THE EFFECT OF REDUCED INTEGRATION IN EIGENVALUE PROBLEMS

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- Mass-lumping vs. Exact integration.
- Singular case: advantage of mass-lumping.
- Regular case: possibility of lower bounds.
- Old results on Finite Differences.
- Non-conforming elements (lower bounds).
- Numerical Examples.
- Adaptive meshes.

$$\begin{aligned} -\Delta u &= \lambda\, u \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

$$\|u\|_0 = 1$$

Solutions: $(\lambda_j, u_j), 0 < \lambda_1 \leq ... \leq \lambda_j \leq ...$

$$u_j \in H^{1+r}(\Omega)$$

$r = 1$ if Ω is convex and $r < \frac{\pi}{\omega}$ (with ω being the largest inner angle of Ω) otherwise.

FINITE ELEMENT APPROXIMATIONS

$$V_h = \{v_h \in H_0^1(\Omega) : \; v_h|_T \in \mathcal{P}_1 \quad \forall T \in \mathcal{T}_h\}$$

EXACT INTEGRATION

$$\int_\Omega \nabla u_h \cdot \nabla v_h = \lambda_h \int_\Omega u_h \, v_h \quad \forall v_h \in V_h$$

MASS-LUMPING

$$\int_\Omega \nabla u_h^{ml} \cdot \nabla v_h = \lambda_h^{ml} \int_\Omega I_h(u_h^{ml} \, v_h) \quad \forall v_h \in V_h$$

I_h : Linear interpolation at the vertices

Spectral convergence with optimal order known in both cases (Babuska-Osborn, Banerjee-Osborn).

$$\|u - u_h\|_1 = O(h^r)$$

$$\|u - u_h\|_0 = O(h^{2r})$$

$$|\lambda - \lambda_h| = O(h^{2r})$$

Lemma: For any $v_h \in V_h$,

$$\int_\Omega \Big(I_h(v_h^2) - v_h^2\Big)$$
$$= \frac{1}{12} \sum_{\ell \in \mathcal{E}_{\rm I}} ((v_h(p_1(\ell)) - v_h(p_2(\ell)))^2 \, |\Omega_\ell|$$

Ω_ℓ : union of two triangles sharing ℓ

In particular,

$$\int_\Omega I_h(v_h^2) \geq \int_\Omega v_h^2$$

Corollary: For any $v_h \in V_h$,

$$0 \leq \int_\Omega I_h(v_h^2) - v_h^2 \leq C h^2 \|\nabla v_h\|_0^2$$

Theorem: $\lambda_{h,j}^{ml} \leq \lambda_{h,j} \quad 1 \leq j \leq N_h$

Proof.

$$\lambda_{h,j} = \min_{V_{h,j}} \max_{v_h \in V_{h,j}} \frac{\int_\Omega |\nabla v_h|^2}{\int_\Omega v_h^2}$$

and,

$$\lambda_{h,j}^{ml} = \min_{V_{h,j}} \max_{v_h \in V_{h,j}} \frac{\int_\Omega |\nabla v_h|^2}{\int_\Omega I_h(v_h^2)}$$

where $V_{h,j}$ denote any subspace of V_h of dimension j.

And from the Lemma,

$$\frac{\int_\Omega |\nabla v_h|^2}{\int_\Omega I_h(v_h^2)} \leq \frac{\int_\Omega |\nabla v_h|^2}{\int_\Omega v_h^2} \qquad \forall v_h \in V_{h,j} \quad \square$$

I will drop the j from now on to simplify notation.

SINGULAR CASE

Theorem: $\|\nabla(u_h^{ml}-u)\|_0 = O(h^r)$, $r<1$, $\Rightarrow \lambda \le \lambda_h^{ml}$ for h small enough.

CONCLUSION: Mass-lumping is better in this case.

The proof uses:

$$\lambda_h^{ml} - \lambda = \|\nabla(u_h^{ml}-u)\|_0^2 - \lambda\|u_h^{ml}-u\|_0^2 \\ -\lambda_h^{ml}\left(\int_\Omega I_h((u_h^{ml})^2) - (u_h^{ml})^2\right)$$

The second and third terms are of higher order than the first one.

REGULAR CASE

Now the first and third terms are of the same order $O(h^2)$. So, we do not know whether

$$\lambda_h^{ml} - \lambda$$

is positive or negative.

From numerical experiments:

$$\lambda_h^{ml} \leq \lambda$$

for reasonable meshes! (but we don't have a proof).

For example: for uniform meshes in a square one can compute explicitly λ_h^{ml} and λ and observe that $\lambda_h^{ml} \leq \lambda$.

REMARK: A trivial example shows that in some cases $\lambda_h^{ml} \geq \lambda$. But, we could not find an example with a "reasonable" mesh such that $\lambda_h^{ml} \geq \lambda$.

OLD RESULTS ON FINITE DIFFERENCES

For uniform meshes mass-lumping corresponds to finite differences.

Forsythe (Pacific J. of Math. (1954, 1955)) proved the following:

Theorem: $\lambda_h^{ml} \leq \lambda - ah^2 + o(h^2)$, with $a > 0$ if Ω is convex.

Forsythe also conjectured that in the singular case $\lambda_h^{ml} \geq \lambda$ for h small enough (which is a particular case of our results).

Other references on lower bounds: Weinberger (Bull. AMS (1955), Comm. Pure Appl. Math (1956) , SIAM, 1974).

Asymptotic lower bounds for the singular case

Non-conforming elements of Crouzeix-Raviart

$$V_h^{NC} = \{v_h \in L^2(\Omega) : v_h|_T \in \mathcal{P}_1,\ \forall T \in \mathcal{T}_h,\ v_h \text{ is continuous at interior midside points and vanishes at boundary midside points}\}$$

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v_h = \lambda_h^{nc} \int_\Omega u_h v_h \quad \forall v_h \in V_h^{NC}$$

Remark: In this case the mass matrix obtained with exact integration is diagonal.

We introduce the "Edge average" interpolant $u_h^{ea} \in V_h^{NC}$ of u:

$$\int_\ell u_h^{ea} = \int_\ell u \quad \forall \text{edge } \ell$$

Then,

$$\int_\Omega \nabla_h (u - u_h^{ea}) \cdot \nabla_h v_h = 0 \quad \forall v_h \in V_h^{NC}$$

Since $\|u\|_0 = \|u_h\|_0 = 1$ we have

$$\lambda_h^{nc} + \lambda = \|\nabla_h u_h\|_0^2 + \|\nabla u\|_0^2$$

$$= \|\nabla_h(u - u_h)\|_0^2 + 2\int_\Omega \nabla_h u_h \cdot \nabla u$$

$$= \|\nabla_h(u - u_h)\|_0^2 + 2\int_\Omega \nabla_h u_h \cdot \nabla u_h^{ea}$$

$$= \|\nabla_h(u - u_h)\|_0^2 + 2\lambda_h^{nc}\int_\Omega u_h u_h^{ea}$$

$$= \|\nabla_h(u - u_h)\|_0^2 + 2\lambda_h^{nc} - \lambda_h^{nc}\|u_h - u_h^{ea}\|_0^2$$

$$+\lambda_h^{nc}\left(\|u_h^{ea}\|_0^2 - \|u\|_0^2\right)$$

Then,

$$\lambda - \lambda_h^{nc} = \|\nabla_h(u - u_h)\|_0^2 - \lambda_h^{nc}\|u_h - u_h^{ea}\|_0^2$$
$$+\lambda_h^{nc}\left(\|u_h^{ea}\|_0^2 - \|u\|_0^2\right)$$

Theorem: $\|\nabla(u_h^{ml} - u)\|_0 = O(h^r)$, $r < 1$, $\Rightarrow \lambda_h^{nc} \leq \lambda$ for h small enough.

Proof. In the above relation the second and third term are of higher order than the first one.
Second term: follows from L^2 estimates

Third term: we use the a-priori estimate

$$\|u\|_{2,p} \leq C\lambda\|u\|_{0,p}$$

for some $p > 1$ (which holds for any polygonal domain).
Then,

$$\left|\int_\Omega (u_h^{ea})^2 - \int_\Omega u^2\right|$$

$$= |\int_\Omega (u_h^{ea} - u)(u_h^{ea} + u)|$$

$$\leq C\|u\|_{0,\infty}\|u_h^{ea} - u\|_{0,p} \leq C(u,\lambda)h^2$$

Non-conforming method with reduced integration

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v_h = \lambda_h^{nc,ms} \int_\Omega I_h(u_h\, v_h)\ \forall v_h \in V_h^{NC}$$

ms: “MASS-SPREADING”

$$\lambda_h^{nc,ms} \leq \lambda_h^{nc}$$

Same proof as in the conforming case.

THEN,

$$\lambda_h^{nc,ms} \leq \lambda$$

For h small enough and u singular.

RELATIONS

$$V_h \subset V_h^{NC} \Rightarrow \lambda_h^{nc} \le \lambda_h \quad \text{and,} \quad \lambda_h^{nc,ms} \le \lambda_h^{ml}$$

$$\lambda_h^{ml} \le \lambda_h$$

$$\lambda_h^{nc,ms} \le \lambda_h^{nc}$$

SINGULAR CASE:

$$\lambda_h^{nc,ms} \le \lambda_h^{nc} \le \lambda \le \lambda_h^{ml}$$

Proved for h small.

REGULAR CASE:

$$\lambda_h^{nc,ms} \le \lambda_h^{ml} \le \lambda$$

No proof of the last inequality! (but is true in all our experiments with reasonable meshes).

CONCLUSIONS

IF u IS SINGULAR: USE MASS LUMPING AND NC TO BOUND BY ABOVE AND BELOW

IF u IS REGULAR: USE EXACT INTEGRATION AND MASS LUMPING TO BOUND BY ABOVE AND BELOW

IF WE DON'T KNOW WE CAN USE EXACT INTEGRATION AND NC WITH REDUCED INTEGRATION

REMARK: When Ω is not convex u may be singular or regular. For example, for the L shape domain, u_1 is singular and u_2 and u_3 are regular.

NUMERICAL EXAMPLES

First eigenvalue for L domain

Uniform refinement

number of nodes	$\lambda_{h,1}$	$\lambda_{h,1}^{ml}$
21	13.199179221542	9.071796769724
65	10.573955451157	9.641425460959
225	9.916549032001	9.693162213551
833	9.728372729312	9.673506476037
3201	9.66981732232	9.65620182015

First eigenvalue for Koch 2 domain

Uniform refinement

number of nodes	$\lambda_{h,1}$	$\lambda_{h,1}^{ml}$
37	46.993282224519	40.401005031470
121	42.121650466929	40.635844194708
433	40.796435658176	40.438418441151
1633	40.39662738666	40.30828543856

First eigenvalue for Koch 3 domain

Uniform refinement

number of nodes	$\lambda_{h,1}$	$\lambda_{h,1}^{ml}$
329	40.94016461357	40.34117804088
1217	40.17948566684	40.03394074483

REGULAR SOLUTIONS

First eigenvalue for a square domain

$$\lambda_1 = \pi^2/2 = 4.93480220054468$$

number of nodes	$\lambda_{h,1}^{ml}$
5	15.789473684211
8	3.123922607067
14	3.091168190991
26	3.903152296554
55	4.646900604880
116	4.742942128240
259	4.888070813057

number of nodes	$\lambda_{h,1}^{ml}$
37	4.442736170666
121	4.810061215139
433	4.903574330061

First eigenvalue for an equilateral triangle

$$\lambda_1 = \frac{16\pi^2}{3} = 52.63789014....$$

number of nodes	$\lambda_{h,1}^{ml}$
15	42.666666666667
45	49.987109344163
153	51.964905805628
561	52.468994312245

number of nodes	$\lambda_{h,1}^{ml}$
28	46.839659383781
91	51.318366941074
325	52.331156996197

First eigenvalue for a circle

$$\lambda_1 = 5.78318596294679...$$

number of nodes	$\lambda_{h,1}^{ml}$
25	4.86961861262
81	5.52128687409
289	5.71540330139
1089	5.76609636891

number of nodes	$\lambda_{h,1}^{ml}$
41	5.469108031446
145	5.698898965742
545	5.761760229781

ADAPTIVE REFINEMENT

First eigenvalue for L domain

Exact integration

number of nodes	$\lambda_{h,1}$
21	13.19917922154
35	11.17861899716
52	10.61960009902
87	10.23370713553
146	9.96803288895
190	9.88390214919
325	9.77175458628
461	9.73564019209
666	9.70629111767
1007	9.68265866024
1420	9.67110120080
2324	9.65833094103

ADAPTIVE REFINEMENT

First eigenvalue for L domain

Mass lumping

number of nodes	$\lambda_{h,1}^{ml}$
21	9.07179676972
35	9.58193479076
60	9.47118381681
97	9.53020788804
152	9.64055123484
218	9.55402279885
345	9.61364966132
478	9.60112464514
687	9.60626373321
993	9.62715585078
1682	9.62863961061
2411	9.63200652455
3268	9.63197587247

NON-CONFORMING ELEMENTS

First eigenvalue for L domain

Uniform refinement

number of nodes	$\lambda_{h,1}^{nc}$
44	9.029162344073
160	9.205405718064
608	9.466269451586

First eigenvalue for Koch 2 domain

Uniform refinement

number of nodes	$\lambda_{h,1}^{nc}$
84	37.00124133068
312	38.84043356529
1200	39.74253482521