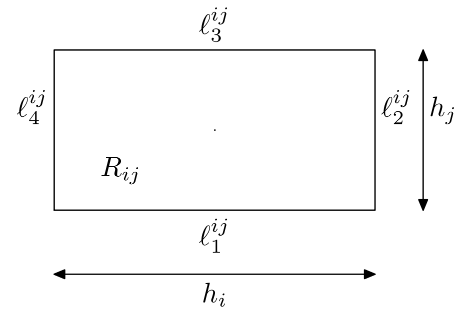



Figure 2.1: General element

In the following lemma we bound the term corresponding to the diffusion part of the equation. The proof uses an argument introduced by Zlamal in [Zl78].

**Lemma 2.3.1.** *Let $u$ be the solution of* (4.1.1). *There exists a constant $C$ such that, for any $\varphi \in V_h$,*

$$\left| \varepsilon \int_\Omega \nabla(u - \Pi u) \cdot \nabla \varphi \, dx_1 \, dx_2 \right| \leq Ch^2 \left\| \varphi \right\|_\varepsilon$$

*Proof.* Let us prove for example,

$$\left| \varepsilon \int_\Omega \frac{\partial(u - \Pi u)}{\partial x_1} \frac{\partial \varphi}{\partial x_1} \, dx_1 \, dx_2 \right| \leq Ch^2 \left\| \varphi \right\|_\varepsilon . \tag{2.3.1}$$

Clearly, analogous arguments apply to estimate the term involving derivatives with respect to $x_2$.
The key observation made in [Zl78] is that, for $p \in \mathcal{P}_2(R_{ij})$ and $\varphi \in \mathcal{Q}_1(R_{ij})$,

$$\int_{R_{ij}} \frac{\partial(p - \Pi_{R_{ij}} p)}{\partial x_1} \frac{\partial \varphi}{\partial x_1} \, dx_1 \, dx_2 = 0.$$

Indeed, this follows easily integrating by the midpoint rule and using that $\dfrac{\partial(p - \Pi_{R_{ij}} p-)}{\partial x_1} \in \mathcal{P}_1(R_{ij})$, vanishes over the segment joining the midpoints of $\ell_1^{i,j}$ and $\ell_3^{i,j}$, and $\dfrac{\partial \varphi}{\partial x_1} = ax_2 + b$ with $a, b \in \mathbb{R}$.
Therefore, for all $p \in \mathcal{P}_2(R_{ij})$ and $\varphi \in \mathcal{Q}_1(R_{ij})$, we have

$$\begin{aligned} \left| \int_{R_{ij}} \frac{\partial(u - \Pi_{R_{ij}} u)}{\partial x_1} \frac{\partial \varphi}{\partial x_1} \, dx_1 \, dx_2 \right| &= \left| \int_{R_{ij}} \frac{\partial[(u-p) - \Pi_{R_{ij}}(u-p)]}{\partial x_1} \frac{\partial \varphi}{\partial x_1} \, dx_1 \, dx_2 \right| \\ &\leq \left\| \frac{\partial[(u-p) - \Pi_{R_{ij}}(u-p)]}{\partial x_1} \right\|_{\mathrm{L}^2(R_{ij})} \left\| \frac{\partial \varphi}{\partial x_1} \right\|_{\mathrm{L}^2(R_{ij})}, \end{aligned}$$

and using (1.3.1) for $w = u - p$, we obtain

$$\left| \int_{R_{ij}} \frac{\partial(u - \Pi_{R_{ij}} u)}{\partial x_1} \frac{\partial \varphi}{\partial x_1} \, dx_1 \, dx_2 \right| \leq C \left\{ h_i \left\| \frac{\partial^2(u-p)}{\partial x_1^2} \right\|_{\mathrm{L}^2(R_{ij})} + h_j \left\| \frac{\partial^2(u-p)}{\partial x_1 \partial x_2} \right\|_{\mathrm{L}^2(R_{ij})} \right\} \left\| \frac{\partial \varphi}{\partial x_1} \right\|_{\mathrm{L}^2(R_{ij})}.$$

Choosing now $p \in \mathcal{P}_2(R_{ij})$ satisfying (1.2.1) and (1.2.2) we obtain,

$$\left| \int_{R_{ij}} \frac{\partial (u - \Pi_{R_{ij}} u)}{\partial x_1} \frac{\partial \varphi}{\partial x_1} \, dx_1 \, dx_2 \right| \\ \leq C \left\{ h_i^2 \left\| \frac{\partial^3 u}{\partial x_1^3} \right\|_{\mathrm{L}^2(R_{ij})} + h_i h_j \left\| \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{\mathrm{L}^2(R_{ij})} + h_j^2 \left\| \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|_{\mathrm{L}^2(R_{ij})} \right\} \left\| \frac{\partial \varphi}{\partial x_1} \right\|_{\mathrm{L}^2(R_{ij})} . \tag{2.3.2}$$

Let us now estimate the right hand side of (2.3.2) over each element according to its position. Since $h_1 = \varepsilon h$, we have

$$\left| \varepsilon \int_{R_{11}} \frac{\partial (u - \Pi_{R_{11}} u)}{\partial x_1} \frac{\partial \varphi}{\partial x_1} \, dx_1 \, dx_2 \right| \leq C h^2 \left\{ \varepsilon^{5/2} \left\| \frac{\partial^3 u}{\partial x_1^3} \right\|_{\mathrm{L}^2(R_{11})} + \varepsilon^{5/2} \left\| \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{\mathrm{L}^2(R_{11})} + \right. \\ \left. + \varepsilon^{5/2} \left\| \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|_{\mathrm{L}^2(R_{11})} \right\} \varepsilon^{1/2} \left\| \frac{\partial \varphi}{\partial x_1} \right\|_{\mathrm{L}^2(R_{11})} .$$

Now, for $j \geq 2$ and any $(x_1, x_2) \in R_{1j}$ we have $h_j \leq h x_2$, which together with $h_1 = \varepsilon h$ gives

$$\left| \varepsilon \int_{R_{1j}} \frac{\partial (u - u_{R_{1j}})}{\partial x_1} \frac{\partial \varphi}{\partial x_1} \, dx_1 \, dx_2 \right| \leq C h^2 \left\{ \varepsilon^{5/2} \left\| \frac{\partial^3 u}{\partial x_1^3} \right\|_{\mathrm{L}^2(R_{1j})} + \varepsilon^{3/2} \left\| x_2 \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{\mathrm{L}^2(R_{1j})} + \right. \\ \left. + \varepsilon^{1/2} \left\| x_2^2 \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|_{\mathrm{L}^2(R_{1j})} \right\} \varepsilon^{1/2} \left\| \frac{\partial \varphi}{\partial x_1} \right\|_{\mathrm{L}^2(R_{1j})} ,$$

analogously, for $i \geq 2$, we obtain

$$\left| \varepsilon \int_{R_{i1}} \frac{\partial (u - \Pi_{R_{i1}} u)}{\partial x_1} \frac{\partial \varphi}{\partial x_1} \, dx_1 \, dx_2 \right| \leq C h^2 \left\{ \varepsilon^{1/2} \left\| x_1^2 \frac{\partial^3 u}{\partial x_1^3} \right\|_{\mathrm{L}^2(R_{i1})} + \varepsilon^{3/2} \left\| x_1 \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{\mathrm{L}^2(R_{i1})} + \right. \\ \left. + \varepsilon^{5/2} \left\| \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|_{\mathrm{L}^2(R_{i1})} \right\} \varepsilon^{1/2} \left\| \frac{\partial \varphi}{\partial x_1} \right\|_{\mathrm{L}^2(R_{i1})} .$$

Finally, for $i, j \geq 2$, using that for any $(x_1, x_2) \in R_{ij}$, $h_i \leq h x_1$ and $h_j \leq h x_2$, we have

$$\left| \varepsilon \int_{R_{ij}} \frac{\partial (u - \Pi_{R_{ij}} u)}{\partial x_1} \frac{\partial \varphi}{\partial x_1} \, dx_1 \, dx_2 \right| \leq \\ \leq C h^2 \left\{ \varepsilon^{1/2} \left\| x_1^2 \frac{\partial^3 u}{\partial x_1^3} \right\|_{\mathrm{L}^2(R_{ij})} + \varepsilon^{1/2} \left\| x_1 x_2 \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{\mathrm{L}^2(R_{ij})} + \right. \\ \left. + \varepsilon^{1/2} \left\| x_2^2 \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|_{\mathrm{L}^2(R_{ij})} \right\} \varepsilon^{1/2} \left\| \frac{\partial \varphi}{\partial x_1} \right\|_{\mathrm{L}^2(R_{ij})} , \forall i, j \geq 2.$$

Therefore, summing over all indices $i, j$, and using the weighted inequalities from Lemma 2.1.1, we obtain (2.3.1). □

Our next goal is to give an estimate for the term corresponding to the convection. We want to apply an argument similar to that used for the diffusion part. With this goal we define, for $u \in H^3(R_{ij})$ and $\varphi \in V_h$,

$$K_{ij}(u,\varphi) = \int_{R_{ij}} \frac{\partial(u - \Pi_{R_{ij}} u)}{\partial x_1} \varphi \, dx_1 \, dx_2 - \frac{h_i^2}{12} \left( \int_{l_2^{i,j}} \frac{\partial^2 u}{\partial x_1^2} \varphi \, dx_2 - \int_{l_4^{i,j}} \frac{\partial^2 u}{\partial x_1^2} \varphi \, dx_2 \right). \quad (2.3.3)$$

In [Zh03] the authors give an explicit expression of $K_{ij}(u, \varphi)$ (see [Zh03], identity (4.28)) which, in particular, implies the result of our next lemma. We will give a more direct proof without making use of that expression.

**Lemma 2.3.2.** *For $p \in \mathcal{P}_2(R_{ij})$ and $\varphi \in \mathcal{Q}_1(R_{ij})$ we have,*

$$K_{ij}(p, \varphi) = 0.$$

*Proof.* Take $p \in \mathcal{P}_2(R_{ij})$ and $\varphi \in \mathcal{Q}_1(R_{ij})$, and define $e = p - \Pi_{R_{ij}} p$. Then, $\frac{\partial e}{\partial x_1}\varphi$ is a polynomial of degree two in $x_1$ and of degree one in $x_2$ which vanishes at the midpoints of $l_1^{i,j}$ and $l_3^{i,j}$. To simplify notation we will write, for any function $f$, $f_{i,j} = f(x_i, y_j)$. Using the Simpson rule in $x$ and the trapezoidal rule in $y$ we obtain,

$$\int_{R_{ij}} \frac{\partial e}{\partial x_1} \varphi \, dx_1 \, dx_2 = \frac{h_i h_j}{12} \left\{ \left( \frac{\partial e}{\partial x_1} \varphi \right)_{i,j-1} + \left( \frac{\partial e}{\partial x_1} \varphi \right)_{i,j} + \left( \frac{\partial e}{\partial x_1} \varphi \right)_{i-1,j-1} + \left( \frac{\partial e}{\partial x_1} \varphi \right)_{i-1,j} \right\}.$$

But, using again that $\frac{\partial e}{\partial x_1}$ vanishes at the midpoints of $l_1^{i,j}$ and $l_3^{i,j}$, we have

$$\left( \frac{\partial e}{\partial x_1} \right)_{i,j-1} = \left( \frac{\partial e}{\partial x_1} \right)_{i,j} = -\left( \frac{\partial e}{\partial x_1} \right)_{i-1,j-1} = -\left( \frac{\partial e}{\partial x_1} \right)_{i-1,j} = \frac{h_i}{2} \frac{\partial^2 e}{\partial x_1^2}$$

and therefore,

$$\begin{aligned}
\int_{R_{ij}} \frac{\partial e}{\partial x_1} \varphi \, dx_1 \, dx_2 &= \frac{h_i^2}{12} \frac{\partial^2 e}{\partial x_1^2} \left\{ \frac{h_j}{2} (\varphi_{i,j-1} + \varphi_{i,j}) - \frac{h_j}{2} (\varphi_{i-1,j-1} + \varphi_{i-1,j}) \right\} \\
&= \frac{h_i^2}{12} \left( \int_{l_2^{i,j}} \frac{\partial^2 p}{\partial x_1^2} \varphi \, dx_2 - \int_{l_4^{i,j}} \frac{\partial^2 p}{\partial x_1^2} \varphi \, dx_2 \right)
\end{aligned}$$

as we wanted to show. □

**Lemma 2.3.3.** *Let $u$ be the solution of* (4.1.1). *There exists a constant $C$ such that, for any $\varphi \in V_h$,*

$$\left| \int_{\Omega} b \cdot \nabla(u - \Pi u) \varphi \, dx_1 \, dx_2 \right| \le C h^2 \log^3(1/\varepsilon) \, \|\varphi\|_{\varepsilon} \, .$$

*Proof.* Let $Pb$ be the piecewise constant approximation of $b$ defined by $Pb|_{R_{ij}} := b^{i,j}$, where $b^{i,j}$ denotes the value of $b$ at the barycenter of $R_{ij}$. We have

$$\int_\Omega b\cdot\nabla(u-\Pi u)\varphi\,dx_1\,dx_2 = \int_\Omega (b-Pb)\cdot\nabla(u-\Pi u)\varphi\,dx_1\,dx_2 + \int_\Omega Pb\cdot\nabla(u-\Pi u)\varphi\,dx_1\,dx_2 \tag{2.3.4}$$

Let us estimate each term on the right-hand side. Since the derivatives of $b$ are bounded we have $\|b-Pb\|_\infty \le Ch|b|_{1,\infty}$ and so, for each element $R_{ij}$,

$$\left|\int_{R_{ij}} (b-Pb)\cdot\nabla(u-\Pi u)\varphi\,dx_1\,dx_2\right| \le Ch\,\|\nabla(u-\Pi u)\|_{\mathrm{L}^2(R_{ij})}\,\|\varphi\|_{\mathrm{L}^2(R_{ij})}\,.$$

Therefore, summing over all indices $i,j$ such that $R_{ij}\subset\Omega_1$, it follows that

$$\left|\int_{\Omega_1} (b-Pb)\cdot\nabla(u-\Pi u)\varphi\,dx_1\,dx_2\right| \le Ch\,\|\nabla(u-\Pi u)\|_{\mathrm{L}^2(\Omega_1)}\,\|\varphi\|_{\mathrm{L}^2(\Omega_1)}\,. \tag{2.3.5}$$

On the other hand, since $\varphi$ vanishes at the boundary of $\Omega$, it follows from the Poincaré inequality (1.1.6) that

$$\|\varphi\|_{\mathrm{L}^2(\Omega_1)} \le Ch_1\left\|\frac{\partial\varphi}{\partial x_1}\right\|_{\mathrm{L}^2(\Omega_1)} \le C\varepsilon\log(1/\varepsilon)\left\|\frac{\partial\varphi}{\partial x_1}\right\|_{\mathrm{L}^2(\Omega_1)} \tag{2.3.6}$$

and therefore, using the estimate

$$\varepsilon^{1/2}\,\|\nabla(u-\Pi u)\|_{\mathrm{L}^2(\Omega_1)} \le Ch,$$

which was proved in [DL06, Theorem 2.1], we obtain from (2.3.5),

$$\left|\int_{\Omega_1} (b-Pb)\cdot\nabla(u-\Pi u)\varphi\,dx_1\,dx_2\right| \le Ch^2\log(1/\varepsilon)\,\|\varphi\|_\varepsilon\,.$$

Clearly, the same argument can be applied to obtain an analogous estimate over $\Omega_2$. Finally, for $R_{ij}\in\Omega_3$, we use a standard interpolation error estimate and (2.2.3) to obtain

$$\left|\int_{\Omega_3} (b-Pb)\cdot\nabla(u-\Pi u)\varphi\,dx_1\,dx_2\right| \le Ch^2\,\|\varphi\|_{\mathrm{L}^2(\Omega_3)}\,.$$

Summing up we conclude that

$$\left|\int_{\Omega} (b-Pb)\cdot\nabla(u-\Pi u)\varphi\,dx_1\,dx_2\right| \le Ch^2\,\|\varphi\|_\varepsilon\,. \tag{2.3.7}$$

Now we estimate the second term on the right hand side of (2.3.4). We have

$$\int_\Omega Pb\cdot\nabla(u-\Pi u)\varphi\,dx_1\,dx_2 =$$
$$= \sum_{i,j=1}^M \int_{R_{ij}} b_1^{i,j}\,\frac{\partial(u-\Pi u)}{\partial x_1}\,\varphi\,dx_1\,dx_2 + \sum_{i,j=1}^M \int_{R_{ij}} b_2^{i,j}\,\frac{\partial(u-\Pi u)}{\partial x_2}\,\varphi\,dx_1\,dx_2$$

and we will estimate the first term on the right hand side (clearly the second one can be handled in an analogous way). From the definition of $K_{ij}$ (2.3.3) it follows that

$$\begin{aligned}\sum_{i,j=1}^{M}\int_{R_{ij}} b_1^{i,j}\frac{\partial(u-\Pi u)}{\partial x_1}\varphi\,dx_1\,dx_2 \\ = \sum_{i,j=1}^{M} b_1^{i,j}K_{ij}(u,\varphi)+\sum_{i,j=1}^{M}\frac{b_1^{i,j}h_i^2}{12}\left(\int_{l_2^{i,j}}\frac{\partial^2 u}{\partial x_1^2}\varphi\,dx_2-\int_{l_4^{i,j}}\frac{\partial^2 u}{\partial x_1^2}\varphi\,dx_2\right).\end{aligned} \tag{2.3.8}$$

Then, it is enough to bound the right hand side of (2.3.8). For the first term we write,

$$K_{ij}(u,\varphi)=K_{1,ij}(u,\varphi)-K_{2,ij}(u,\varphi)$$

with

$$K_{1,ij}(u,\varphi)=\int_{R_{ij}}\frac{\partial(u-\Pi u)}{\partial x_1}\varphi\,dx_1\,dx_2$$

and

$$K_{2,ij}(u,\varphi)=\frac{h_i^2}{12}\left(\int_{l_2^{i,j}}\frac{\partial^2 u}{\partial x_1^2}\varphi\,dx_2-\int_{l_4^{i,j}}\frac{\partial^2 u}{\partial x_1^2}\varphi\,dx_2\right).$$

From Lemma 2.3.2 we know that, for any $p\in\mathcal{P}_2(R_{ij})$,

$$K_{ij}(u,\varphi)=K_{ij}(u-p,\varphi)=K_{1,ij}(u-p,\varphi)-K_{2,ij}(u-p,\varphi).$$

Now, taking $p\in\mathcal{P}_2(R_{ij})$ satisfying (1.2.1) and (1.2.2) and using the interpolation error estimate (1.3.1) for $w=u-p$ we obtain,

$$\begin{aligned}&|K_{1,ij}(u-p,\varphi)|\le\\ &\le C\left\{h_i^2\left\|\frac{\partial^3 u}{\partial x_1^3}\right\|_{\mathrm{L}^2(R_{ij})}+h_ih_j\left\|\frac{\partial^3 u}{\partial x_1^2\partial x_2}\right\|_{\mathrm{L}^2(R_{ij})}+h_j^2\left\|\frac{\partial^3 u}{\partial x_1\partial x_2^2}\right\|_{\mathrm{L}^2(R_{ij})}\right\}\|\varphi\|_{\mathrm{L}^2(R_{ij})}.\end{aligned}$$

On the other hand, using now (1.1.3) for $w=\frac{\partial^2 u}{\partial x_1^2}$, (1.1.4), and again (1.2.1), we get

$$|K_{2,ij}(u-p,\varphi)|\le C\left\{h_i^2\left\|\frac{\partial^3 u}{\partial x_1^3}\right\|_{\mathrm{L}^2(R_{ij})}+h_ih_j\left\|\frac{\partial^3 u}{\partial x_1^2\partial x_2}\right\|_{\mathrm{L}^2(R_{ij})}\right\}\|\varphi\|_{\mathrm{L}^2(R_{ij})}.$$

In conclusion we have,

$$|K_{ij}(u,\varphi)|\le C\left\{h_i^2\left\|\frac{\partial^3 u}{\partial x_1^3}\right\|_{\mathrm{L}^2(R_{ij})}+h_ih_j\left\|\frac{\partial^3 u}{\partial x_1^2\partial x_2}\right\|_{\mathrm{L}^2(R_{ij})}+h_j^2\left\|\frac{\partial^3 u}{\partial x_1\partial x_2^2}\right\|_{\mathrm{L}^2(R_{ij})}\right\}\|\varphi\|_{\mathrm{L}^2(R_{ij})}. \tag{2.3.9}$$

Now we are ready to estimate the first term on the right hand side of (2.3.8). Setting

$$I_s:=\sum_{i,j:R_{ij}\subset\Omega_s} b_1^{i,j}K_{ij}(u,\varphi),\qquad s=1,2,3,$$

we have

$$\sum_{i,j=1}^{M} b_1^{i,j} K_{ij}(u,\varphi) = I_1 + I_2 + I_3.$$

From (2.3.9), using the Cauchy-Schwarz inequality we obtain,

$$|I_1| \le C \left\{ \sum_{i,j:R_{ij}\subset\Omega_1} \left( h_i^4 \left\| \frac{\partial^3 u}{\partial x_1^3} \right\|^2_{L^2(R_{ij})} + h_i^2 h_j^2 \left\| \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|^2_{L^2(R_{ij})} + \right.\right.$$
$$\left.\left. + h_j^4 \left\| \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|^2_{L^2(R_{ij})} \right) \right\}^{\frac{1}{2}} \|\varphi\|_{L^2(\Omega_1)}$$

and therefore, using now the Poincaré inequality (1.1.6),

$$|I_1| \le C \log(1/\varepsilon) \left\{ \sum_{i,j:R_{ij}\subset\Omega_1} \left( \varepsilon h_i^4 \left\| \frac{\partial^3 u}{\partial x_1^3} \right\|^2_{L^2(R_{ij})} + \varepsilon h_i^2 h_j^2 \left\| \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|^2_{L^2(R_{ij})} + \right.\right.$$
$$\left.\left. + \varepsilon h_j^4 \left\| \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|^2_{L^2(R_{ij})} \right) \right\}^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \left\| \frac{\partial \varphi}{\partial x} \right\|_{L^2(\Omega_1)}.$$

Now, for $R_{i1} \subset \Omega_1$ we have $h_i \le c_1 \varepsilon h \log(1/\varepsilon)$ and $h_1 = \varepsilon h$, then

$$\varepsilon h_i^4 \left\| \frac{\partial^3 u}{\partial x_1^3} \right\|^2_{L^2(R_{i1})} + \varepsilon h_i^2 h_j^2 \left\| \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|^2_{L^2(R_{i1})} + \varepsilon h_j^4 \left\| \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|^2_{L^2(R_{i1})}$$
$$\le h^4 \log^4(1/\varepsilon) \left( \varepsilon^5 \left\| \frac{\partial^3 u}{\partial x_1^3} \right\|^2_{L^2(R_{i1})} + \varepsilon^5 \left\| \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|^2_{L^2(R_{i1})} + \varepsilon^5 \left\| \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|^2_{L^2(R_{i1})} \right).$$

If $R_{ij} \subset \Omega_1$, with $j \ge 2$, we use that $h_i \le c_1 \varepsilon h \log(1/\varepsilon)$ and that $h_j \le h x_2$ for all $(x_1, x_2) \in R_{ij}$, obtaining

$$\varepsilon h_i^4 \left\| \frac{\partial^3 u}{\partial x_1^3} \right\|^2_{L^2(R_{ij})} + \varepsilon h_i^2 h_j^2 \left\| \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|^2_{L^2(R_{ij})} + \varepsilon h_j^4 \left\| \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|^2_{L^2(R_{ij})}$$
$$\le h^4 \log^4(1/\varepsilon) \left( \varepsilon^5 \left\| \frac{\partial^3 u}{\partial x_1^3} \right\|^2_{L^2(R_{ij})} + \varepsilon^3 \left\| x_2 \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|^2_{L^2(R_{ij})} + \varepsilon \left\| x_2^2 \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|^2_{L^2(R_{ij})} \right).$$

Therefore,

$$|I_1| \le C h^2 \log^2(1/\varepsilon) \left( \varepsilon^5 \left\| \frac{\partial^3 u}{\partial x_1^3} \right\|^2_{0,\Omega_1} + \varepsilon^5 \left\| \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|^2_{0,\Omega_1} \right.$$
$$\left. + \varepsilon^5 \left\| \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|^2_{0,\Omega_1} + \varepsilon^3 \left\| y \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|^2_{0,\Omega_1} + \varepsilon \left\| y^2 \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|^2_{0,\Omega_1} \right) \varepsilon^{\frac{1}{2}} \left\| \frac{\partial \varphi}{\partial x} \right\|_{L^2(\Omega_1)},$$

and consequently, using the weighted estimates from Lemma 2.1.1, we obtain

$$|I_1| \leq Ch^2 \log^3(1/\varepsilon) \, \|\varphi\|_\varepsilon .$$

An analogous argument can be used to estimate $I_2$. Finally, for $I_3$, using (2.2.3) and (2.3.9), we arrive at

$$|I_3| \leq Ch^2 \, \|\varphi\|_{0,\Omega_3} .$$

Therefore, we conclude that

$$\left| \sum_{i,j=1}^{M} b_1^{i,j} K_{ij}(u,\varphi) \right| \leq Ch^2 \log^3(1/\varepsilon) \, \|\varphi\|_\varepsilon .$$

To finish the proof it remains only to estimate the second term in (2.3.8). Observe that, for $1 \leq i \leq M-1$, we have $l_2^{i,j} = l_4^{i+1,j}$. Therefore,

$$\begin{aligned} b_1^{i,j} h_i^2 &\left( \int_{l_2^{i,j}} \frac{\partial^2 u}{\partial x_1^2} \varphi \, dx_2 - \int_{l_4^{i,j}} \frac{\partial^2 u}{\partial x_1^2} \varphi \, dx_2 \right) + b_1^{i+1,j} h_{i+1}^2 \left( \int_{l_2^{i+1,j}} \frac{\partial^2 u}{\partial x_1^2} \varphi \, dx_2 - \int_{l_4^{i+1,j}} \frac{\partial^2 u}{\partial x_1^2} \varphi \, dx_2 \right) \\ &= -b_1^{i,j} h_i^2 \int_{l_4^{i,j}} \frac{\partial^2 u}{\partial x_1^2} \varphi \, dx_2 + \\ &\qquad + \left( b_1^{i,j} h_i^2 - b_1^{i+1,j} h_{i+1}^2 \right) \int_{l_2^{i,j}} \frac{\partial^2 u}{\partial x_1^2} \varphi \, dx_2 + b_1^{i+1,j} h_{i+1}^2 \int_{l_2^{i+1,j}} \frac{\partial^2 u}{\partial x_1^2} \varphi \, dx_2 \end{aligned}$$

From the definition of the mesh it follows that $h_{i+1}^2 - h_i^2 = h^2 h_i (x_{i-1} + x_i)$, $h_{i+1} \leq 2h_i$, and $x_{i-1} + x_i \leq 3x_{i-1}$, then

$$\left| b_1^{i,j} h_i^2 - b_1^{i+1,j} h_{i+1}^2 \right| \leq Ch^2 h_i x_{i-1}.$$

Thus, since $\varphi$ vanishes on edges contained in $\partial\Omega$, we have

$$\begin{aligned} \sum_{i=1}^{M} \sum_{j=1}^{M} b_1^{i,j} \, h_i^2 &\left( \int_{l_2^{i,j}} \frac{\partial^2 u}{\partial x_1^2} \varphi \, dx_2 - \int_{l_4^{i,j}} \frac{\partial^2 u}{\partial x_1^2} \varphi \, dx_2 \right) \leq \\ &\leq C \sum_{i=1}^{M-1} \sum_{j=1}^{M} Ch^2 h_i x_{i-1} \int_{l_2^{i,j}} \frac{\partial^2 u}{\partial x_1^2} \varphi dy \leq C \sum_{i=1}^{M-1} \sum_{j=1}^{M} Ch^2 h_i x_{i-1} \left\| \frac{\partial^2 u}{\partial x_1^2} \right\|_{0,l_2^{i,j}} \|\varphi\|_{0,l_2^{i,j}} \end{aligned}$$

and using the inequalities (1.1.3) and (1.1.4) we obtain,

$$\begin{aligned} \sum_{i=1}^{M} \sum_{j=1}^{M} b_1^{i,j} \, h_i^2 &\left( \int_{l_2^{i,j}} \frac{\partial^2 u}{\partial x_1^2} \varphi \, dx_2 - \int_{l_4^{i,j}} \frac{\partial^2 u}{\partial x_1^2} \varphi \, dx_2 \right) \\ &\leq Ch^2 \sum_{i=1}^{M-1} \sum_{j=1}^{M} h_i x_{i-1} \left( h_i^{-1/2} \left\| \frac{\partial^2 u}{\partial x_1^2} \right\|_{\mathrm{L}^2(R_{ij})} + h_i^{1/2} \left\| \frac{\partial^3 u}{\partial x_1^3} \right\|_{\mathrm{L}^2(R_{ij})} \right) h_i^{-1/2} \|\varphi\|_{\mathrm{L}^2(R_{ij})} \\ &\leq Ch^2 \sum_{i=1}^{M-1} \sum_{j=1}^{M} x_{i-1} \left( \left\| \frac{\partial^2 u}{\partial x_1^2} \right\|_{\mathrm{L}^2(R_{ij})} + h_i \left\| \frac{\partial^3 u}{\partial x_1^3} \right\|_{\mathrm{L}^2(R_{ij})} \right) \|\varphi\|_{\mathrm{L}^2(R_{ij})} . \end{aligned}$$

Now, for $R_{ij} \subset \Omega_1$, using the Poincaré inequality (1.1.6) and the weighted inequalities from Lemma 2.1.1, we have

$$\begin{aligned}
&\sum_{R_{ij}\subset\Omega_1} x_{i-1}\left(\left\|\frac{\partial^2 u}{\partial x_1^2}\right\|_{\mathrm{L}^2(R_{ij})} + h_i\left\|\frac{\partial^3 u}{\partial x_1^3}\right\|_{\mathrm{L}^2(R_{ij})}\right)\|\varphi\|_{\mathrm{L}^2(R_{ij})} \\
&\quad\le \sum_{R_{ij}\subset\Omega_1} c_1\varepsilon\log(1/\varepsilon)\left(\left\|\frac{\partial^2 u}{\partial x_1^2}\right\|_{\mathrm{L}^2(R_{ij})} + c_1\varepsilon\log(1/\varepsilon)\left\|\frac{\partial^3 u}{\partial x_1^3}\right\|_{\mathrm{L}^2(R_{ij})}\right)\varepsilon\log(1/\varepsilon)\left\|\frac{\partial\varphi}{\partial x_1}\right\|_{\mathrm{L}^2(\Omega_1)} \\
&\quad\le C\log^3(1/\varepsilon)\sum_{R_{ij}\subset\Omega_1}\left(\varepsilon^{3/2}\left\|\frac{\partial^2 u}{\partial x_1^2}\right\|_{\mathrm{L}^2(R_{ij})} + \varepsilon^{5/2}\left\|\frac{\partial^3 u}{\partial x_1^3}\right\|_{\mathrm{L}^2(R_{ij})}\right)\varepsilon^{1/2}\left\|\frac{\partial\varphi}{\partial x_1}\right\|_{\mathrm{L}^2(\Omega_1)} \\
&\quad\le C\log^3(1/\varepsilon)\,\|\varphi\|_{\varepsilon}.
\end{aligned}$$

A similar argument can be used to estimate the sum over $R_{ij} \subset \Omega_2$. Finally, for $\Omega_3$, using (2.2.3) we have

$$\sum_{R_{ij}\subset\Omega_3} x_{i-1}\left(\left\|\frac{\partial^2 u}{\partial x_1^2}\right\|_{\mathrm{L}^2(R_{ij})} + h_i\left\|\frac{\partial^3 u}{\partial x_1^3}\right\|_{\mathrm{L}^2(R_{ij})}\right)\|\varphi\|_{\mathrm{L}^2(R_{ij})} \le C\,\|\varphi\|_{\varepsilon}.$$

Collecting all the estimates we obtain

$$\sum_{i=1}^{M}\sum_{j=1}^{M} b_1^{i,j} h_i^2\left(\int_{l_2^{i,j}}\frac{\partial^2 u}{\partial x_1^2}\varphi\,dx_2 - \int_{l_4^{i,j}}\frac{\partial^2 u}{\partial x_1^2}\varphi\,dx_2\right) \le Ch^2\log^3(1/\varepsilon)\,\|\varphi\|_{\varepsilon}$$

concluding the proof of the lemma. □

In the next Lemma we give an estimate for the reaction term of the equation. This estimate follows immediately from results in [DL06].

**Lemma 2.3.4.** *Let $u$ be the solution of* (4.1.1). *There exists a constant $C$ such that, for any $\varphi \in V_h$,*

$$\left|\int_\Omega c(u-\Pi u)\varphi\,dx_1\,dx_2\right| \le Ch^2\,\|\varphi\|_{\varepsilon}.$$

*Proof.* From [DL06, Theorem 2.1] we know that $\|u-\Pi u\|_{\mathrm{L}^2(\Omega)} \le Ch^2$, hence

$$\begin{aligned}
\left|\int_\Omega c(u-\Pi u)\varphi\,dx_1\,dx_2\right| &\le C\,\|u-\Pi u\|_{\mathrm{L}^2(\Omega)}\,\|\varphi\|_{\mathrm{L}^2(\Omega)} \\
&\le Ch^2\,\|\varphi\|_{\mathrm{L}^2(\Omega)} \le Ch^2\,\|\varphi\|_{\varepsilon}.
\end{aligned}$$

□

We can now state and prove our main result.

**Theorem 2.3.5.** *Let $u$ be the solution of* (4.1.1), *$u_h \in V_h$ its finite element approximation and $\Pi u \in V_h$ its Lagrange interpolation. There exists a constant C such that,*

$$\|u_h - \Pi u\|_{\varepsilon} \le Ch^2\log^3(1/\varepsilon).$$

*Proof.* From (2.2.1) and the error equation $\mathcal{B}(u - u_h, u_h - \Pi u) = 0$, we have

$$\beta \left\| u_h - \Pi u \right\|_{\varepsilon}^2 \leq \mathcal{B}(u_h - \Pi u, u_h - \Pi u) = \mathcal{B}(u - \Pi u, u_h - \Pi u).$$

But, from Lemmas 2.3.1, 2.3.3 and 2.3.4, we have

$$\mathcal{B}(u - \Pi u, u_h - \Pi u) \leq C \log^3(1/\varepsilon) h^2 \left\| u_h - \Pi u \right\|_{\varepsilon},$$

and therefore the theorem is proved. □

An immediate consequence of the previous Theorem combined with the interpolation results from Lemma 2.2.2 is the optimal order convergence in the $\mathrm{L}^2$-norm.

**Corollary 2.3.6.** *Let $u$ be the solution of* (4.1.1) *and $u_h \in V_h$ its finite element approximation. There exists a constant C such that,*

$$\| u - u_h \|_{\mathrm{L}^2(\Omega)} \leq C \log^3(1/\varepsilon) h^2.$$

*Proof.* The result follows immediately from the interpolation error estimate $\| u - \Pi u \|_{\mathrm{L}^2(\Omega)} \leq C h^2$ proved in [DL06, Theorem 2.1] and the estimate given in Theorem 2.3.5. □

We end this section giving error estimates in terms of the number of nodes.

**Corollary 2.3.7.** *Let $u$ be the solution of* (4.1.1) *and $u_h \in V_h$ its finite element approximation. If $N$ is the number of nodes in $\mathcal{T}_h$, then there exists a constant $C$ such that,*

$$\| u_h - \Pi u \|_{\varepsilon} \leq C \frac{\log^5(1/\varepsilon)}{N}$$

*and*

$$\| u - u_h \|_{\mathrm{L}^2(\Omega)} \leq C \frac{\log^5(1/\varepsilon)}{\sqrt{N}}.$$

*Proof.* The results follow from Theorem 2.3.5, Corollary 2.3.6 and the estimate

$$h \leq C \frac{\log(1/\varepsilon)}{\sqrt{N}}.$$

which was proved in [DL06, Corollary 2.3]. □

## 2.4 A higher order approximation by postprocessing

As it is known (see for example [DMR92]), superconvergence results of the type of Theorem 2.3.5 can be used to improve the numerical approximation by some local postprocessing. In this section we construct a higher order approximation $u_h^*$ of the solution of (4.1.1), obtained from the computed finite element approximation $u_h \in V_h$ by a simple local procedure.
For simplicity we consider now the meshes defined as indicated in Remark 2.2.1. In this way the lengths in each direction of neighbor elements are comparable and this simplifies the analysis.

We define the postprocessed solution $u_h^*$ as in [DMR92]. We repeat the construction given in that paper for the sake of completeness. Assume that the mesh $\mathcal{T}_h$ is a refinement of a coarser mesh formed by elements $S_{ij}$ which are as in Figure 2.2 (note that we assume that $\mathcal{T}_h$ contains an even number of elements). We define $u_h^* = I_2 u_h$, where $I_2 u_h$ is the biquadratic interpolation of $u_h$ on $S_{ij}$, over the nine nodes indicated in Figure 2, i.e., the vertices of the elements of the original mesh.

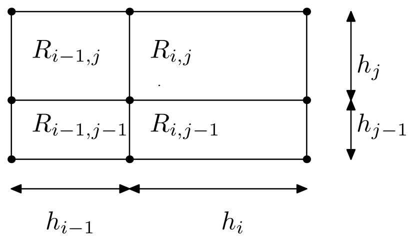



Figure 2.2: Reference element for the $Q_2$-interpolation and region $S_{ij}$

We want to show that $u_h^*$ is a higher order approximation in the $\varepsilon$-norm. We will need the following estimates for the biquadratic interpolation.

**Lemma 2.4.1.** *Let $u$ be the solution of* (4.1.1) *and $I_2 u$ the piecewise biquadratic interpolation of $u$ on the mesh made with the elements $S_{ij}$ and using the nodes corresponding to the vertices of the original mesh (as indicated in Figure 2.2). There exists a constant $C$ such that*

$$\|u - I_2 u\|_\varepsilon \leq C h^2 \tag{2.4.1}$$

*Proof.* The inequality is an easy consequence of the weighted a priori estimates given in Lemma 2.1.1 and the following error estimates for the interpolation operator $I_2$. Let $H_i$ and $H_j$ be the lengths of the element $S_{ij}$ along the directions of the $x_1$ and $x_2$ axis respectively. Then, for $w \in H^3(S_{ij})$, we have

$$\|w - I_2 w\|_{\mathrm{L}^2(S_{ij})} \leq C \left\{ H_i^3 \left\| \frac{\partial^3 w}{\partial x_1^3} \right\|_{\mathrm{L}^2(S_{ij})} + H_j^3 \left\| \frac{\partial^3 w}{\partial x_2^3} \right\|_{\mathrm{L}^2(S_{ij})} \right\}, \tag{2.4.2}$$

$$\left\| \frac{\partial (w - I_2 w)}{\partial x_1} \right\|_{\mathrm{L}^2(S_{ij})} \leq C \left\{ H_i^2 \left\| \frac{\partial^3 w}{\partial x_1^3} \right\|_{\mathrm{L}^2(S_{ij})} + H_j^2 \left\| \frac{\partial^3 w}{\partial x_1 \partial x_2^2} \right\|_{\mathrm{L}^2(S_{ij})} \right\}, \tag{2.4.3}$$

and

$$\left\| \frac{\partial (w - I_2 w)}{\partial x_2} \right\|_{\mathrm{L}^2(S_{ij})} \leq C \left\{ H_i^2 \left\| \frac{\partial^3 w}{\partial x_1^2 \partial x_2} \right\|_{\mathrm{L}^2(S_{ij})} + H_j^2 \left\| \frac{\partial^3 w}{\partial x_2^3} \right\|_{\mathrm{L}^2(S_{ij})} \right\} \tag{2.4.4}$$

where the constant $C$ is independent of the element $S_{ij}$ and $w$. For the standard biquadratic interpolation, these inequalities are proved in [Ap99, Theorem 2.7]. The only difference between our case and that considered in [Ap99], is that we are not using the usual interpolation nodes. Indeed, our interpolation nodes on $S_{ij}$ are $(\xi_k, \xi_l)$, with $k = i-2, i-1, i$ and $l = j-2, j-1, j$, i.e., we have moved a little bit the nodes usually located at edge mid-points and barycenter of the elements. However, it follows from the

definition of the meshes $\mathcal{T}_h$ that the ratios $(\xi_i - \xi_{i-1})/(\xi_{i-1} - \xi_{i-2})$ are uniformly bounded from below and above. Using this fact, it is not difficult to see that the arguments used in [Ap99] can be adapted to our case.
Let us now prove (2.4.1). Since $H_1 \le Ch\varepsilon$, using (2.4.3) for the element $S_{11}$ we obtain

$$\left\|\frac{\partial(u-I_2u)}{\partial x_1}\right\|^2_{\mathrm{L}^2(S_{11})} \le Ch^4\left\{\varepsilon^4\left\|\frac{\partial^3 u}{\partial x_1^3}\right\|^2_{\mathrm{L}^2(S_{11})} + \varepsilon^4\left\|\frac{\partial^3 u}{\partial x_1 \partial x_2^2}\right\|^2_{\mathrm{L}^2(S_{11})}\right\}.$$

Now, for $S_{i1}, i > 1$, using now $H_1 \le Ch\varepsilon$ and $H_i \le Chx_1$ for all $(x_1, x_2) \in S_{i1}$, we have

$$\left\|\frac{\partial(u-I_2u)}{\partial x_1}\right\|^2_{\mathrm{L}^2(S_{i1})} \le Ch^4\left\{\left\|x_1^2\frac{\partial^3 u}{\partial x_1^3}\right\|^2_{\mathrm{L}^2(S_{i1})} + \varepsilon^4\left\|\frac{\partial^3 u}{\partial x_1 \partial x_2^2}\right\|^2_{\mathrm{L}^2(S_{i1})}\right\}.$$

Analogously, for $S_{1j}, j > 1$, we use that $H_1 \le Ch\varepsilon$ and $K_j \le Chx_2$ for all $(x_1, x_2) \in S_{1j}$ to obtain

$$\left\|\frac{\partial(u-I_2u)}{\partial x_1}\right\|^2_{\mathrm{L}^2(S_{1j})} \le Ch^4\left\{\varepsilon^4\left\|\frac{\partial^3 u}{\partial x_1^3}\right\|^2_{\mathrm{L}^2(S_{1j})} + \left\|x_2^2\frac{\partial^3 u}{\partial x_1 \partial x_2^2}\right\|^2_{\mathrm{L}^2(S_{1j})}\right\}.$$

Finally, for $i, j > 1$, we have $H_i \le Chx_1$ and $H_j \le Chx_2$ for all $(x_1, x_2) \in S_{ij}$, and so,

$$\left\|\frac{\partial(u-I_2u)}{\partial x_1}\right\|^2_{\mathrm{L}^2(S_{ij})} \le Ch^4\left\{\left\|x_1^2\frac{\partial^3 u}{\partial x_1^3}\right\|^2_{\mathrm{L}^2(S_{ij})} + \left\|x_2^2\frac{\partial^3 u}{\partial x_1 \partial x_2^2}\right\|^2_{\mathrm{L}^2(S_{ij})}\right\}.$$

Therefore, multiplying by $\varepsilon$, summing up, and using the a priori estimates from Lemma 2.1.1, we obtain

$$\varepsilon^{\frac{1}{2}}\left\|\frac{\partial(u-I_2u)}{\partial x_1}\right\|_{\mathrm{L}^2(\Omega)} \le Ch^2.$$

In a similar way, using now (2.4.4) and (2.4.2), we can prove

$$\varepsilon^{\frac{1}{2}}\left\|\frac{\partial(u-I_2u)}{\partial x_2}\right\|_{\mathrm{L}^2(\Omega)} \le Ch^2 \qquad \text{and} \qquad \|u - I_2u\|_{\mathrm{L}^2(\Omega)} \le Ch^2.$$

Therefore, (2.4.1) holds. □

**Lemma 2.4.2.** *There exists a constant $C$ such that, for any $\varphi \in V_h$,*

$$\|I_2\varphi\|_\varepsilon \le C\|\varphi\|_\varepsilon. \tag{2.4.5}$$

*Proof.* It is easy to see that the Lagrange basis functions corresponding to $I_2$ are bounded independently of $h$. Indeed, this follows from the fact that the ratios $h_i/h_{i-1}$ are uniformly bounded. Consequently we have

$$\|I_2\varphi\|_{L^\infty(S_{ij})} \le C\|\varphi\|_{L^\infty(S_{ij})}.$$

Therefore, using the Schwarz inequality and an inverse inequality

$$\|\varphi\|_{L^\infty(S_{ij})} \le \frac{C}{|S_{ij}|^{\frac{1}{2}}}\|\varphi\|_{\mathrm{L}^2(S_{ij})},$$

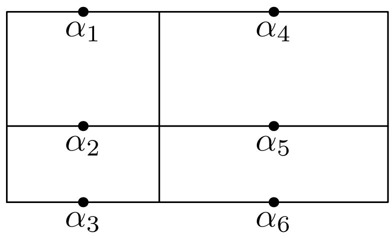



Figure 2.3: Interpolation points for $Q_{12}$ on $S_{ij}$

we obtain

$$\|I_2\varphi\|_{L^2(S_{ij})} \leq C\|\varphi\|_{L^2(S_{ij})}.$$

On the other hand, for $\varphi \in V_h$, $\frac{\partial(I_2\varphi)}{\partial x_1}$ can be seen as a Lagrange type interpolation of $\frac{\partial\varphi}{\partial x_1}$. Indeed, $\frac{\partial(I_2\varphi)}{\partial x_1}$ is the unique polynomial in $Q_{12}$ (the space of polynomials of degree one in the $x_1$ variable and two in the $x_2$ variable) such that

$$\frac{\partial\varphi}{\partial x_1}(\alpha_j) = \frac{\partial(I_2\varphi)}{\partial x_1}(\alpha_j), \qquad j = 1, \cdots, 6$$

where the points $\alpha_j$ are those indicated in Figure 2.3.
Then, using again that $h_i/h_{i-1}$ are uniformly bounded and an inverse inequality, we obtain

$$\left\|\frac{\partial(I_2\varphi)}{\partial x_1}\right\|_{L^2(S_{ij})} \leq C\left\|\frac{\partial\varphi}{\partial x_1}\right\|_{L^2(S_{ij})}.$$

Analogously, we can prove

$$\left\|\frac{\partial(I_2\varphi)}{\partial x_2}\right\|_{L^2(S_{ij})} \leq C\left\|\frac{\partial\varphi}{\partial x_2}\right\|_{L^2(S_{ij})}$$

and so, the Lemma is proved. □

We can now give the main result of this section.

**Theorem 2.4.3.** *Let $u$ be the solution of* (4.1.1), *$u_h \in V_h$ its finite element approximation and $u_h^* = I_2u_h$. There exists a constant $C$ such that,*

$$\|u - u_h^*\|_\varepsilon \leq C\log^3(1/\varepsilon)h^2.$$

*If $N$ is the number of nodes in $\mathcal{T}_h$, it holds*

$$\|u - u_h^*\|_\varepsilon \leq C\frac{\log^5(1/\varepsilon)}{N}.$$

*Proof.* Since $I_2\Pi u = I_2u$, we have

$$\|u - u_h^*\|_\varepsilon \leq \|u - I_2u\|_\varepsilon + \|I_2(\Pi u - u_h)\|_\varepsilon$$

and therefore, using (2.4.1), (2.4.5) and Theorem 3.3.8, we conclude the proof of the first inequality. For the second inequality we use (2.2.2). □

## 2.5 Numerical Experiments

We end this Chapter with some numerical results. We consider problem (4.1.1) with

$$b = (1-2\varepsilon)(-1,-1)\ , \quad c = 2(1-\varepsilon),$$

and the right hand side given by

$$f(x_1,x_2) = -\left[x_1 - \left(\frac{1-e^{-\frac{x_1}{\varepsilon}}}{1-e^{-\frac{1}{\varepsilon}}}\right) + x_2 - \left(\frac{1-e^{-\frac{x_2}{\varepsilon}}}{1-e^{-\frac{1}{\varepsilon}}}\right)\right] e^{x_1+x_2}.$$

In this case the exact solution is

$$u(x_1,x_2) = \left[\left(x_1 - \frac{1-e^{-\frac{x_1}{\varepsilon}}}{1-e^{-\frac{1}{\varepsilon}}}\right)\left(x_2 - \frac{1-e^{-\frac{x_2}{\varepsilon}}}{1-e^{-\frac{1}{\varepsilon}}}\right)\right] e^{x_1+x_2},$$

In the next two tables we present the results for $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-6}$ respectively. Recall that $N$ denotes the number of nodes.

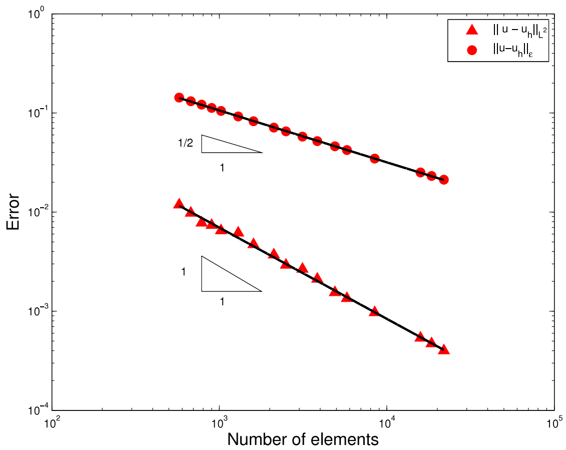
| N | h | $\|u-u_h\|_{L^2}$ | $\|u-u_h\|_{\varepsilon}$ | $\|\Pi u-u_h\|_{\varepsilon}$ | $\|u-u_h^*\|_{\varepsilon}$ |
|---|---|---|---|---|---|
| 625 | 0.375 | 0.011851 | 0.142881 | 0.020495 | 0.040664 |
| 729 | 0.345 | 0.009791 | 0.131127 | 0.016457 | 0.033664 |
| 841 | 0.320 | 0.007778 | 0.121252 | 0.012387 | 0.027535 |
| 961 | 0.295 | 0.007373 | 0.112401 | 0.011523 | 0.024435 |
| 1089 | 0.275 | 0.006511 | 0.104864 | 0.009772 | 0.021133 |
| 1369 | 0.240 | 0.006161 | 0.092214 | 0.009391 | 0.017947 |
| 1681 | 0.215 | 0.004717 | 0.082466 | 0.006485 | 0.013545 |
| 2209 | 0.185 | 0.003710 | 0.071073 | 0.004709 | 0.010019 |
| 2601 | 0.170 | 0.002924 | 0.065163 | 0.003068 | 0.007807 |
| 3249 | 0.150 | 0.002663 | 0.057742 | 0.003036 | 0.006598 |
| 3969 | 0.135 | 0.002113 | 0.051935 | 0.002022 | 0.005047 |
| 5041 | 0.120 | 0.001546 | 0.046078 | 0.002146 | 0.004320 |
| 5929 | 0.110 | 0.001349 | 0.042269 | 0.001863 | 0.003686 |
| 8649 | 0.090 | 0.000971 | 0.034644 | 0.001374 | 0.002580 |
| 16129 | 0.065 | 0.000538 | 0.025071 | 0.000995 | 0.001581 |
| 18769 | 0.060 | 0.000472 | 0.023150 | 0.000752 | 0.001267 |
| 22201 | 0.055 | 0.000400 | 0.021228 | 0.000661 | 0.001089 |

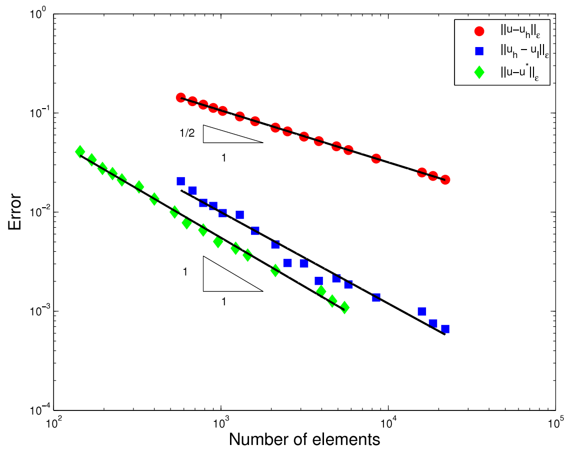
Table 2.1: $\varepsilon = 10^{-3}$

| N | h | $\|u-u_h\|_{L^2}$ | $\|u-u_h\|_\varepsilon$ | $\|\Pi u-u_h\|_\varepsilon$ | $\|u-u_h^*\|_\varepsilon$ |
|---|---|---|---|---|---|
| 2025 | 0.390 | 0.013595 | 0.148975 | 0.023704 | 0.045450 |
| 2401 | 0.355 | 0.010646 | 0.135035 | 0.018168 | 0.036312 |
| 2601 | 0.340 | 0.009248 | 0.128921 | 0.015445 | 0.032214 |
| 2809 | 0.325 | 0.008836 | 0.123478 | 0.014559 | 0.029923 |
| 3249 | 0.300 | 0.006961 | 0.113833 | 0.010700 | 0.024212 |
| 4489 | 0.250 | 0.005232 | 0.095092 | 0.007160 | 0.016828 |
| 5625 | 0.220 | 0.004565 | 0.084096 | 0.006050 | 0.013562 |
| 8649 | 0.175 | 0.002745 | 0.066857 | 0.003300 | 0.008193 |
| 10201 | 0.160 | 0.002444 | 0.061215 | 0.002998 | 0.007048 |
| 11449 | 0.150 | 0.002220 | 0.057456 | 0.002254 | 0.005992 |
| 16129 | 0.125 | 0.001728 | 0.047970 | 0.002055 | 0.004523 |
| 18769 | 0.115 | 0.001511 | 0.044188 | 0.001305 | 0.003569 |
| 20449 | 0.110 | 0.001407 | 0.042271 | 0.001372 | 0.003378 |
| 22201 | 0.105 | 0.001376 | 0.040404 | 0.001088 | 0.002998 |
| 29929 | 0.090 | 0.001031 | 0.034640 | 0.001019 | 0.002348 |
| 33489 | 0.085 | 0.000979 | 0.032738 | 0.001211 | 0.002330 |
| 37249 | 0.080 | 0.000903 | 0.030837 | 0.000694 | 0.001795 |

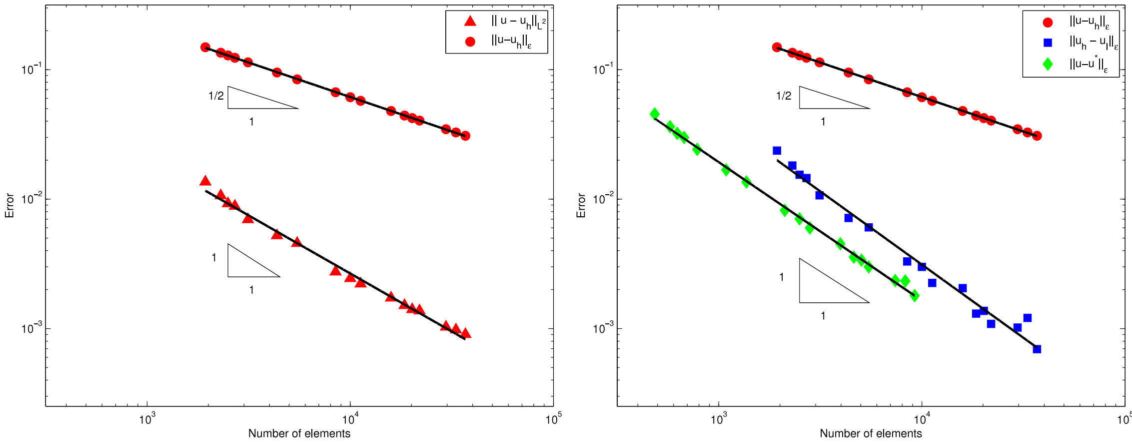
Table 2.2: $\varepsilon = 10^{-6}$

With these numerical results we have computed the order of the different errors in terms of $N$. The computed orders, for the case $\varepsilon = 10^{-3}$, are shown in the following pictures. The picture in the left shows the order of the errors $\|u-u_h\|_{L^2}$ and $\|u-u_h\|_\varepsilon$, and that in the right the different errors in the $\varepsilon$-norm.
Observe that, the order of $\|u-u_h\|_{L^2(\Omega)}$ is 0.93741, which essentially agrees with that predicted by the theory which is 1. Similarly, the orders shown in the second picture agree with the theoretical ones.






Finally, we show analogous pictures for the case $\varepsilon = 10^{-6}$. Again, the estimated orders agree with those given by the theory.



||u − u_h||_L²
||u−u_h||_ε
1/2
1
1
1
Error
Number of elements
10^−1
10^−2
10^−3
10^3
10^4
10^5
||u−u_h||_ε
||u_h − u_I||_ε
||u−u*||_ε
1/2
1
1
1
Error
Number of elements

# Chapter 3

# Reaction-Diffusion problem

## 3.1 Introduction

In this chapter we discuss the numerical approximation of the singularly perturbed reaction-diffusion equation. As for the convection-diffusion case, information on the behavior of the solution can be used to design a priori well adapted meshes.
In [DL05] it was considered the use of graded meshes for singularly perturbed model problems and an almost optimal order error estimate with a constant independent of the singular perturbation parameter $\varepsilon$ was obtained. Also, optimal order for the interpolation error was obtained when an averaged interpolation is considered over the graded meshes.
The main difference between the meshes defined in [DL05] and those considered in [DL06] and discussed in the previous chapter, is that the first one are independent of $\varepsilon$ while those in [DL06] are $\varepsilon$-dependent.
The aim of this chapter is to prove that a similar superconvergence estimates are valid for the $\varepsilon$-independent graded meshes considered in [DL05]. Beside that, we prove that the interpolation error is of optimal order when the Lagrange interpolation is used. These two results allow us to obtain an almost optimal order for the $L^2-$norm of the error.
In order to present the ideas, we first consider the one dimensional case avoiding technical details. In this case, the main issue is to prove that the error between the exact solution $u$ and the Lagrange interpolation $\Pi u$ is of almost optimal order in the $L^2-$norm. Once we have this result, the superconvergence result is straightforward.
The two dimensional case needs more technical results. We need to prove some new weighted a priori estimates for the solution of the reaction-diffusion problem. Also, we will make use of some weighted Poincaré type inequalities, which were proved in section 1.1.
The rest of the chapter is organized as follow. In section 3.2 we analyze the one dimensional case for the reaction-diffusion problem. In section 3.2.1 we present the weak formulation of the reaction-diffusion problem, recall the graded meshes introduced in [DL05] and the finite element approximation over these meshes. In section 3.2.2 we estimate the interpolation error between the exact solution of the problem and the Lagrange interpolation. In section 3.2.3 we prove the superconvergence theorem.
In section 3.3 we analyze the two dimensional case for the reaction-diffusion problem. Here we prove some new weighted estimates for the exact solution of the problem. In

section 3.3.1 we set the weak formulation of the problem, recall the graded meshes introduced in [DL05] and the finite element approximation over these meshes. In section 3.3.2 we estimate the interpolation error between the exact solution of the problem and the Lagrange interpolation. We need to introduce some anisotropic norms and prove weighted estimations for the Lagrange interpolation. In section 3.3.3 we prove the superconvergence theorem and state the results in term of the number of nodes of the graded meshes. Finally, in section 3.4 we present some numerical experiments.

## 3.2 Case 1D

We consider the model problem,

$$\begin{aligned} -\varepsilon^2 u'' + cu = f \quad & \text{in } (0,1) \\ u(0) = u(1) = 0 & \end{aligned} \tag{3.2.1}$$

where $\varepsilon$ is a small positive parameter and $c$ is a positive constant.
For the solution of problem (3.2.1), we have the following a priori estimate.

$$\left|u^{(k)}\right| \leq C\{1 + \varepsilon^{-k} e^{-\frac{x}{\varepsilon}} + \varepsilon^{-k} e^{-\frac{1-x}{\varepsilon}}\} \tag{3.2.2}$$

(see [RST08]). As a consequence of the previous Lemma, we have the following estimates for the exact solution of problem (3.2.1):

**Lemma 3.2.1.** *Let $I = [0,1]$ and $u$ the solution of Problem* (3.2.1)*. Then, there exists a constant $C$ such that*

$$\|u'\|_{L^1(I)} \leq C \tag{3.2.3}$$

$$\varepsilon^a \|x^b u'\|_{L^2(I)} \leq C \qquad si\ a+b \geq \frac{1}{2} \tag{3.2.4}$$

$$\varepsilon^a \|x^b u''\|_{L^2(I)} \leq C \qquad si\ a+b \geq \frac{3}{2} \tag{3.2.5}$$

### 3.2.1 Weak formulation and finite element approximation

The standard weak formulation of Problem (3.2.1) is given by: find $u \in \mathrm{H}^1_0(0,1)$ such that

$$\mathcal{B}(u,\varphi) = \int_0^1 f\,\varphi\,dx, \quad \forall \varphi \in \mathrm{H}^1_0(0,1), \tag{3.2.6}$$

where the bilinear form $\mathcal{B}$ is defined as

$$\mathcal{B}(u,\varphi) = \int_0^1 (\varepsilon^2 u'\varphi' + cu\varphi)\,dx.$$

We denote $\|\cdot\|_\varepsilon$ the norm associated with the bilinear form $\mathcal{B}$, i.e.,

$$\|\varphi\|_\varepsilon^2 := \mathcal{B}(\varphi,\varphi) = \varepsilon^2 \left\|\varphi'\right\|^2_{\mathrm{L}^2(0,1)} + \|\varphi\|^2_{\mathrm{L}^2(0,1)}$$

Let us recall the definition of the graded meshes introduced in [DL06]. Let $h, \gamma \in (0,1)$ be fixed. We consider the partition $\{x_i\}_{i=0}^{M}$ of the interval $[0, 1/2]$ given by

$$\begin{cases} x_0 &= 0 \\ x_1 &= h^{\frac{1}{1-\gamma}} \\ x_{i+1} &= x_i + h x_i^{\gamma} \quad \text{for } 1 \le i \le M-2 \\ x_M &= 1/2 \end{cases} \tag{3.2.7}$$

where $M$ is such that $x_{M-1} < 1/2$ and $x_{M-1} + h x_{M-1}^{\gamma} \ge 1/2$, and $x_M = 1/2$. If $1/2 - x_{M-1} < x_{M-1} - x_{M-2}$ we modify the definition of $x_{M-1}$ by taking $x_{M-1} = (1/2 + x_{M-2})/2$.
By symmetry, we define a partition on the interval $[1/2, 1]$, thus obtaining the partition $\{x_i\}_{i=0}^{2M}$ of the interval $[0,1]$. We call $\mathcal{T}_{h,\gamma}$ such partition. Calling $I_i = [x_{i-1}, x_i]$, we have the finite-dimensional subspace of $\mathrm{H}_0^1(0,1)$:

$$V_h = \left\{ \varphi \in \mathrm{H}_0^1(0,1) : \varphi\,|_{I_i} \in \mathcal{P}_1(I_i), 1 \le i \le 2M \right\}$$

The finite element approximation of problem (3.2.1) is: find $u_h \in V_h$ such that

$$\mathcal{B}(u_h, \varphi) = \int_0^1 f\varphi, \quad \forall \varphi \in V_h.$$

### 3.2.2 Interpolation and convergence error

In this section we prove that for the Lagrange interpolation, the $\mathrm{L}^2-$norm of the error is of optimal order and recall the convergence of the method for the graded meshes.
Let $I = [a, b]$ be an interval of the real line. For each continuous function $w$ over $I$ we define the linear interpolant $\Pi_I w \in \mathcal{P}_1(I)$ which satisfies $\Pi_I w(a) = w(a)$ and $\Pi_I w(b) = w(b)$. Given a partition of $I$: $a = x_0 < x_1 < \cdots < x_N = b$, we call $I_i = [x_{i-1}, x_i]$. Then, we define the Lagrange interpolant $\Pi w$ of $w$ over $I$ as the unique piecewise linear function which satisfies

$$(\Pi w)|_{I_i} = \Pi_{I_i} w$$

**Lemma 3.2.2.** *Let $I = [a, b]$ and $0 \le \alpha < 1/2$. If $w \in \mathrm{H}^1(I)$ and $\Pi_I w$ is its linear interpolation over $I$, it holds*

$$\|w - \Pi_I w\|_{\mathrm{L}^2(I)} \le \frac{C}{(1-2\alpha)^{1/2}} |I|^{1-\alpha} \left\| (x-a)^{\alpha} (w - \Pi_I w)' \right\|_{\mathrm{L}^2(I)}$$

*Proof.* First, we consider the reference element $\hat{I} = [0,1]$ and $w \in \mathrm{H}^1(\hat{I})$. As $(w - \Pi_{\hat{I}} w)(0) = 0$ we have

$$\begin{aligned} \left\| w - \Pi_{\hat{I}} w \right\|_{\mathrm{L}^2(\hat{I})}^2 &= \int_0^1 (w - \Pi_{\hat{I}} w)^2(x)dx \\ &= \int_0^1 \left[ \int_0^x (w - \Pi_{\hat{I}} w)'(t)dt \right]^2 dx \\ &\le \int_0^1 \left[ \int_0^1 \left| (w - \Pi_{\hat{I}} w)'(t) \right| dt \right]^2 dx \\ &\le \int_0^1 \left[ \int_0^1 t^{-\alpha} t^{\alpha} \left| (w - \Pi_{\hat{I}} w)'(t) \right| dt \right]^2 dx \end{aligned}$$

Now, we apply Cauchy-Schwartz inequality to obtain

$$\left\|w - \Pi_{\hat{I}} w\right\|^2_{\mathrm{L}^2(\hat{I})} \leq \int_0^1 \left[\frac{1}{(1-2\alpha)^{1/2}} \int_0^1 t^{2\alpha} \left|(w - \Pi_{\hat{I}} w)'(t)\right|^2 dt\right]^2 dx$$
$$\leq \frac{1}{1-2\alpha} \left\|x^\alpha (w - \Pi_{\hat{I}} w)'\right\|^2_{\mathrm{L}^2(\hat{I})}$$

The result follows by scaling arguments. □

**Lemma 3.2.3.** *Let $u$ be the solution of Problem* (3.2.1) *and $\Pi u$ its Lagrange interpolation over $I = [0,1]$. Then, for $\alpha \geq 0$ it holds*

$$\left\|x^\alpha (\Pi u)'\right\|_{\mathrm{L}^2(I_1)} \leq \frac{C}{1+2\alpha}$$

*Proof.* We have

$$\Pi u(x)|_{I_1} = [u(x_1) - u(0)]x + u(0)$$

and

$$(\Pi u)'(x)|_{I_1} = u(x_1) - u(0) = \int_0^{x_1} u'(t)dt$$

Using that $\|u'\|_{L^1(I)} \leq C$ from Lemma 3.2.1 we obtain

$$|(\Pi u)'(x)|_{I_1}| \leq \int_0^1 |u'(t)|dt = \left\|u'\right\|_{L^1(I)} \leq C$$

and then

$$\left\|x^\alpha (\Pi u)'\right\|^2_{\mathrm{L}^2(I_1)} = \int_0^{x_1} x^{2\alpha} |(\Pi u)'(x)|^2 dx$$
$$\leq C \int_0^1 x^{2\alpha} dx \leq \frac{C}{1+2\alpha}$$

□

**Theorem 3.2.4.** *Let $u$ be the solution of problem* (3.2.1), *$\Pi u$ be its Lagrange interpolation and suppose that $3/4 \leq \gamma < 1$. There exists a constant $C$, independent of $\varepsilon$ and $h$, such that*

$$\|u - \Pi u\|_{\mathrm{L}^2(I)} \leq C \log(1/\varepsilon)^{1/2} h^2$$

*Proof.* We first note that it is enough to prove the estimate in $[0, 1/2]$. The idea of the proof is to estimate in a different way the $\mathrm{L}^2-$norm of the interpolation error on $I_1$ and over the rest of the interval $\hat{I}$. We use Lemma 3.2.2 for $I_1$ and a standard interpolation inequality for the rest of the interval. So for any $\alpha < 1/2$ we have:

$$\|u - \Pi u\|^2_{\mathrm{L}^2(I)} = \sum_{i=1}^N \|u - \Pi_{I_i} u\|^2_{\mathrm{L}^2(I_i)}$$
$$\leq \|u - \Pi_{I_1} u\|^2_{\mathrm{L}^2(I_1)} + \sum_{i=2}^N \|u - \Pi_{I_i} u\|^2_{\mathrm{L}^2(I_i)}$$
$$\leq \frac{C}{1-2\alpha} |I_1|^{2-2\alpha} \left\|x^\alpha (u - \Pi_{I_1} u)'\right\|_{\mathrm{L}^2(I_1)} + C \sum_{i=2}^N |I_i|^4 \|u''\|^2_{\mathrm{L}^2(I_i)}. \quad (3.2.8)$$

From the mesh definition we observe that $h_i \leq h x_{i-1}^{\gamma}$ for $i \geq 2$. Then

$$\begin{aligned} |I_i|^4 \left\|u''\right\|^2_{\mathrm{L}^2(I_i)} &\leq h^4 x_{i-1}^{4\gamma} \int_{x_{i-1}}^{x_i} \left|u''\right|^2 dx \\ &\leq h^4 \int_{x_{i-1}}^{x_i} x^{4\gamma} \left|u''\right|^2 dx \\ &\leq h^4 \left\|x^{2\gamma} u''\right\|_{\mathrm{L}^2(I_i)} \end{aligned}$$

We multiply (3.2.8) by $\varepsilon^{-2\beta}\varepsilon^{2\beta}$ (where $\beta$ is a constant to be determined later) and we obtain:

$$\|u - \Pi u\|^2_{\mathrm{L}^2(I)} \leq \varepsilon^{-2\beta} \left\{ \frac{C}{1-2\alpha} h^{2\frac{1-\alpha}{1-\gamma}} \varepsilon^{2\beta} \|x^{\alpha}(u - \Pi_{I_1} u)'\|^2_{\mathrm{L}^2(I_1)} + C h^4 \sum_{i=2}^{N} \varepsilon^{2\beta} \|x^{2\gamma} u''\|^2_{\mathrm{L}^2(I_i)} \right\} \tag{3.2.9}$$

Now take $\beta > 0$ and $0 < \alpha < 1/2$ as

$$\beta = 1/2 - \alpha = \frac{1}{\log \frac{1}{\varepsilon}}.$$

So

$$\varepsilon^{-\beta} = e, \qquad \text{and} \qquad \frac{1}{1-2\alpha} = \frac{1}{2}\log(1/\varepsilon),$$

and it follows from $3/4 \leq \gamma < 1$ that

$$\frac{1-\alpha}{1-\gamma} \geq 2.$$

We also have

$$2\frac{1-\alpha}{1-\gamma} \geq 4, \qquad 2\beta + \alpha \geq 1/2, \qquad \text{and} \qquad 2\beta + 2\gamma \geq 3/2.$$

With this choice of $\alpha$ and $\beta$, we can express the inequality (3.2.9) as

$$\|u - \Pi u\|^2_{\mathrm{L}^2(I)} \leq C \log(1/\varepsilon) h^4 \left\{ \varepsilon^{2\beta} \|x^{\alpha}(u - \Pi_{I_1} u)'\|^2_{\mathrm{L}^2(I_1)} + \sum_{i=2}^{N} \varepsilon^{2\beta} \|x^{2\gamma} u''\|^2_{\mathrm{L}^2(I_i)} \right\}$$

and we know from Lemmas 3.2.1 and 3.2.3 that the terms inside the brackets are bounded by a constant C.

□

**Theorem 3.2.5** (Corollary 4.5, DL05). *Let $u$ be the solution of Problem* (3.2.1) *and $u_h \in V_h$ its finite element approximation obtain using the mesh defined in* (3.2.7). *Then,*

$$\|u - u_h\|_{\varepsilon} \leq C h \log(1/\varepsilon)$$

*with a constant $C$ independent of $h$ and $\varepsilon$.*

### 3.2.3 Superconverge for graded meshes

In this section, we prove superconvergence error estimates for the solution of problem (3.2.1) over the graded meshes. Precisely, if $u_h$ is the finite element solution and $\Pi u$ is the Lagrange interpolation of the exact solution $u$, we prove that $\|u_h - \Pi u\|_\varepsilon$ is of higher order than $\|u - u_h\|_\varepsilon$.
We will use the following two Lemmas.

**Lemma 3.2.6.** *Let $u$ be the solution of* (3.2.1). *Then for any $\varphi \in V_h$ it holds*

$$\varepsilon^2 \int_0^1 (u - \Pi u)' \varphi' \, dx = 0$$

*Proof.* We integrate by parts over each interval:

$$\begin{aligned}\varepsilon^2 \int_0^1 (u - \Pi u)' \varphi' \, dx &= \varepsilon^2 \sum_{i=1}^{M} \int_{x_{i-1}}^{x_i} (u - \Pi_{I_i} u)' \varphi' \, dx = \\ &= \varepsilon^2 \sum_{i=1}^{M} \left[ \varphi' (u - \Pi_{I_i} u)|_{x_{i-1}}^{x_i} - \int_{x_{i-1}}^{x_i} (u - \Pi_{I_i} u)(x) \varphi''(x) \, dx \right]\end{aligned}$$

The desire result follows using that $\varphi''(x) \equiv 0$ and $(u - \Pi_{I_i} u)(x)|_{x_{i-1}}^{x_i} = 0$. □

**Lemma 3.2.7.** *Let $u$ be the solution of* (3.2.1). *Then for any $\varphi \in V_h$ it holds*

$$\left| \int_0^1 (u - \Pi u) \varphi dx \right| \le C h^2 \log(1/\varepsilon)^{1/2} \|\varphi\|_\varepsilon$$

*Proof.* By Theorem 3.2.4

$$\left| \int_0^1 (u - \Pi u) \varphi dx \right| \le \|u - \Pi u\|_{L^2(I)} \|\varphi\|_{L^2(I)} \le C h^2 \log(1/\varepsilon)^{1/2} \|\varphi\|_\varepsilon$$

□

We can now state and prove the main result of this Section.

**Theorem 3.2.8.** *Let $u$ be the solution of Problem* (3.2.1), *$u_h \in V_h$ its finite element approximation and $\Pi u \in V_h$ its Lagrange interpolation. Suppose that $3/4 \le \gamma < 1$. Then, there exists a constant $C$ such that,*

$$\|u_h - \Pi u\|_\varepsilon \le C h^2 \log(1/\varepsilon)^{1/2}$$

*Proof.* Let $\eta$ be the coercive constant, then,

$$\eta \|u_h - \Pi u\|_\varepsilon^2 \le \mathcal{B}(u_h - \Pi u, u_h - \Pi u) = \mathcal{B}(u - \Pi u, u_h - \Pi u)$$

Using that $u_h - \Pi u \in V_h$ and Lemmas 3.2.6 and 3.2.7 we obtain

$$\mathcal{B}(u - \Pi u, u_h - \Pi u) \le C h^2 \log(1/\varepsilon)^{1/2} \|u_h - \Pi u\|_\varepsilon$$

and the Theorem is proved. □

## 3.3 Case 2D

We consider the problem

$$\begin{aligned} -\varepsilon^2 \Delta u + u = f \qquad & \text{in } \Omega \\ u = 0 \qquad & \text{on } \partial\Omega \end{aligned} \tag{3.3.1}$$

where $\Omega = (0,1)^2$ and $\varepsilon$ is a small positive parameter.
We will assume $f \in \mathcal{C}^2([0,1]^2)$ and that it satisfies the compatibility conditions

$$f(0,0) = f(1,0) = f(0,1) = f(1,1) = 0.$$

It is known that under these hypotheses $u \in \mathcal{C}^4(\Omega) \cap \mathcal{C}^2(\overline{\Omega})$. We have the following pointwise estimates for the solution $u$ of problem (3.3.1) (see [LW00], Lemma 4.1): if $0 \le k \le 4$ then

$$\left| \frac{\partial^k u}{\partial x_1^k}(x_1,x_2) \right| \le C \left( 1 + \varepsilon^{-k} e^{-x_1/\varepsilon} + \varepsilon^{-k} e^{-(1-x_1)/\varepsilon} \right) \tag{3.3.2}$$

$$\left| \frac{\partial^k u}{\partial x_2^k}(x_1,x_2) \right| \le C \left( 1 + \varepsilon^{-k} e^{-x_2/\varepsilon} + \varepsilon^{-k} e^{-(1-x_2)/\varepsilon} \right), \tag{3.3.3}$$

Let $d(t) = \min\{t, 1-t\}$ be the distance between $t$ and the boundary of the interval $[0,1]$.

**Lemma 3.3.1.** *If $u$ is the solution of problem* (3.3.1)*, there exists a constant $C$ independent of $\varepsilon$ such that*

*(i) if* $0 \le k \le 4, a+b \ge k - 1/2,\ a \ge 0, b > -1/2$ *then*

$$\varepsilon^a \left\| d(x_1)^b \frac{\partial^k u}{\partial x_1^k} \right\|_{\mathrm{L}^2(\Omega)} \le C, \qquad \varepsilon^a \left\| d(x_2)^b \frac{\partial^k u}{\partial x_2^k} \right\|_{\mathrm{L}^2(\Omega)} \le C,$$

*(ii) if* $a+b \ge 1,\ a \ge 1/2, b > -1/2$ *then*

$$\varepsilon^a \left\| d(x_2)^b \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{\mathrm{L}^2(\Omega)} \le C, \qquad \varepsilon^a \left\| d(x_1)^b \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{\mathrm{L}^2(\Omega)} \le C,$$

*(iii) if* $a+b > 7/4, b > 1/2, c \ge 3/4$ *then*

$$\varepsilon^a \left\| d(x_1)^b d(x_2)^c \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{\mathrm{L}^2(\Omega)} \le C,$$

*(iv) if* $a+c \ge 5/2, a \ge 3/4, c > 1/2$ *then*

$$\varepsilon^a \left\| d(x_2)^c \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|_{\mathrm{L}^2(\Omega)} \le C.$$

*Proof.* It is easy to check that the inequalities involving pure derivatives follow from the pointwise estimates (3.3.2) and (3.3.3).
To prove the estimates for cross derivatives, the idea is to reduce them to known pointwise estimates for the pure derivatives by integrating by parts as many times as necessary. As an example we prove $(iii)$, the other inequalities can be proved in an analogous way. If $D(t) = t(1-t)$ then we clearly have $D(t) \leq d(t) \leq 2D(t)$. So, it is enough to show that

$$\varepsilon^a \left\| D(x_1)^b D(x_2)^c \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{L^2(\Omega)} \leq C, \qquad \text{for } a+b > 7/4, b > 1/2, c \geq 3/4.$$

We integrate by parts with respect to the variables $x_1$ and $x_2$ separately and use that $D(0) = D(1) = 0$ . So for $b > 1/2$ and $c > 0$ we obtain

$$\int_0^1 \int_0^1 \left( D(x_1)^b D(x_2)^c \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right)^2 dx_1\, dx_2 =$$
$$= \int_0^1 \int_0^1 \left[ 2b(2b-1) D(x_1)^{2b-2} D'(x_1)^2 \frac{\partial^2 u}{\partial x_1^2} + 2b D(x_1)^{2b-1} D''(x_1) \frac{\partial^2 u}{\partial x_1^2} + 4b D(x_1)^{2b-1} D'(x_1) \frac{\partial^3 u}{\partial x_1^3} + D(x_1)^{2b} \frac{\partial^4 u}{\partial x_1^4} \right]$$
$$\times \left[ (-2c) D(x_2)^{2c-1} D'(x_2) \frac{\partial u}{\partial x_2} - D(x_2)^{2c} \frac{\partial^2 u}{\partial x_2^2} \right] dx_1\, dx_2.$$

So, using the Cauchy-Schwarz inequality and that $|D'(t)| \leq 1$ and $|D''(t)| = 2$, we obtain

$$\varepsilon^{2a} \left\| D(x_1)^b D(x_2)^c \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|^2_{L^2(\Omega)} \leq$$
$$\leq C \left\{ \varepsilon^{2a} \left[ \left\| D(x_1)^{2b-2} \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2(\Omega)} + \left\| D(x_1)^{2b-1} \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2(\Omega)} + \left\| D(x_1)^{2b-1} \frac{\partial^3 u}{\partial x_1^3} \right\|_{L^2(\Omega)} + \left\| D(x_1)^{2b} \frac{\partial^4 u}{\partial x_1^4} \right\|_{L^2(\Omega)} \right] \right\}$$
$$\times \left[ \left\| D(x_2)^{2c-1} \frac{\partial u}{\partial x_2} \right\|_{L^2(\Omega)} + \left\| D(x_2)^{2c} \frac{\partial^2 u}{\partial x_2^2} \right\|_{L^2(\Omega)} \right].$$

The first factor, that involves norms of pure derivatives in $x_1$, is bounded if $2a + 2b \geq 7/2$, that is, $a + b \geq 7/4$. The second factor, involving pure derivatives in $x_2$, is bounded if $2c \geq 3/2$. Therefore we conclude the proof. □

The following anisotropic norms will be used to estimate the $L^2$-norm of the interpolation error for the reaction-diffusion problem. For $v : R \to \mathbb{R}$, where $R$ is the rectangle $R = l_1 \times l_2$, define

$$\|v\|_{\infty\times 1,R} := \Big\| \|v(x_1,\cdot)\|_{L^1(l_2)} \Big\|_{L^\infty(l_1)} \qquad \|v\|_{1\times\infty,R} := \Big\| \|v(\cdot,x_2)\|_{L^1(l_1)} \Big\|_{L^\infty(l_2)}$$

The next result is a straightforward consequence of the pointwise estimates (3.3.2) and (3.3.3).

**Lemma 3.3.2.** *If $u$ is the solution of problem* (3.3.1), *there exists a constant $C$ such that*

$$\left\| \frac{\partial u}{\partial x_1} \right\|_{1\times\infty,\Omega} \leq C \qquad \text{and} \qquad \left\| \frac{\partial u}{\partial x_2} \right\|_{\infty\times 1,\Omega} \leq C$$

### 3.3.1 Weak formulation and finite element approximation

The standard weak formulation of Problem (3.3.1) is given by: find $u \in \mathrm{H}_0^1(\Omega)$ such that

$$\mathcal{B}(u, \varphi) = \int_\Omega f\, \varphi \, dx, \quad \forall \varphi \in \mathrm{H}_0^1(\Omega),$$

where

$$\mathcal{B}(u, \varphi) = \int_\Omega (\varepsilon^2 \nabla u \cdot \nabla \varphi + u\varphi)\, dx. \tag{3.3.4}$$

We denote with $\|\cdot\|_\varepsilon$ the norm associated with the bilinear form $\mathcal{B}$, i.e., $\|\varphi\|_\varepsilon^2 := \mathcal{B}(\varphi, \varphi)$. Given a finite-dimensional subspace $V_h$ of $\mathrm{H}_0^1(\Omega)$, the finite element approximation is given by: find $u_h \in V_h$ such that

$$\mathcal{B}(u_h, \varphi) = \int_\Omega f \varphi \, dx, \quad \forall \varphi \in V_h$$

Let us recall the definition of the graded meshes introduced in [DL05]. Let $h, \gamma \in (0, 1)$ be fixed. As for the one dimensional case, we consider the partition $\{\xi_i\}_{i=0}^M$ of the interval $[0, 1/2]$ given by

$$\begin{cases} \xi_0 &= 0 \\ \xi_1 &= h^{\frac{1}{1-\gamma}} \\ \xi_{i+1} &= \xi_i + h\xi_i^\gamma \quad \text{for } 1 \le i \le M-2 \\ \xi_M &= 1/2 \end{cases}$$

where $M$ is such that $\xi_{M-1} < 1/2$ and $\xi_{M-1} + h\xi_{M-1}^\gamma \ge 1/2$, and $\xi_M = 1/2$. If $1/2 - \xi_{M-1} < \xi_{M-1} - \xi_{M-2}$ we modify the definition of $\xi_{M-1}$ by taking $\xi_{M-1} = (1/2 + \xi_{M-2})/2$.
By symmetry, we define a partition on the interval $[1/2, 1]$, thus obtaining the partition $\{\xi_i\}_{i=0}^{2M}$ of the interval $[0, 1]$.
For $1 \le i, j \le 2M$ let $R_{ij} = [\xi_{i-1}, \xi_i] \times [\xi_{j-1}, \xi_j]$. Then the graded mesh is $\mathcal{T}_{h,\gamma} = \{R_{ij}\}_{i,j=1}^{2M}$ in $\Omega = [0, 1]^2$. Also we set $h_i = \xi_i - \xi_{i-1}$.
Then, we have the finite-dimensional subspace

$$V_h = \left\{ \varphi \in \mathrm{H}_0^1(\Omega) : \varphi\,|_{R_{ij}} \in \mathcal{Q}_1(R_{ij}), 1 \le i, j \le 2M \right\}$$

Our next goal is to obtain interpolation error estimates for the solution $u$ of problem (3.3.1). It is clear that, by symmetry, it is enough to prove the estimates in $\widetilde{\Omega} = [0, 1/2]^2$.
We will use the splitting of $\widetilde{\Omega}$ as $\widetilde{\Omega} = \Omega_{11} \cup \Omega_{12} \cup \Omega_{21} \cup \Omega_{22}$, where $\Omega_{11}, \Omega_{12}, \Omega_{21}$ and $\Omega_{22}$ are the closed sets with disjoint interiors defined by

$$\begin{aligned} \Omega_{11} &= R_{11} \\ \Omega_{12} &= \bigcup \{R_{1j}, j \ge 2\} \\ \Omega_{21} &= \bigcup \{R_{i1}, i \ge 2\} \\ \Omega_{22} &= \bigcup \{R_{ij}, i, j \ge 2\} \end{aligned}$$

### 3.3.2 Interpolation error estimates

**Lemma 3.3.3.** *Let $S = [0,1]^2$ be the reference element and $\alpha > -1/2$. Then for all $u \in H^2(S)$, we have*

$$\left|\frac{\partial(\Pi_S u)}{\partial x_1}\right| \le \left\|\frac{\partial u}{\partial x_1}\right\|_{1\times\infty,S}, \quad \left|\frac{\partial(\Pi_S u)}{\partial x_2}\right| \le \left\|\frac{\partial u}{\partial x_2}\right\|_{\infty\times 1,S}$$

$$\left\|x_1^\alpha \frac{\partial(\Pi_S u)}{\partial x_1}\right\|_{\mathrm{L}^2(S)} \le \frac{1}{1+2\alpha}\left\|\frac{\partial u}{\partial x_1}\right\|_{1\times\infty,S}, \quad \left\|x_2^\alpha \frac{\partial(\Pi_S u)}{\partial x_2}\right\|_{\mathrm{L}^2(S)} \le \frac{1}{1+2\alpha}\left\|\frac{\partial u}{\partial x_2}\right\|_{\infty\times 1,S}$$

*Proof.* Since $u$ and $\Pi_I u$ agrees at the vertices, we have

$$\begin{aligned}\frac{\partial(\Pi_I u)}{\partial x_1} &= x_2[u(1,1)-u(0,1)] + (1-x_2)[u(1,0)-u(0,0)]\\ &= x_2\int_0^1 \frac{\partial u}{\partial x_1}(t,1)dt + (1-x_2)\int_0^1 \frac{\partial u}{\partial x_1}(t,0)dt\end{aligned}$$

Then

$$\begin{aligned}\left|\frac{\partial(\Pi_I u)}{\partial x_1}\right| &\le x_2\int_0^1 \left|\frac{\partial u}{\partial x_1}(t,1)\right| dt + (1-x_2)\int_0^1 \left|\frac{\partial u}{\partial x_1}(t,0)\right| dt\\ &\le x_2\left\|\frac{\partial u}{\partial x_1}(\cdot,1)\right\|_{L^1(l_1)} + (1-x_2)\left\|\frac{\partial u}{\partial x_1}(\cdot,0)\right\|_{L^1(l_1)}\\ &\le \left\|\left\|\frac{\partial u}{\partial x_1}(\cdot,x_2)\right\|_{L^1(l_1)}\right\|_{L^\infty(l_2)}\\ &\le \left\|\frac{\partial u}{\partial x_1}\right\|_{1\times\infty,S},\end{aligned}$$

and therefore,

$$\begin{aligned}\left\|x_1^\alpha \frac{\partial(\Pi_S u)}{\partial x_1}\right\|^2_{\mathrm{L}^2(S)} &= \int_0^1\int_0^1 x_1^{2\alpha}\left|\frac{\partial(\Pi_S u)}{\partial x_1}\right|^2 dx_1\,dx_2\\ &\le \int_0^1\int_0^1 x_1^{2\alpha}\left\|\frac{\partial u}{\partial x_1}\right\|^2_{1\times\infty,S} dx_1\,dx_2\\ &\le \frac{1}{1+2\alpha}\left\|\frac{\partial u}{\partial x_1}\right\|^2_{1\times\infty,S}\end{aligned}$$

Similar arguments prove the remaining inequalities. □

**Theorem 3.3.4.** *Let $u$ be the solution of problem* (3.3.1), $\Pi u$ *be its Lagrange interpolation over the graded mesh $\mathcal{T}_h$ defined in the previous section, and suppose that $3/4 \le \gamma < 1$. There exists a constant $C$, independent of $\varepsilon$ and $h$, such that*

$$\|u - \Pi u\|_{\mathrm{L}^2(\Omega)} \le C\log(1/\varepsilon)^{1/2} h^2.$$

*Proof.* It is enough to obtain the estimate replacing $\Omega$ by $\widetilde{\Omega}$. We decompose the error as

$$\|u-\Pi u\|^2_{L^2(\widetilde{\Omega})} = \|u-\Pi u\|^2_{L^2(\Omega_{11}\cup\Omega_{12})} + \|u-\Pi u\|^2_{L^2(\Omega_{21})} + \|u-\Pi u\|^2_{L^2(\Omega_{22})}$$

with $\Omega_{ij}$ as in the previous section.
For $R_{ij}\subset\Omega_{11}\cup\Omega_{12}$, since $u$ vanishes on $l_j=\{(0,x_2),\xi_{j-1}\le x_2\le\xi_j\}$, we use inequality (1.3.3) of Lemma 1.3.1 to obtain

$$\|u-\Pi u\|^2_{L^2(R_{ij})} \le \frac{C}{1-2\alpha} h_i^{2-2\alpha}\left\{\left\|x_1^\alpha\frac{\partial u}{\partial x_1}\right\|^2_{L^2(R_{ij})} + \left\|x_1^\alpha\frac{\partial(\Pi u)}{\partial x_1}\right\|^2_{L^2(R_{ij})}\right\}$$

(recall that $(\Pi u)|_{R_{ij}}=\Pi_{R_{ij}}u$)) Since $h_1=h^{\frac{1}{1-\gamma}}$, multiplying and dividing by $\varepsilon^\beta$ (where $\beta$ is a constant to be determined later) we have for $j\ge 1$ and $\alpha<1/2$:

$$\begin{aligned}\|u-\Pi u\|^2_{L^2(R_{1j})} &\le \frac{C}{1-2\alpha} h^{2\frac{1-\alpha}{1-\gamma}}\left\{\left\|x_1^\alpha\frac{\partial u}{\partial x_1}\right\|^2_{L^2(R_{1j})} + \left\|x_1^\alpha\frac{\partial(\Pi u)}{\partial x_1}\right\|^2_{L^2(R_{1j})}\right\}\\ &\le \frac{C}{1-2\alpha} h^{2\frac{1-\alpha}{1-\gamma}}\varepsilon^{-2\beta}\left\{\varepsilon^{2\beta}\left\|x^\alpha\frac{\partial u}{\partial x_1}\right\|^2_{L^2(R_{1j})} + \varepsilon^{2\beta}\frac{1}{1+2\alpha}\left\|\frac{\partial u}{\partial x_1}\right\|^2_{1\times\infty,R_{1j}}\right\}\end{aligned} \tag{3.3.5}$$

Now take $\beta>0$ and $0<\alpha<1/2$ as

$$\beta = 1/2-\alpha = \frac{1}{\log\frac{1}{\varepsilon}}. \tag{3.3.6}$$

So

$$\varepsilon^{-\beta}=e, \qquad \text{and} \qquad \frac{1}{1-2\alpha}=\frac{1}{2}\log(1/\varepsilon),$$

and it follows from $3/4\le\gamma<1$ that

$$\frac{1-\alpha}{1-\gamma}\ge 2.$$

We also have

$$2\frac{1-\alpha}{1-\gamma}\ge 4, \qquad \beta+\alpha=1/2, \qquad \text{and} \qquad \beta+2\gamma\ge 3/2. \tag{3.3.7}$$

With this choice of $\alpha$ and $\beta$, we know from the weighted inequalities of Lemma 3.3.1 that the first term inside the brackets in (3.3.5) is bounded by a constant C. The second term is also bounded in $\Omega$, because of Lemma 3.3.2. Then, summing over all $R_{ij}\in\Omega_{11}\cup\Omega_{12}$ we have

$$\|u-\Pi u\|^2_{L^2(\Omega_{11}\cup\Omega_{12})}\le C\log(1/\varepsilon)h^4 \tag{3.3.8}$$

With an analogous argument, we estimate the error for $R_{ij}\in\Omega_{21}$.
For $R_{ij}\in\Omega_{22}$, we use the standard estimate

$$\|u-\Pi u\|_{L^2(R_{ij})}\le C\left\{h_i^2\left\|\frac{\partial^2 u}{\partial x_1^2}\right\|_{L^2(R_{ij})} + h_j^2\left\|\frac{\partial^2 u}{\partial x_2^2}\right\|_{L^2(R_{ij})}\right\}$$

and the fact that $h_i \leq h\xi_{i-1}^{\gamma}, h_j \leq h\xi_{j-1}^{\gamma}$ over $\Omega_{22}$. Multiplying by $\varepsilon^{\beta}\varepsilon^{-\beta}$, it follows

$$\begin{aligned}\|u - \Pi_I u\|_{L^2(R_{ij})} &\leq C\left\{h^2\xi_{i-1}^{2\gamma}\left\|\frac{\partial^2 u}{\partial x_1^2}\right\|_{L^2(R_{ij})} + h^2\xi_{j-1}^{2\gamma}\left\|\frac{\partial^2 u}{\partial x_2^2}\right\|_{L^2(R_{ij})}\right\}\\ &\leq C\varepsilon^{-\beta}h^2\left\{\varepsilon^{\beta}\left\|x_1^{2\gamma}\frac{\partial^2 u}{\partial x_1^2}\right\|_{L^2(R_{ij})} + \varepsilon^{\beta}\left\|x_2^{2\gamma}\frac{\partial^2 u}{\partial x_2^2}\right\|_{L^2(R_{ij})}\right\}\end{aligned}$$

As $\beta + 2\gamma \geq 3/2$ for $\gamma \geq 3/4$, using the weighted inequalities from Lemma 3.3.1, we obtain

$$\|u - \Pi u\|_{L^2(\Omega_{22})} \leq Ch^2 \tag{3.3.9}$$

Collecting inequalities (3.3.8) and (3.3.9) we obtain the desired estimate. □

**Theorem 3.3.5** (Corollary 4.5, DL05). *Let $u$ be the solution of Problem* (3.2.1) *and $u_h \in V_h$ its finite element approximation obtain using the mesh defined in* (3.2.7) *. Then,*

$$\|u - u_h\|_{\varepsilon} \leq Ch\log(1/\varepsilon)^{1/2}$$

*with a constant $C$ independent of $h$ and $\varepsilon$.*

### 3.3.3 Superconvergence for graded meshes

**Lemma 3.3.6.** *Let $u$ be the solution of* (3.3.1) *and suppose that $3/4 \leq \gamma < 1$. Then, there exists a constant $C$ such that, for any $v \in V_h$,*

$$\left|\varepsilon^2\int_{\Omega}\nabla(u - \Pi u)\cdot\nabla v\,dx_1\,dx_2\right| \leq C\log(1/\varepsilon)^{1/2}h^2\|v\|_{\varepsilon}$$

*Proof.* As in the previous theorem it is enough to prove the estimate in $\widetilde{\Omega}$. We use again the decomposition $\widetilde{\Omega} = \Omega_{11}\cup\Omega_{12}\cup\Omega_{21}\cup\Omega_{22}$. In $\Omega_{11} = R_{11}$ we have

$$\begin{aligned}\left|\varepsilon^2\int_{R_{11}}\frac{\partial(u-\Pi u)}{\partial x_1}\frac{\partial v}{\partial x_1}\,dx_1\,dx_2\right| &\leq C\varepsilon^2\left\|\frac{\partial(u-\Pi u)}{\partial x_1}\right\|_{L^2(R_{11})}\left\|\frac{\partial v}{\partial x_1}\right\|_{L^2(R_{11})}\\ &\leq C\varepsilon\left\|\frac{\partial(u-\Pi u)}{\partial x_1}\right\|_{L^2(R_{11})}\|v\|_{\varepsilon}\end{aligned}$$

From Lemma 1.3.2, using that $h_1 = h^{\frac{1}{1-\gamma}}, \xi_0 = 0$, we have

$$\begin{aligned}\varepsilon^2\left\|\frac{\partial(u-\Pi u)}{\partial x_1}\right\|^2_{L^2(R_{11})} &\leq \frac{C\varepsilon^2}{1-2\alpha}\left\{h^{2\frac{1-\alpha}{1-\gamma}}\left\|x_1^{\alpha}\frac{\partial^2 u}{\partial x_1^2}\right\|^2_{L^2(R_{11})} + h^{2\frac{1-\alpha}{1-\gamma}}\left\|x_1^{\alpha}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|^2_{L^2(R_{11})}\right\}\\ &\leq \frac{C\,h^{2\frac{1-\alpha}{1-\gamma}}\varepsilon^{-2\beta}}{1-2\alpha}\left\{\varepsilon^{2(1+\beta)}\left\|x_1^{\alpha}\frac{\partial^2 u}{\partial x_1^2}\right\|^2_{L^2(R_{11})} + \varepsilon^{2(1+\beta)}\left\|x_1^{\alpha}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|^2_{L^2(R_{11})}\right\}\end{aligned}$$

Choosing $\alpha$ and $\beta$ as in (3.3.6), and using the weighted inequalities from Lemma 3.3.1 we have

$$\left|\varepsilon^2 \int_{R_{11}} \frac{\partial(u-\Pi u)}{\partial x_1}\frac{\partial v}{\partial x_1}\,dx_1\,dx_2\right| \le \frac{C\,h^{\frac{1-\alpha}{1-\gamma}}\varepsilon^{-\beta}}{(1-2\alpha)^{1/2}}\left\{\varepsilon^{1+\beta}\left\|x_1^\alpha \frac{\partial^2 u}{\partial x_1^2}\right\|_{\mathrm{L}^2(R_{11})} + \varepsilon^{1+\beta}\left\|x_1^\alpha \frac{\partial^2 u}{\partial x_1 \partial x_2}\right\|_{\mathrm{L}^2(R_{11})}\right\}\|v\|_\varepsilon$$
$$\le C\log(1/\varepsilon)^{1/2}h^2\,\|v\|_\varepsilon \tag{3.3.10}$$

Let $R_{ij} \in \Omega_{22}$. We have $h_i \le h\xi_{i-1}^\gamma$ and $h_j \le h\xi_{j-1}^\gamma$. We use a standard inequality (see for example [Ap99]) and multiply and divide by $\varepsilon^\beta$ as before to obtain

$$\left|\varepsilon^2 \int_{R_{ij}} \frac{\partial(u-\Pi u)}{\partial x_1}\frac{\partial v}{\partial x_1}\,dx_1\,dx_2\right|$$
$$\le C\varepsilon^2\left\{h_i^2\left\|\frac{\partial^3 u}{\partial x_1^3}\right\|_{\mathrm{L}^2(R_{ij})} + h_i h_j\left\|\frac{\partial^3 u}{\partial x_1^2\partial x_2}\right\|_{\mathrm{L}^2(R_{ij})} + h_j^2\left\|\frac{\partial^3 u}{\partial x_1\partial x_2^2}\right\|_{\mathrm{L}^2(R_{ij})}\right\}\left\|\frac{\partial v}{\partial x_1}\right\|_{\mathrm{L}^2(R_{ij})}$$
$$\le Ch^2\varepsilon^{-\beta}\varepsilon^{1+\beta}\left\{\left\|x_1^{2\gamma}\frac{\partial^3 u}{\partial x_1^3}\right\|_{\mathrm{L}^2(R_{ij})} + \left\|x_1^\gamma x_2^\gamma\frac{\partial^3 u}{\partial x_1^2\partial x_2}\right\|_{\mathrm{L}^2(R_{ij})} + \left\|x_2^{2\gamma}\frac{\partial^3 u}{\partial x_1\partial x_2^2}\right\|_{\mathrm{L}^2(R_{ij})}\right\}\|v\|_\varepsilon$$

Then, choosing $\beta$ as in (3.3.6), and taking into account $\gamma \ge 3/4$, we have

$$1+\beta+2\gamma \ge 5/2 \quad \text{and} \quad 1+\beta+\gamma \ge 7/4,$$

then from Lemma 3.3.1 we have

$$\left|\varepsilon^2 \int_{\Omega_{22}} \frac{\partial(u-\Pi u)}{\partial x_1}\frac{\partial v}{\partial x_1}\,dx_1\,dx_2\right| \le Ch^2\,\|v\|_\varepsilon \tag{3.3.11}$$

For the rest of the mesh, we use an argument introduced by Zlamal in [Zl78]. We know that for $p \in \mathcal{P}_2(R_{ij})$ and $v \in \mathcal{Q}_1(R_{ij})$,

$$\int_{R_{ij}} \frac{\partial(p-\Pi p)}{\partial x_1}\frac{\partial v}{\partial x_1}\,dx_1\,dx_2 = 0.$$

Then, for every $p \in \mathcal{P}_2(R_{ij})$ and $v \in \mathcal{Q}_1(R_{ij})$, we have

$$\left|\int_{R_{ij}} \frac{\partial(u-\Pi u)}{\partial x_1}\frac{\partial v}{\partial x_1}\,dx_1\,dx_2\right| = \left|\int_{R_{ij}} \frac{\partial[(u-p)-\Pi(u-p)]}{\partial x_1}\frac{\partial v}{\partial x_1}\,dx_1\,dx_2\right|$$
$$\le \left\|\frac{\partial[(u-p)-\Pi(u-p)]}{\partial x_1}\right\|_{\mathrm{L}^2(R_{ij})}\left\|\frac{\partial v}{\partial x_1}\right\|_{\mathrm{L}^2(R_{ij})},$$

Now, we use Lemmas 1.3.2 and 1.2.1 to obtain, for $0 < \alpha < 1$,

$$\begin{aligned}\left\| \frac{\partial[(u-p)-\Pi(u-p)]}{\partial x_1} \right\|^2_{L^2(R_{ij})} &\leq \frac{C}{1-2\alpha}\left\{ h_i^{2-2\alpha}\left\|(x_1-\xi_{i-1})^\alpha \frac{\partial^2(u-p)}{\partial x_1^2}\right\|^2_{L^2(R_{ij})} + h_i^{-2\alpha}h_j^2\left\|(x_1-\xi_{i-1})^\alpha\frac{\partial^2(u-p)}{\partial x_1\partial x_2}\right\|^2_{L^2(R_{ij})}\right\} \\ &\leq \frac{C}{1-2\alpha}\left\{ h_i^{2-2\alpha}\left\|(x_1-\xi_{i-1})^{\alpha+1}\frac{\partial^3 u}{\partial x_1^3}\right\|^2_{L^2(R_{ij})} + h_i^{-2\alpha}h_j^2\left\|(x_1-\xi_{i-1})^{\alpha+1}\frac{\partial^3 u}{\partial x_1^2\partial x_2}\right\|^2_{L^2(R_{ij})} + \right. \\ &\qquad \left. + h_i^{-2-2\alpha}h_j^4\left\|(x_1-\xi_{i-1})^{\alpha+1}\frac{\partial^3 u}{\partial x_1\partial x_2^2}\right\|^2_{L^2(R_{ij})}\right\}\end{aligned}$$

Then for $R_{ij}\subset\Omega_{12}$ or $R_{ij}\subset\Omega_{21}$ we have

$$\begin{aligned}\left|\int_{R_{ij}} \frac{\partial(u-\Pi u)}{\partial x_1}\frac{\partial v}{\partial x_1}\,dx_1\,dx_2\right| &\leq \frac{C}{(1-2\alpha)^{1/2}}\left\{ h_i^{1-\alpha}\left\|(x_1-\xi_{i-1})^{\alpha+1}\frac{\partial^3 u}{\partial x_1^3}\right\|_{L^2(R_{ij})} + \right. \\ &\left. + h_i^{-\alpha}h_j\left\|(x_1-\xi_{i-1})^{\alpha+1}\frac{\partial^3 u}{\partial x_1^2\partial x_2}\right\|_{L^2(R_{ij})} + h_i^{-1-\alpha}h_j^2\left\|(x_1-\xi_{i-1})^{\alpha+1}\frac{\partial^3 u}{\partial x_1\partial x_2^2}\right\|_{L^2(R_{ij})}\right\}\left\|\frac{\partial v}{\partial x_1}\right\|_{L^2(R_{ij})}\end{aligned}$$

For $R_{1j}, j\geq 2$ we have $\xi_{i-1}=0$, and then,

$$\begin{aligned}\left|\varepsilon^2\int_{R_{ij}} \frac{\partial(u-\Pi u)}{\partial x_1}\frac{\partial v}{\partial x_1}\,dx_1\,dx_2\right| &\leq \frac{C\varepsilon^2}{(1-2\alpha)^{1/2}}\left\{ h_1^{1-\alpha}\left\|x_1^{\alpha+1}\frac{\partial^3 u}{\partial x_1^3}\right\|_{L^2(R_{ij})} + h_1^{-\alpha}h_j\left\|x_1^{\alpha+1}\frac{\partial^3 u}{\partial x_1^2\partial x_2}\right\|_{L^2(R_{ij})} + \right. \\ &\left. + h_1^{-1-\alpha}h_j^2\left\|x_1^{\alpha+1}\frac{\partial^3 u}{\partial x_1\partial x_2^2}\right\|_{L^2(R_{ij})}\right\}\left\|\frac{\partial v}{\partial x_1}\right\|_{L^2(R_{ij})}\end{aligned}$$

Now, we analyze each term inside the brackets in the right hand side of the previous inequality. We take $\alpha$ and $\beta$ as in (3.3.6). From the definition of the mesh, we know that $h_1 = h^{\frac{1}{1-\gamma}}$, and then, the first term can be written as $h^{\frac{1-\alpha}{1-\gamma}}\left\|x_1^{\alpha+1}\frac{\partial^3 u}{\partial x_1^3}\right\|_{L^2(R_{ij})}$, where $\frac{1-\alpha}{1-\gamma}\geq 2$. For the second term, since $x_1 = h_1$, we can multiply and divide by $h_1^s$ with $s>0$, use the definition of $h_1$ and that $h_j \leq h\xi_{j-1}^\gamma$ to obtain

$$h_1^{-\alpha}h_j\left\|x_1^{\alpha+1}\frac{\partial^3 u}{\partial x_1^2\partial x_2}\right\|_{L^2(R_{ij})} = h_1^{s-\alpha}h_j\left\|\frac{x_1^{\alpha+1}}{h_1^s}\frac{\partial^3 u}{\partial x_1^2\partial x_2}\right\|_{L^2(R_{ij})} \leq h^{\frac{s-\alpha}{1-\gamma}}h\left\|x_1^{\alpha+1-s}x_2^\gamma\frac{\partial^3 u}{\partial x_1^2\partial x_2}\right\|_{L^2(R_{ij})}$$

We choose $s$ such that

$$\frac{s-\alpha}{1-\gamma}=1,$$

that is, $s=\alpha+1-\gamma$. Then the second term can be bounded by $h^2\left\|x_1^\gamma x_2^\gamma\frac{\partial^3 u}{\partial x_1^2\partial x_2}\right\|_{L^2(R_{ij})}$.

For the third term, we use that $x_1\leq h_1$ and $h_j\leq h\xi_{j-1}^\gamma$ and so

$$h_1^{-1-\alpha}h_j^2\left\|x_1^{\alpha+1}\frac{\partial^3 u}{\partial x_1\partial x_2^2}\right\|_{L^2(R_{ij})} \leq h^2\left\|x_2^{2\gamma}\frac{\partial^3 u}{\partial x_1\partial x_2^2}\right\|_{L^2(R_{ij})}$$

Collecting all the estimates, we have for $R_{1j}, j \geq 2$, after multiplying and dividing by $\varepsilon^{2\beta}$

$$\left| \varepsilon^2 \int_{R_{ij}} \frac{\partial(u-\Pi u)}{\partial x_1} \frac{\partial v}{\partial x_1}\, dx_1\, dx_2 \right| \leq \frac{Ch^2 \varepsilon^{-2\beta}}{(1-2\alpha)^{1/2}} \varepsilon^{1+2\beta} \left\{ \left\| x_1^{\alpha+1} \frac{\partial^3 u}{\partial x_1^3} \right\|_{L^2(R_{ij})} + \left\| x_1^{\gamma} x_2^{\gamma} \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{L^2(R_{ij})} + \left\| x_2^{2\gamma} \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|_{L^2(R_{ij})} \right\} \|v\|_\varepsilon$$

Then, choosing $\alpha$ and $\beta$ as in (3.3.6) and using Lemma 3.3.1, we have

$$\left| \varepsilon^2 \int_{\Omega_{12}} \frac{\partial(u-\Pi u)}{\partial x_1} \frac{\partial v}{\partial x_1}\, dx_1\, dx_2 \right| \leq \frac{C}{(1-2\alpha)^{1/2}} h^2 \|v\|_\varepsilon \tag{3.3.12}$$

An analogous argument works for $R_{i1}, i \geq 2$, and therefore,

$$\left| \varepsilon^2 \int_{\Omega_{21}} \frac{\partial(u-\Pi u)}{\partial x_1} \frac{\partial v}{\partial x_1}\, dx_1\, dx_2 \right| \leq \frac{C}{(1-2\alpha)^{1/2}} h^2 \|v\|_\varepsilon . \tag{3.3.13}$$

Collecting inequalities (3.3.10), (3.3.11), (3.3.12) and (3.3.13) we obtain

$$\left| \varepsilon^2 \int_{\widetilde{\Omega}} \frac{\partial(u-\Pi u)}{\partial x_1} \frac{\partial v}{\partial x_1}\, dx_1\, dx_2 \right| \leq C \log(1/\varepsilon)^{1/2} h^2 \|v\|_\varepsilon$$

concluding the proof. □

**Lemma 3.3.7.** *Let $u$ be the solution of* (3.3.1) *and suppose that $3/4 \leq \gamma < 1$ Then, there exists a constant $C$ such that, for any $v \in V_h$,*

$$\left| \int_\Omega (u-\Pi u) v\, dx_1\, dx_2 \right| \leq C \log(1/\varepsilon)^{1/2} h^2 \|v\|_\varepsilon$$

*Proof.* From Theorem 3.3.4 we know that $\|u-\Pi u\|_{L^2(\Omega)} \leq C \log(1/\varepsilon) h^2$, hence

$$\begin{aligned} \left| \int_\Omega (u-\Pi u) v\, dx_1\, dx_2 . \right| &\leq C \|u-\Pi u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\leq C \log(1/\varepsilon)^{1/2} h^2 \|v\|_\varepsilon . \end{aligned}$$

□

We can now state and prove the main result of this section.

**Theorem 3.3.8.** *Let $u$ be the solution of* (3.3.1)*, $u_h \in V_h$ its finite element approximation and $\Pi u \in V_h$ its Lagrange interpolation. Suppose that $3/4 \leq \gamma < 1$. Then, there exists a constant C such that,*

$$\|u_h - \Pi u\|_\varepsilon \leq C h^2 \log(1/\varepsilon)^{1/2}$$

*Proof.* From the error equation $\mathcal{B}(u-u_h, u_h-\Pi u)=0$, we have

$$\|u_h-\Pi u\|_\varepsilon^2=\mathcal{B}(u_h-\Pi u, u_h-\Pi u)=\mathcal{B}(u-\Pi u, u_h-\Pi u).$$

But, from Lemmas 3.3.6 and 3.3.7, we have

$$\mathcal{B}(u-\Pi u, u_h-\Pi u)\le C\log(1/\varepsilon)^{1/2}h^2\,\|u_h-\Pi u\|_\varepsilon\,,$$

and therefore the theorem is proved. □

An immediate consequence of the last Theorem combined with the interpolation result proved in Theorem 3.3.4 is the optimal order convergence in the $\mathrm{L}^2$-norm.

**Corollary 3.3.9.** *Let $u$ be the solution of* (3.3.1) *and $u_h\in V_h$ its finite element approximation. Suppose that $3/4\le\gamma<1$. Then, there exists a constant $C$ such that,*

$$\|u-u_h\|_{\mathrm{L}^2(\Omega)}\le C\log(1/\varepsilon)^{1/2}h^2$$

We end this section by stating the error estimates in terms of the number of nodes. It can be seen (see the proof of Corollary 4.5 in [DL05]) that there exists a constant $C$ depending on $\gamma$ such that

$$h\le C\frac{\log N}{\sqrt{N}}$$

**Corollary 3.3.10.** *Let $u$ be the solution of* (3.3.1)*, $u_h\in V_h$ its finite element approximation, and $\Pi u$ its Lagrange interpolation. Suppose that $3/4\le\gamma<1$. Then, if $N$ is the number of nodes in $\mathcal{T}_h$ then, there exists a constant $C$ such that,*

$$\|u_h-\Pi u\|_\varepsilon\le C\log(1/\varepsilon)^{1/2}\frac{(\log N)^2}{N}$$

*and*

$$\|u-u_h\|_{\mathrm{L}^2(\Omega)}\le C\log(1/\varepsilon)^{1/2}\frac{(\log N)^2}{N}$$

*Proof.* The results follow from Theorem 3.3.8, Corollary 3.3.9 and the estimate

$$h\le C\frac{\log(N)}{\sqrt{N}}.$$

which was proved in [DL05, Corollary 4.5]. □

## 3.4 Numerical experiments

We end the paper with some numerical results. We consider the problem

$$-\varepsilon^2\Delta u+u=f$$

where

$$f(x_1,x_2)=(-2)\frac{1-e^{-1/\sqrt{2}\varepsilon}}{1-e^{-\sqrt{2}/\varepsilon}}\left(e^{-\frac{x_1}{\sqrt{2}\varepsilon}}+e^{-\frac{(1-x_1)}{\sqrt{2}\varepsilon}}+e^{-\frac{x_2}{\sqrt{2}\varepsilon}}+e^{-\frac{(1-x_2)}{\sqrt{2}\varepsilon}}\right)+4$$

Calling

$$u_0(t) = (-2)\frac{1 - e^{-1/\sqrt{2}\varepsilon}}{1 - e^{-\sqrt{2}/\varepsilon}}(e^{-\frac{t}{\sqrt{2}\varepsilon}} + e^{-\frac{(1-t)}{\sqrt{2}\varepsilon}}) + 2$$

The exact solution of this equation is

$$u(x_1, x_2) = u_0(x_1)u_0(x_2)$$

In Tables 3.1 and 3.2 we present the results for the graduation parameter $\gamma = 0.75$, $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-6}$ respectively. Recall that $N$ denotes the number of nodes. The approximate orders in terms of $N$ given in the tables are computed at each step comparing the errors between two following meshes.

| N | h | $\|u - u_h\|_{L^2}$ | order | $\|u - u_h\|_\varepsilon$ | order | $\|\Pi u - u_h\|_\varepsilon$ | order |
|---|---|---|---|---|---|---|---|
| 3721 | 0.11 | 8.5422e-004 | – | 4.1474e-002 | – | 6.0837e-003 | – |
| 4489 | 0.10 | 7.1497e-004 | 0.94832 | 3.8073e-002 | 0.45612 | 5.1118e-003 | 0.92769 |
| 5625 | 0.09 | 5.8695e-004 | 0.87463 | 3.4604e-002 | 0.42341 | 4.2104e-003 | 0.85993 |
| 7225 | 0.08 | 4.7039e-004 | 0.88437 | 3.1067e-002 | 0.43074 | 3.3834e-003 | 0.87349 |
| 9409 | 0.07 | 3.6558e-004 | 0.95436 | 2.7459e-002 | 0.46743 | 2.6351e-003 | 0.94642 |

Table 3.1: $\gamma = 0.75$ $\varepsilon = 10^{-2}$

| N | h | $\|u - u_h\|_{L^2}$ | order | $\|u - u_h\|_\varepsilon$ | order | $\|\Pi u - u_h\|_\varepsilon$ | order |
|---|---|---|---|---|---|---|---|
| 3721 | 0.11 | 5.2081e-002 | – | 8.2358e-002 | – | 3.2041e-002 | – |
| 4489 | 0.10 | 4.3042e-002 | 1.0159 | 6.8075e-002 | 1.0150 | 2.6468e-002 | 1.0182 |
| 5625 | 0.09 | 3.4862e-002 | 0.9344 | 5.5159e-002 | 0.9327 | 2.1416e-002 | 0.9389 |
| 7225 | 0.08 | 2.7471e-002 | 0.9519 | 4.3507e-002 | 0.9480 | 1.6861e-002 | 0.9554 |
| 9409 | 0.07 | 2.0353e-002 | 1.1355 | 3.2304e-002 | 1.1273 | 1.2688e-002 | 1.0766 |

Table 3.2: $\gamma = 0.75$ $\varepsilon = 10^{-6}$

Observe that the orders agree with those predicted by the theory.

With the next numerical example we want to show that some restriction in the parameter $\gamma$ is really necessary in order to have supercloseness (recall that for our proofs we needed $3/4 \leq \gamma < 1$). It is interesting to observe that for almost optimal order convergence the restriction $\gamma \geq 1/2$ was enough (see [DL05]). In Table 3.3 we present the results for $\varepsilon = 10^{-6}$ and the graduation given by $\gamma = 0.60$. It is observed that the order is deteriorated, indeed, it is close to $0.5$.

| N | h | $\|u-u_h\|_{L^2}$ | order | $\|u-u_h\|_\varepsilon$ | order | $\|\Pi u-u_h\|_\varepsilon$ | order |
|---|---|---|---|---|---|---|---|
| 1225 | 0.11 | 2.7244e-001 | – | 4.3076e-001 | – | 1.6832e-001 | – |
| 1521 | 0.10 | 2.4187e-001 | 0.54991 | 3.8243e-001 | 0.54991 | 1.4938e-001 | 0.55173 |
| 1849 | 0.09 | 2.1205e-001 | 0.67389 | 3.3528e-001 | 0.67389 | 1.3091e-001 | 0.67563 |
| 2401 | 0.08 | 1.8303e-001 | 0.56323 | 2.8940e-001 | 0.56323 | 1.1297e-001 | 0.56434 |
| 3025 | 0.07 | 1.5491e-001 | 0.72217 | 2.4493e-001 | 0.72217 | 9.5585e-002 | 0.72321 |

Table 3.3: $\gamma = 0.60 \qquad \varepsilon = 10^{-6}$

Finally, we present some comparisons with the well known Shishkin meshes. An advantage of the graded meshes considered here is that they are independent of the singular perturbation parameter $\varepsilon$, and therefore, the same mesh can be used for different values of $\varepsilon$. This can be of interest, for example, in numerical approximation of systems of equations involving different order diffusion parameters. On the other hand, we have observed in numerical experiments that Shishkin meshes designed for a given value of $\varepsilon$ do not give good approximation for larger values of $\varepsilon$. Indeed, this can be seen in Table 3.4 where we give the values of $\|\Pi u - u_h\|_\varepsilon$ for several values of $\varepsilon$ using both kind of meshes with the same number of nodes. The graded mesh is generated using $\gamma = 0.75$ and the Shishkin one corresponds to $\varepsilon = 10^{-6}$.

| $\varepsilon$ | Graded mesh | Shishkin mesh |
|---|---|---|
| $10^{-1}$ | 0.004038859190331 | 0.003773049703335 |
| $10^{-2}$ | 0.002635114829374 | 0.097598906440701 |
| $10^{-3}$ | 0.002326352901515 | 0.419801187583572 |
| $10^{-4}$ | 0.001995679630450 | 0.909830363131721 |
| $10^{-5}$ | 0.003944055357608 | 0.554766167574668 |
| $10^{-6}$ | 0.012687752835709 | 0.001683543678915 |

Table 3.4: $\|\Pi u - u_h\|_\varepsilon$ for both kind of meshes with 9409 nodes.

# Chapter 4

# A posteriori error estimator and adaptivity

## 4.1 Introduction

In the previous chapters we have shown that, for some simple singularly perturbed model problems presenting boundary layers, quasi-optimal order error estimates with constants depending only weakly on the perturbation parameter $\varepsilon$ can be obtained if appropriate a-priori adapted meshes are used. In practice, for more complicated problems, the adapted meshes have to be obtained by using some kind of adaptivity based on a-posteriori error estimators. The goal of this chapter is to analyze this kind of approach for convection-diffusion problems.
Consider the model problem

$$\begin{aligned} -\varepsilon\Delta u + b\cdot\nabla u + cu &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{4.1.1}$$

For our subsequent arguments it will be convenient to write the problem in the form

$$\begin{aligned} -\mathrm{div}(\varepsilon\nabla u - bu) + cu &= f && \text{en } \Omega \\ u &= 0 && \text{en } \partial\Omega \end{aligned} \tag{4.1.2}$$

Observe that, when $b$ is constant, both problems are equivalent. On the other hand, if $b$ is not constant, one can pass from (4.1.1) to (4.1.2) by simply changing the reaction coefficient $c$.
In recent years several methods to obtain so-called guaranteed error estimators have been developed. Some of these methods are based on flux reconstruction using a mesh which is dual to the original triangulation used to compute the approximate solution (see for example [CFPV09, LW04, Vo11] and their references). In order to apply some of these ideas to problem (4.1.2) we need to construct an approximation $\mathbf{t}_h$ to $\mathbf{t} := -(\varepsilon\nabla u - bu)$. For the a posteriori error analysis we will need $\mathbf{t}_h \in \mathbf{H}(\mathrm{div},\Omega) := \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \nabla\cdot\mathbf{v} \in L^2(\Omega)\}$. Using the approximation given by the computed solution $u_h$, namely, $\varepsilon\nabla u_h - bu_h$, we will construct $\mathbf{t}_h$ such that it belongs to a Raviart-Thomas space associated with a mesh dual to the original one. Then, using $\mathbf{t}_h$ we introduce the a posteriori error estimator and prove that it gives an upper bound for the error.

In the last part of the chapter we deal with adaptive anisotropic refinement. We use the constructed $\mathbf{t}_h$ to obtain an approximation to the Hessian matrix of $u$ which, combined with the code FreeFem++, can be used to define an adaptive procedure. We present some numerical results showing the good behavior of the method.
We work in this chapter with partitions $\mathcal{T}_h$ which for all $h > 0$ consists of triangles $T$ (resp.rectangles $R$) such that $\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} T$ (resp. $\overline{\Omega} = \bigcup_{R\in\mathcal{T}_h} R$) and which are conforming, *i.e.*, if $T, K \in \mathcal{T}_h, T \neq K$, then $T \cap K$ is either an empty set or a common edge, or vertex of $T$ and $K$.
Let $v_i, 1 \leq i \leq N$, denote the vertices of the triangulation. With each vertex $v_i$ we associate a region $\Omega_i$, consisting of those triangles or rectangles which have $v_i$ as a vertex. We know construct a dual partition $\mathcal{D}_h$ of $\Omega$ such that $\overline{\Omega} = \bigcup_{D\in\mathcal{D}_h} D$ and such that each vertex $v_i$ of the partition $\mathcal{T}_h$ is in exactly one $D \in \mathcal{D}_h$. The elements in the dual mesh are called dual volume or boxes, and are constructed as follows.
In the case of a triangular partition, for each vertex $v_i$, we consider all the triangles $T \in \Omega_i$. Then, the dual volume or box $D_i$ associated to $v_i$ is the polygon which has these triangle barycenters and the midpoints of the edges passing trough $v_i$ as vertices. Note that

$$\frac{|D_i|}{|\Omega_i|} = \frac{1}{3}, \qquad \forall 1 \leq i \leq N.$$

We use the notation $\mathcal{D}_h^{\text{int}}$ ($\mathcal{D}_h^{\text{ext}}$) to denote the dual volumes associated with interior (exterior) vertices.
In the case of rectangular partition, for each vertex $v_i$, we consider all the rectangles $R \in \Omega_i$. Then, the dual volume or box $D_i$ associated to $v_i$ is the rectangle which has as vertices the barycentres of the rectangular sharing $v_i$. Note that in this case

$$\frac{|D_i|}{|\Omega_i|} = \frac{1}{4}, \qquad \forall 1 \leq i \leq N.$$

We use the notation $\mathcal{D}_h^{\text{int}}$ ($\mathcal{D}_h^{\text{ext}}$) to denote the dual volumes associated with interior (exterior) vertices.
Finally, in order to define our a posteriori error estimator, we need another partition $\mathcal{S}_h$ of $\Omega$ which is obtained dividing each volume $D_i$ as shown in Figure 4.1.

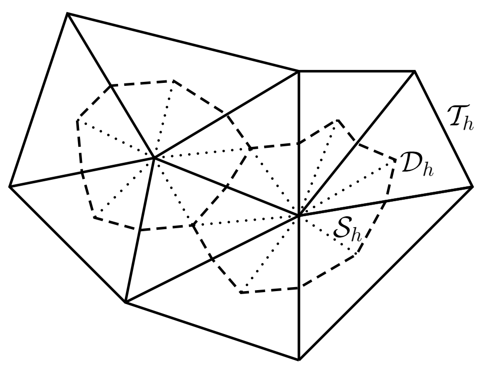



Figure 4.1: Original mesh $\mathcal{T}_h$, the associated dual partition $\mathcal{D}_h$, and the fine mesh $\mathcal{S}_h$

For an element in $\mathcal{T}_h$, $\mathbf{n}$ denotes its exterior normal vector and we employ the notation $\mathbf{n}_e$ for a normal vector of a side $e \in \mathcal{E}_h$, whose orientation is chosen arbitrarily but fixed for interior sides and coinciding with the exterior normal of $\Omega$ for exterior sides. For a function $\varphi$ and a side $e \in \mathcal{E}_h^{\text{int}}$ shared by $T, K \in \mathcal{T}_h$ we define the average operator $\{\!\{\cdot\}\!\}$ by

$$\{\!\{\varphi\}\!\} := \frac{1}{2}(\varphi|_T)|_e + \frac{1}{2}(\varphi|_K)|_e, \tag{4.1.3}$$

whereas for $e \in \mathcal{E}_h^{\text{ext}}$, $\{\!\{\varphi\}\!\} := \varphi|_e$. We use the same type of notation also for the meshes $\mathcal{D}_h$ and $\mathcal{S}_h$.

We denote with $\mathcal{P}_1(\mathcal{T}_h)$ (resp. $\mathcal{Q}_1(\mathcal{T}_h)$) the space of continuous piecewise linear (resp. bilinear) polynomials associated with $\mathcal{T}_h$ when the mesh is made of triangles (resp. rectangles). And $\mathcal{P}_0(\mathcal{D}_h)$ is the space of piecewise constant functions associated with $\mathcal{D}_h$.

For problem (4.1.2) we define the bilinear form $\mathcal{B}$ by

$$\mathcal{B}(u, \varphi) := \int_\Omega (\varepsilon \nabla u - bu) \cdot \nabla \varphi + cu\varphi,$$

where $u, \varphi \in \mathrm{H}_0^1(\Omega)$.

The weak formulation for this problem is then to find $u \in \mathrm{H}_0^1(\Omega)$ such that

$$\mathcal{B}(u, \varphi) = \int_\Omega f\varphi \qquad \forall \varphi \in \mathrm{H}_0^1(\Omega). \tag{4.1.4}$$

and the standard finite element approximation in a space $V_h$ is given by: find $u_h \in V_h$ such that

$$\mathcal{B}(u_h, \varphi) = \int_\Omega fv, \qquad \forall \varphi \in V_h$$

We are going to consider $V_h = \mathcal{P}_1(\mathcal{T}_h) \cap \mathrm{H}_0^1(\Omega)$ or $V_h = \mathcal{Q}_1(\mathcal{T}_h) \cap \mathrm{H}_0^1(\Omega)$. For simplicity in our analysis, we are going to consider $b$ constant on $\Omega$.

## 4.2 A posteriori error estimator for $\mathcal{P}_1$ approximation

To construct our error estimator we are going to use the Raviart-Thomas space of order one associated with the mesh $\mathcal{S}_h$, which is a subspace of $\mathbf{H}(\mathrm{div}, \Omega)$, defined locally on a triangle $S$ by

$$\mathbf{RT}_1(S) = \mathcal{P}_1(S)^2 + (x_1, x_2)\mathcal{P}_1(S).$$

It is known that there exists an interpolation operator which is defined locally by

$$\Pi_T : \mathrm{H}^1(S)^2 \to \mathbf{RT}_k,$$

such that for $\mathbf{v} \in \mathrm{H}^1(\Omega)^2$,

$$\int_{l_i} \Pi_T \mathbf{v} \cdot \mathbf{n}_i p_1 ds = \int_{l_i} \mathbf{v} \cdot \mathbf{n}_i p_1 ds, \qquad \forall p_1 \in \mathcal{P}_1(l_i).i = 1, 2, 3.$$

$$\int_T \Pi_T \mathbf{v} \cdot \mathbf{p}_0 dx = \int_T \mathbf{v} \cdot \mathbf{p}_0 dx, \quad \forall \mathbf{p}_0 \in \mathcal{P}_0^2(S)$$

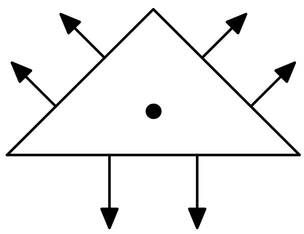


Figure 4.2: Degrees of freedom for $RT_1$

where $l_i, i = 1, 2, 3$ are the edges of the triangle $S$. See for example [BBDDFF08]
Figure 4.2 shows the degrees of freedom.
Therefore, if $u_h$ is the solution of the discrete problem and $S$ is a triangle of $S_h$, there exists a unique function $\mathbf{t}_h \in \mathbf{RT}_1(S)$ such that:

$$\int_{l_i} \mathbf{t}_h \cdot \mathbf{n}_i p_1 ds = \int_{l_i} \{\!\{-(\varepsilon\nabla u_h - bu_h)\cdot \mathbf{n}_i\}\!\} p_1 ds, \qquad \forall p_1 \in \mathcal{P}_1(l_i).i = 1, 2, 3.$$
$$\int_S (\mathbf{t}_h)_1 dx = \int_S -(\varepsilon\nabla u_h - bu_h)_1 dx,$$
$$\int_S (\mathbf{t}_h)_2 dx = \int_S -(\varepsilon\nabla u_h - bu_h)_2 dx.$$

Since $\mathbf{t}_h \cdot \mathbf{n}_e$ restricted to an edge is linear, it follows that

$$\mathbf{t}_h \cdot \mathbf{n}_e = \{\!\{-(\varepsilon\nabla u_h - bu_h)\cdot \mathbf{n}_e\}\!\}, \quad \forall e \in \mathcal{E}_h.$$

In particular, if $e \in \partial D$ for a $D \in \mathcal{D}_h$, we have $\mathbf{t}_h \cdot \mathbf{n}_e = -(\varepsilon\nabla u_h - bu_h)\cdot \mathbf{n}_e$.
In [BR87] it was proved the following

**Lemma 4.2.1.** *Given a vertex $v_j$ of the triangulation, let $\varphi_j \in \mathcal{P}^1(\mathcal{T}_h)$ be the associated basis function and $D_j$ be the dual volume associated with $v_j$. Then, for $u \in \mathcal{P}_1(\mathcal{T}_h)$ it holds*

$$-\int_{\partial D_j} u \cdot \mathbf{n} ds = \int_{\Omega_j} \nabla u \cdot \nabla \varphi_j dx \tag{4.2.1}$$

*where $\mathbf{n}$ is the outward pointing normal.*

We are going to use the projection $P_{\mathcal{D}_h} : \mathrm{L}^2(\Omega) \to \mathcal{P}_0(\mathcal{D}_h)$ which is defined in terms of a weighted mean value of $v$ on $D$:

$$P_{\mathcal{D}_h}(v)|_{D_j} := P_{\mathcal{D}_j}(v) := \frac{1}{|D_j|}\int_{\Omega_j} v\varphi_j \, dx$$

where $\varphi_j$ is the nodal $\mathcal{P}_1$ conforming basis function associated with the vertex $v_j$ of the triangulation $\mathcal{T}_h$. If $v$ is constant on each element $T \in \mathcal{T}_h$, then $P_D(v) = v_D$, where $v_D$ denotes the average of $v$ over $D$.
In the next lemma we are going to use the following relationship between the meshes $\mathcal{T}_h$ and $\mathcal{D}_h$: if $v$ is constant over each $T \in \mathcal{T}_h$, then

$$\int_{D_j} v dx = \int_{\Omega_j} v\phi_j dx \tag{4.2.2}$$

**Lemma 4.2.2.** *For $D \in \mathcal{D}_h$ it holds*

$$\int_D div(\mathbf{t}_h) = \int_D P_D(f - cu_h)$$

*Proof.* Given $D \in \mathcal{D}_h$, let $\varphi_j$ the nodal basis function associated to its central vertex and $\Omega_j$ the support of $\varphi_j$. Since $\mathbf{t}_h \cdot \mathbf{n}_e = -(\varepsilon \nabla u_h - bu_h) \cdot \mathbf{n}_e$ for $e \in \partial D$, we have

$$\begin{aligned}\int_D \operatorname{div} \mathbf{t}_h &= \int_{\partial D} \mathbf{t}_h \cdot \mathbf{n} \\ &= \int_{\partial D} -\varepsilon \nabla u_h \cdot \mathbf{n} + \int_{\partial D} bu_h \cdot \mathbf{n}\end{aligned}$$

Using Lemma 4.2.1, that $\operatorname{div}(bu_h)$ is constant over each triangle $T$, and (4.2.2) we obtain:

$$\begin{aligned}\int_D \operatorname{div} \mathbf{t}_h &= \int_{\Omega_j} \varepsilon \nabla u_h \cdot \nabla \varphi_j + \int_D \operatorname{div}(bu_h) \\ &= \int_{\Omega_j} \varepsilon \nabla u_h \cdot \nabla \varphi_j + \int_{\Omega_j} \operatorname{div}(bu_h)\varphi_j \\ &= \int_{\Omega_j} (\varepsilon \nabla u_h - bu_h) \cdot \nabla \varphi_j \\ &= \int_{\Omega_j} (f - cu_h)\varphi_j \\ &= \int_D P_D(f - cu_h)\end{aligned}$$

□

For $D \in \mathcal{D}_h$, we define

$$\|v\|_{\varepsilon,D}^2 := \varepsilon \|\nabla v\|_{\mathrm{L}^2(D)}^2 + \|v\|_{\mathrm{L}^2(D)}^2$$

If $D \in \mathcal{D}_h^{\mathrm{int}}$, we will use the Poincaré inequality

$$\|\varphi - \varphi_D\|_{\mathrm{L}^2(D)}^2 \leq C_{P,D} h_D^2 \|\nabla \varphi\|_{\mathrm{L}^2(D)}^2$$

and for $\varphi \in \mathrm{H}_0^1(\Omega)$ and $D \in \mathcal{D}_h^{\mathrm{ext}}$ (since vanishes on a part of $\partial D$ of positive measure) we have the following Friedrich inequality

$$\|\varphi\|_{\mathrm{L}^2(D)}^2 \leq C_{F,D} h_D^2 \|\nabla \varphi\|_{\mathrm{L}^2(D)}^2 .$$

**Lemma 4.2.3.** *Given $\varphi \in \mathrm{H}_0^1(\Omega)$ it holds:*

$$\|\varphi - \varphi_D\|_{\mathrm{L}^2(D)}^2 \leq m_D^2 \|\varphi\|_{\varepsilon,D}^2$$

*where*

$$\begin{aligned}m_D^2 &= \min\{\varepsilon^{-1} C_{P,D} h_D^2, 1\}, \quad D \in \mathcal{D}_h^{\mathrm{int}} \\ m_D^2 &= \min\{\varepsilon^{-1} C_{F,D} h_D^2, 1\}, \quad D \in \mathcal{D}_h^{\mathrm{ext}}\end{aligned}$$

*Proof.* As $\varphi_D$ is the $\mathrm{L}^2-$projection over the constant functions on $D$, we have:

$$\|\varphi - \varphi_D\|^2_{\mathrm{L}^2(D)} \leq \|\varphi\|^2_{\mathrm{L}^2(D)} \leq \|\varphi\|^2_{\varepsilon,D}$$

On the other hand, as the integral of $\varphi - \varphi_D$ vanishes over $D$ we can use the Poincaré inequality and obtain:

$$\begin{aligned}\|\varphi - \varphi_D\|^2_{\mathrm{L}^2(D)} &\leq C_{P,D} h_D^2 \|\nabla\varphi\|^2_{\mathrm{L}^2(D)} = C_{P,D} h_D^2 \varepsilon^{-1}\varepsilon \|\nabla\varphi\|^2_{\mathrm{L}^2(D)} \\ &\leq \varepsilon^{-1} C_{P,D} h_D^2 \|\varphi\|^2_{\varepsilon,D}\end{aligned}$$

for interior volumes.
If $D \in \mathcal{D}_h^{\mathrm{ext}}$ , we have

$$\|\varphi\|^2_{\mathrm{L}^2(D)} \leq \|\varphi\|^2_{\varepsilon,D}$$

and using the Friedrich inequality:

$$\|\varphi\|^2_{\mathrm{L}^2(D)} \leq C_{F,D} h_D^2 \|\nabla\varphi\|^2_{\mathrm{L}^2(D)} \leq \varepsilon^{-1} C_{F,D} h_D^2 \|\varphi\|^2_{\varepsilon,D}$$

□

We define the estimator in the following way:

$$\begin{aligned}\eta_{1,D} &:= m_D \|f - \mathrm{div}\mathbf{t}_h - c u_h\|_{\mathrm{L}^2(D)} \\ \eta_{2,D} &:= m_D \left\|\varepsilon^{1/2}\nabla u_h + \varepsilon^{-1/2}(\mathbf{t}_h - b u_h)\right\|_{\mathrm{L}^2(D)} \\ \eta_{3,D} &:= \|(f - c u_h)_D - P_D(f - c u_h)\|_{\mathrm{L}^2(D)}\end{aligned}$$

where

$$\begin{aligned}m_D &= \min\{\varepsilon^{-1/2} C_{P,D}^{1/2} h_D, 1\}, \quad D \in \mathcal{D}_h^{\mathrm{int}} \\ m_D &= \min\{\varepsilon^{-1/2} C_{F,D}^{1/2} h_D, 1\}, \quad D \in \mathcal{D}_h^{\mathrm{ext}}\end{aligned}$$

**Theorem 4.2.4.**

$$\|u - u_h\|_\varepsilon \leq \frac{1}{\mu}\left\{\sum_{D\in\mathcal{D}_h} (\eta_{1,D} + \eta_{2,D} + \eta_{3,D})^2\right\}^{1/2}$$

*where* $\mu = \min\{c\}$.

*Proof.* We first notice that according to the definition of the $\varepsilon-$norm,

$$\mu \|u - u_h\|_\varepsilon \leq \mathcal{B}\left(u - u_h, \frac{u - u_h}{\|u - u_h\|_\varepsilon}\right).$$

As $\varphi := (u - u_h)/|||u - u_h||| \in \mathrm{H}_0^1$, we have $\mathcal{B}(u, \varphi) = (f, \varphi)$ by (4.1.4).

Then

$$\begin{aligned}
\mu \|u-u_h\|_\varepsilon &= \mathcal{B}(u-u_h,\varphi) = \mathcal{B}(u,\varphi) - \mathcal{B}(u_h,\varphi) \\
&= \int_\Omega f\varphi dx - \int_\Omega (\varepsilon\nabla u_h - bu_h)\nabla\varphi dx - \int_\Omega cu_h\varphi dx \\
&= \int_\Omega f\varphi dx - \int_\Omega (\varepsilon\nabla u_h - bu_h + \mathbf{t}_h)\nabla\varphi + \int_\Omega \mathbf{t}_h\nabla\varphi dx - \int_\Omega cu_h\varphi dx \\
&= \int_\Omega f\varphi dx - \int_\Omega (\varepsilon\nabla u_h - bu_h + \mathbf{t}_h)\nabla\varphi dx - \int_\Omega \mathrm{div}\mathbf{t}_h\varphi dx - \int_\Omega cu_h\varphi dx \\
&= \int_\Omega (f - \mathrm{div}\mathbf{t}_h - cu_h)\varphi dx - \int_\Omega (\varepsilon\nabla u_h - bu_h + \mathbf{t}_h)\nabla\varphi dx
\end{aligned}$$

We estimate each integral separately.
Let $D \in \mathcal{D}_h^{\mathrm{int}}$. From Lemma 4.2.2 we have that $P_D(f-cu_h) - \mathrm{div}\mathbf{t}_h = 0$. On the other hand, $\int_D (f-cu_h)_D dx = \int_D (f-cu_h)dx$. Then,

$$\begin{aligned}
\int_D (f - \mathrm{div}\mathbf{t}_h - cu_h)\varphi_D dx &= \int_D \left[f - cu_h - P_D(f-cu_h)\right]\varphi_D\, dx \\
&= \int_D \left[(f-cu_h)_D - P_D(f-cu_h)\right]\varphi_D\, dx
\end{aligned}$$

Now, writting

$$\int_D (f - \mathrm{div}\mathbf{t}_h - cu_h)\varphi\, dx = \int_D (f - \mathrm{div}\mathbf{t}_h - cu_h)(\varphi-\varphi_D) + \int_D (f - \mathrm{div}\mathbf{t}_h - cu_h)\varphi_D\, dx$$

we have

$$\int_D (f - \mathrm{div}\mathbf{t}_h - cu_h)\varphi\, dx = \int_D (f - \mathrm{div}\mathbf{t}_h - cu_h)(\varphi-\varphi_D) + \int_D \left[(f-cu_h)_D - P_D(f-cu_h)\right]\varphi_D\, dx$$

Then,

$$\begin{aligned}
\mu \|u-u_h\|_\varepsilon \le & \sum_{D\in\mathcal{D}_h^{\mathrm{int}}} \left\{ \int_D (f - \mathrm{div}\mathbf{t}_h - cu_h)(\varphi-\varphi_D) - \int_D (\varepsilon\nabla u_h - bu_h + \mathbf{t}_h)\nabla\varphi\, dx + \right. \\
& \left. + \int_D \left[(f-cu_h)_D - P_D(f-cu_h)\right]\varphi_D\, dx \right\} \\
& + \sum_{\mathcal{D}_h^{\mathrm{ext}}} \left\{ \int_D (f - \mathrm{div}\mathbf{t}_h - cu_h)\varphi\, dx - \int_D (\varepsilon\nabla u_h - bu_h + \mathbf{t}_h)\nabla\varphi\, dx \right\} \\
\le & \sum_{D\in\mathcal{D}_h^{\mathrm{int}}} \left\{ \|f - \mathrm{div}\mathbf{t}_h - cu_h\|_{\mathrm{L}^2(D)} \|\varphi-\varphi_D\|_{\mathrm{L}^2(D)} + \|\varepsilon\nabla u_h - bu_h + \mathbf{t}_h\|_{\mathrm{L}^2(D)} \|\nabla\varphi\|_{\mathrm{L}^2(D)} + \right. \\
& \left. + \|(f-cu_h)_D - P_D(f-cu_h)\|_{\mathrm{L}^2(D)} \|\varphi_D\|_{\mathrm{L}^2(D)} \right\} \\
& + \sum_{\mathcal{D}_h^{\mathrm{ext}}} \left\{ \|f - \mathrm{div}\mathbf{t}_h - cu_h\|_{\mathrm{L}^2(D)} \|\varphi\|_{\mathrm{L}^2(D)} + \|\varepsilon\nabla u_h - bu_h + \mathbf{t}_h\|_{\mathrm{L}^2(D)} \|\nabla\varphi\|_{\mathrm{L}^2(D)} \right\}
\end{aligned}$$

Using the result of Lemma 4.2.3 and the definition of the estimators we obtain

$$\begin{aligned}
\mu \|u-u_h\|_\varepsilon \leq \sum_{D\in\mathcal{D}_h^{\text{int}}} & \Big\{ m_D \|f - \text{div}\mathbf{t}_h - cu_h\|_{\mathrm{L}^2(D)} + \left\| \varepsilon^{1/2}\nabla u_h + \varepsilon^{-1/2}(\mathbf{t}_h - bu_h)\right\|_{\mathrm{L}^2(D)} + \\
& \|\Pi_D(f-cu_h) - P_Q(f-cu_h)\|_{\mathrm{L}^2(D)}\Big\} \|\varphi\|_{\varepsilon,D} + \\
& + \sum_{\mathcal{D}_h^{\text{ext}}} \Big\{ m_D \|f - \text{div}\mathbf{t}_h - cu_h\|_{\mathrm{L}^2(D)} + \left\| \varepsilon^{1/2}\nabla u_h + \varepsilon^{-1/2}(\mathbf{t}_h - bu_h)\right\|_{\mathrm{L}^2(D)} \Big\} \|\varphi\|_{\varepsilon,D} \\
\leq & \sum_{D\in\mathcal{D}_h} (\eta_1+\eta_2+\eta_3) \|\varphi\|_{\varepsilon,D} \\
\leq & \left\{ \sum_{D\in\mathcal{D}_h} (\eta_1+\eta_2+\eta_3)^2 \right\}^{1/2} \left\{ \sum_{D\in\mathcal{D}_h} \|\varphi\|^2_{\varepsilon,D} \right\}^{1/2} \\
\leq & \left\{ \sum_{D\in\mathcal{D}_h} (\eta_1+\eta_2+\eta_3)^2 \right\}^{1/2}
\end{aligned}$$

□

## 4.3 A posteriori error estimator for $\mathcal{Q}_1$ approximation

Similar ideas can be used for rectangular partitions.
For nonnegative integers $k, m$ we call $\mathcal{Q}_{k,m}$ the space of polynomials of the form

$$q(x_1,x_2) = \sum_{i=0}^{k}\sum_{j=0}^{m} a_{ij} x_1^i x_2^j$$

then, the $\mathbf{RT}_1(K)$ is given by

$$\mathbf{RT}_1(K) = \mathcal{Q}_{2,1}(K) \times \mathcal{Q}_{1,2}(K)$$

It is known that there exists an interpolation operator for $\mathbf{v} \in \mathrm{H}^1(K)^2$

$$\Pi_K : \mathrm{H}^1(K)^2 \to \mathbf{RT}_1(K)$$

given by the following degrees of freedom

$$\begin{aligned}
\int_{l_i} \Pi_K\mathbf{v}\cdot\mathbf{n}_i p_1 ds &= \int_{l_i} \mathbf{v}\cdot\mathbf{n}_i p_1 ds, \qquad \forall p_1 \in \mathcal{P}_1(l_i). i = 1,2,3,4. \\
\int_K \Pi_K\mathbf{v}\cdot\phi_1 dx &= \int_K \mathbf{v}\cdot\phi dx, \quad \forall\phi \in \mathcal{Q}_{0,1}(K)\times\mathcal{Q}_{1,0}(K)
\end{aligned}$$

Figure 4.5 shows the degrees of freedom.

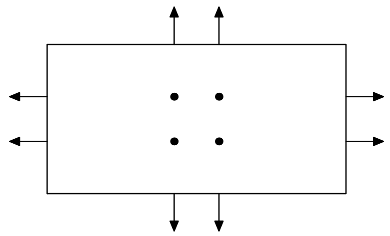


Figure 4.3: Degrees of freedom for $RT_1$

If $u_h$ is the solution of the discrete problem, as in the case of $\mathcal{P}_1$ approximation, we define $\mathbf{t}_h \in \mathbf{RT}_1(\mathcal{K}_h)$ by

$$\int_{l_i} \mathbf{t}_h \cdot \mathbf{n}_i p_1 ds = \int_{l_i} \{\!\{-(\varepsilon\nabla u_h - bu_h)\cdot \mathbf{n}_i\}\!\} p_1 ds, \qquad \forall p_1 \in \mathcal{P}_1(l_i).i = 1,2,3,4.$$
$$\int_K \mathbf{t}_h \cdot \phi dx = \int_K -(\varepsilon\nabla u_h - bu_h)\cdot \phi dx, \quad \forall \phi \in \mathcal{Q}_{0,1}(K)\times \mathcal{Q}_{1,0}(K)$$

where $l_i, i = 1,2,3,4$ are the edges of the rectangle $K$, $\forall K \in \mathcal{K}_h$.
We can define a posteriori error estimator as in the $\mathcal{P}_1$ case. However, the extension of Theorem 4.2.4 to this case is not straightforward and requires some further research. Anyway we will show that the estimator can be used for an adaptive procedure.

## 4.4 Numerical examples

### 4.4.1 $\mathcal{Q}_1$ approximation

First we consider an adaptive procedure for rectangular meshes. This is the simpler case because we already know the orientation of the mesh. Our goal is to show that similar meshes to those considered in previous chapters can be obtained by a posteriori adaptive procedures. In particular, we obtain the good order of convergence.
We consider problem (4.1.2), with

$$b = (1-2\varepsilon)(-1,-1)\,, \quad c = 2(1-\varepsilon),$$

and the right hand side given by

$$f(x_1,x_2) = -\left[x_1 - \left(\frac{1-e^{-\frac{x_1}{\varepsilon}}}{1-e^{-\frac{1}{\varepsilon}}}\right) + x_2 - \left(\frac{1-e^{-\frac{x_2}{\varepsilon}}}{1-e^{-\frac{1}{\varepsilon}}}\right)\right] e^{x_1+x_2}.$$

In this case the exact solution is

$$u(x_1,x_2) = \left[\left(x_1 - \frac{1-e^{-\frac{x_1}{\varepsilon}}}{1-e^{-\frac{1}{\varepsilon}}}\right)\left(x_2 - \frac{1-e^{-\frac{x_2}{\varepsilon}}}{1-e^{-\frac{1}{\varepsilon}}}\right)\right] e^{x_1+x_2},$$

The adaptive procedure is as follows. Let $V_K$ be the value of $\mathbf{t}_h + \varepsilon\nabla u_h - bu_h$ at the barycenter of the subelement $K$ of $R$, and $W_R$ the average of the $V_K$ , for $K \in R$.

Then we use as local error estimator

$$\eta_R := \varepsilon^{-1/2} \left\| \mathbf{t}_h + \varepsilon \nabla u_h - b u_h \right\|_{\mathrm{L}^2(R)}$$

and mark rectangles $R$ such that

$$\eta_R \geq \theta \max_{R' \in \mathcal{T}_h} \{\eta_{R'}\}$$

where $\theta$ is a parameter. In this case, we use $\theta = 1/2$.

If a rectangle is marked, we compute the angle between $W_R$ and the $x_1-$axis. If the angle is lower than $\pi/6$, we split the rectangle $R$ in the $x_1$ direction. If the angle is greater of $\pi/6$, we split the rectangle in the $x_2$ direction. If the angle is between $\pi/6$ and $\pi/3$, we do not refine.

We show the mesh obtained.

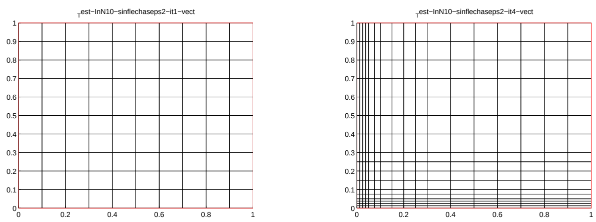


Figure 4.4: Initial mesh and mesh after 3 iterations

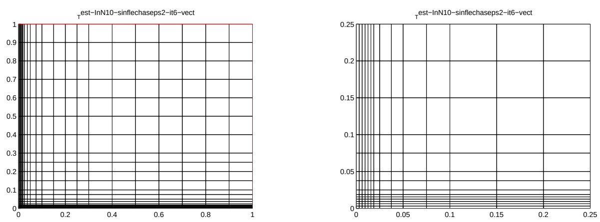


Figure 4.5: Mesh after 6 iterations

In the next table, we show the error against the number of elements.

| $N$ | $\|u-u_h\|_{L^2}$ | orden local | $\|u-u_h\|_\varepsilon$ | orden local |
|---|---|---|---|---|
| 100 | 0.1978640 | – | 9.15395788350 | – |
| 169 | 0.0839770 | 1.6332 | 8.2852698 | 0.2789 |
| 225 | 0.0263058 | 4.0557 | 6.3791625 | 0.9561 |
| 289 | 0.0070909 | 5.2370 | 3.8809885 | 1.9934 |
| 361 | 0.0025778 | 4.5486 | 2.1017833 | 2.7582 |
| 484 | 0.0020551 | 0.7728 | 1.1452654 | 2.0700 |
| 841 | 0.0019866 | 0.0613 | 0.6175016 | 1.1164 |
| 2209 | 0.0019409 | 0.0241 | 0.3384223 | 0.6198 |
| 7921 | 0.0004963 | 1.0679 | 0.1755448 | 0.5164 |
| 23409 | 0.0001245 | 1.2758 | 0.0943480 | 0.5737 |

Table 4.1: $\mathcal{Q}_1$ approximation error, $\varepsilon = 0.01$.

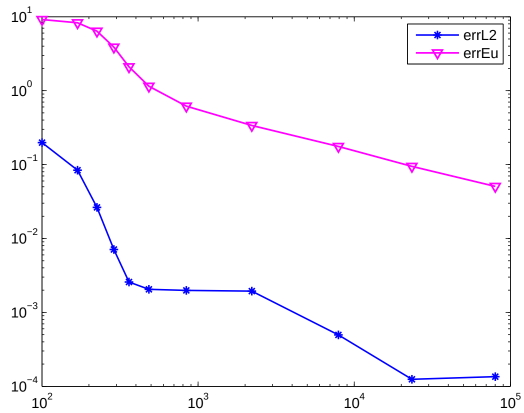



Figure 4.6: Order of $\mathcal{Q}_1$ error approximation

### 4.4.2 $\mathcal{P}_1$ approximation

We make an adaptive anisotropic procedure based on an approximation of the Hessian of the exact solution.
Given a symmetric, positive definite matrix $\mathcal{M}$, it defines a dot product:

$$\langle x, y\rangle_{\mathcal{M}} = x^t \mathcal{M} y, \forall x, y \in \mathbb{R}^2.$$

which induces a norm and a distance:

$$\|x\|_{\mathcal{M}} = \sqrt{\langle x, y\rangle_{\mathcal{M}}} = \sqrt{x^t \mathcal{M} x}, \qquad \forall x \in \mathbb{R}^2$$
$$d_{\mathcal{M}}(x, y) = \|x-y\|_{\mathcal{M}} = \sqrt{(x-y)^t \mathcal{M}(x-y)}, \qquad \forall x, y \in \mathbb{R}^2$$

As $\mathcal{M}$ is symmetric positive definite, there exists an orthonormal basis of $\mathbb{R}^2$ of eigenvectors and we can express $\mathcal{M} = VDV^t$, where $D$ is a diagonal matrix with positive entries

$\{\lambda_1, \lambda_2\}$, and $V$ is an orthogonal matrix which columns are the normalized eigenvectors of $\mathcal{M}$. In particular, we are using that $V^{-1} = V^t$.
How is the unit ball in the metric given by $\mathcal{M}$? Let $c = (c_1, c_2)$ be a point in $\mathbb{R}^2$ and $B^1_{\mathcal{M}}(C)$ the unit ball centered in $c$, then:

$$\begin{aligned} B^1_{\mathcal{M}}(c) &= \{x \in \mathbb{R}^2 : (x-c)^t M (x-c) = 1\} \\ &= \{x \in \mathbb{R}^2 : (x-c)^t V D V^t (x-c) = 1\} \\ &= \{x \in \mathbb{R}^2 : (V^t(x-c))^t D V^t (x-c) = 1\} \end{aligned}$$

Let $\bar{x} := V^t x$ the coordinates of $x$ in the eigenvector basis. Then, we can rewrite the last expression in this basis and obtain:

$$\begin{aligned} B^1_{\mathcal{M}}(c) &= \{\bar{x} \in \mathbb{R}^2 : (\bar{x}-\bar{c})^t D (\bar{x}-\bar{c}) = 1\} \\ &= \{\bar{x} \in \mathbb{R}^2 : \lambda_1(\bar{x}_1-\bar{c}_1)^2 + \lambda_2(\bar{x}_2-\bar{c}_2)^2 = 1\} \\ &= \{\bar{x} \in \mathbb{R}^2 : \left(\frac{\bar{x}_1-\bar{c}_1}{h_1}\right)^2 + \left(\frac{\bar{x}_2-\bar{c}_2}{h_2}\right)^2 = 1\} \end{aligned}$$

where $h_i = \frac{1}{\sqrt{\lambda_i}}, i = 1, 2$.
That is, the unit ball is an ellipse centered at $c$ with axis aligned with the eigenvectors of $\mathcal{M}$ of size $h_i = \frac{1}{\sqrt{\lambda_i}}$.
A usual way to obtain anisotropic triangulations to approximate in an efficient way a function $u$ which has anisotropic behavior is to use a metric related to the Hessian matrix, and to try to construct meshes with triangles which are equilateral in this metric.
In practice, the Hessian of the exact solution is not known, and therefore, some approximation has to be used. Since we have

$$\nabla u \approx \varepsilon^{-1}(bu_h - \mathbf{t}_h)$$

we can obtain an approximation of the second derivatives of $u$ taking derivatives of the right hand side.
We introduced this information as data in the code FreeFem++ in order to make an adaptive anisotropic refinement.
First, we consider the same example as in the previous subsection. In Figure 4.7 we show the mesh that we obtain after three iterations.

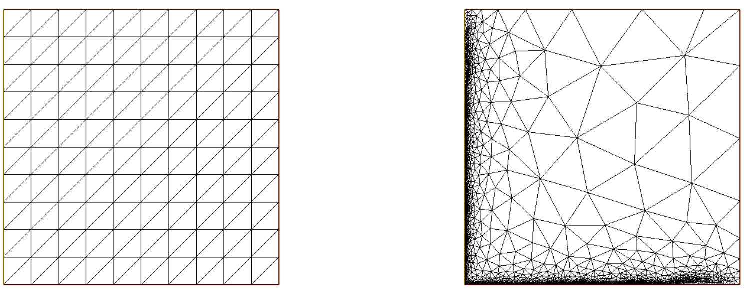


Figure 4.7: Initial mesh and mesh obtained after 3 iterations

Finally, we present the "double ramp" example which has apart of the boundary layer an interior layer. In this case, we take $\Omega = [0,1]^2 \setminus [1/2,1]^2$, $b = [-5,0], c = 0$ and $f = 1$.

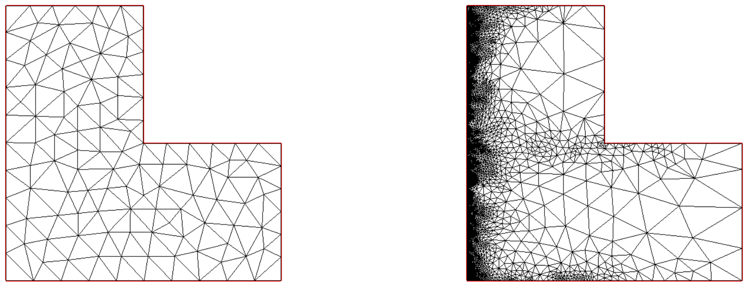


Figure 4.8: Initial mesh and anisotropic mesh after 3 iterations.

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