Error estimates for an average interpolation on anisotropic $Q_1$ elements

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January 16, 2004
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Joint work with Ariel Lombardi
- The classic Hardy inequality and its dual inequality.

- Their application in error estimates for functions in weighted Sobolev spaces.

- Advantages over compactness arguments.

- Example of application in 1D. Graded meshes for singularly perturbed problems.

- Error estimates for narrow elements in 2D and 3D. Necessity of Average Interpolants.

- The generalized Hardy inequality in 2D and 3D.

- Estimates for average interpolants in anisotropic rectangular elements (in 2D and 3D).

- Applications.

- Numerical examples.
We are interested in estimates with weighted norms on the right hand side.

REASONS: to approximate singular functions or functions with large derivatives.

THE USE OF HARDY INEQUALITY

THE 1D CASE:

CLASSIC HARDY INEQUALITY:

\[ \left\| \frac{v}{d} \right\|_{L^2(a,b)} \leq 2\left\| v' \right\|_{L^2(a,b)} \]

\( v \in H_0^1(a,b) \quad \text{d}(x) \quad \text{distance to the boundary of (a, b)} \)
DUAL INEQUALITY:

\[ u \in H^1(a, b) \quad , \quad \int_a^b u = 0 \]

then,

\[ \|u\|_{L^2(a,b)} \leq 2\|du'\|_{L^2(a,b)} \]

**Proof:** Define \( v \in H^1_0(a, b) \)

\[ v(x) = -\int_a^x u(y)dy \]

Using the Hardy inequality for \( v \), we have

\[ \|u\|_{L^2}^2 = \int_a^b u'(x)v(x)dx \]

\[ \leq \left\| \frac{v}{d} \right\|_{L^2} \|du'\|_{L^2} \leq 2\|u\|_{L^2}\|du'\|_{L^2} \]
ERROR ESTIMATE FOR THE DERIVATIVE:

\[ I = (a, b) \quad u_I \in \mathcal{P}_1 \] Lagrange interpolation

\[ u \in H^2(I) \quad 0 \leq \alpha \leq 1 \]

\[ \|(u - u_I)'\|_{L^2(I)} \leq 2|I|^{1-\alpha}\|d^\alpha u''\|_{L^2(I)} \]

Proof: Use that

\[ \int_I (u - u_I)' = 0 \]

and the DUAL HARDY INEQUALITY.
REMARKS: Estimates of this kind can be proved by compactness arguments. However, our method has the following advantages:

1- In the n-dimensional case we obtain explicit information on the dependence of the constants on the geometry of the elements. This is important in our analysis for anisotropic elements.

2- Our argument gives better results: One can not obtain the case $\alpha = 1$ by compactness. This case is of interest in some applications.

$$H^{1,d} = \{ v \in L^2(I) : dv' \in L^2(I) \}$$

with norm

$$\| u \|_{H^{1,d}} = \| u \|_{L^2}^2 + \| du' \|_{L^2}^2$$
The inclusion $H^{1,d} \subset L^2$ IS NOT COMPACT!

**EXAMPLE** (Ariel Lombardi): $I = (0,1)$

Consider the sequence

$$u_n(x) = \begin{cases} 
  nx & 0 < x < \frac{1}{n} \\
  2 - nx & \frac{1}{n} \leq x < \frac{2}{n} \\
  0 & \frac{2}{n} \leq x < 1 
\end{cases}$$

and $w_n = \sqrt{n} u_n$. Then, $\|w_n\|_{H^{1,d}}^2 = \frac{10}{3}$. If $H^{1,d} \subset L^2$ is compact, there exists a subsequence $w_n$ such that

$$w_n \to w \quad \text{in} \quad L^2$$

but $w_n(x) \to 0$ $\forall x \in I$ and so $w = 0$. But,

$$\|w_n\|_{L^2}^2 = \frac{2}{3}$$

**CONTRADICTION!**
APPLICATIONS

GRADED MESHES: AN EXAMPLE IN 1D

CONVECTION-DIFFUSION EQUATION:

\[-\varepsilon u'' - b(x)u' + c(x)u = f \text{ in } (0, 1)\]
\[u(0) = u(1) = 0\]

\[b(x) \geq b_0 > 0 \quad \forall \ x \in (0, 1)\]

There is a boundary layer at \( x = 0 \).

GRADED MESH:

\[x_0 = 0 < x_1 < \cdots < x_N\]

\[u_I \text{ piecewise } \mathcal{P}_1 \text{ Lagrange interpolation}\]
Error estimate for the first interval \((0, x_1)\):

\[
\varepsilon \| (u - u_I)' \|^2_{L^2(0, x_1)} \leq 4\varepsilon \| xu'' \|^2_{L^2(0, x_1)}
\]

\[
\leq 4\varepsilon^{-2\beta} x_1^{2(1-\alpha)} \varepsilon^{1+2\beta} \| x^\alpha u'' \|^2_{L^2(0, x_1)}
\]

**REMARK:** We will use this estimate for \(\alpha < 1\), but it is important to have a constant independent of \(\alpha\).

Choose:

\[
\beta = 1 - \alpha = \frac{1}{\log(\frac{1}{\varepsilon})}
\]

So, \(\varepsilon^{-\beta} = e\) and then,

\[
\varepsilon \| (u - u_I)' \|^2_{L^2(0, x_1)} \leq C x_1^{2(1-\alpha)} \varepsilon^{1+2\beta} \| x^\alpha u'' \|^2_{L^2(0, x_1)}
\]
Take $h > 0$ and $x_1 \leq h^{\frac{1}{1-\alpha}}$. Then,
\[
\varepsilon \| (u - u_I)' \|_{L^2(0,x_1)}^2 \leq C h^2 \varepsilon^{1+2\beta} \| x^\alpha u'' \|_{L^2(0,x_1)}^2
\]

Error estimate for the other intervals $(x_j, x_{j+1})$:
\[
\varepsilon \| (u - u_I)' \|_{L^2(x_j,x_{j+1})}^2 \leq 4 \varepsilon^{-2\beta} (x_{j+1} - x_j)^2 \varepsilon^{1+2\beta} \| u'' \|_{L^2(x_j,x_{j+1})}^2
\]

Now choose $x_j$ such that:
\[
x_{j+1} \leq x_j + hx_j^\alpha
\]
Then,
\[
\varepsilon \|(u - u_I)'\|^2_{L^2(x_j, x_{j+1})} \leq C h^2 x_j^{2 \alpha} \varepsilon^{1+2\beta} \|u''\|^2_{L^2(x_j, x_{j+1})}
\]
\[
\leq C h^2 \varepsilon^{1+2\beta} \|x^{\alpha} u''\|^2_{L^2(x_j, x_{j+1})}
\]

**WEIGHTED A PRIORI ESTIMATE:**
\[
\varepsilon^{1+2\beta} \|x^{\alpha} u''\|^2_{L^2} \leq C
\]
if \( \alpha \geq 0, \beta \geq 0, \alpha + \beta = 1 \)

Consequently,
\[
\varepsilon \|(u - u_I)'\|^2_{L^2(0,1)} \leq C h^2
\]
with \( C \) independent of \( \varepsilon \).
\[ N: \text{Number of nodes in graded mesh} \implies h \sim \frac{\log N}{N} \]

Therefore,

\[ \varepsilon \| (u - u_I)' \|^2_{L^2(0,1)} \leq C \frac{\log N}{N} \]

Similar weighted estimates, but with different powers of \( d(x) \), can be proved for the \( L^2 \) interpolation error.

**\( L^2 \)- ERROR ESTIMATE:**

\[ ||u - u_I||_{L^2(I)} \leq \frac{C}{1 - 2\alpha} |I|^{1-\alpha} ||d^\alpha u'||_{L^2(I)} \]

for \( 0 \leq \alpha < \frac{1}{2} \).
The following example shows that the estimate is not true for \(\alpha > \frac{1}{2}\):

\[
\begin{align*}
u_n(x) &= \begin{cases} 
  nx & \text{if } 0 \leq x \leq \frac{1}{n} \\
  1 & \text{if } \frac{1}{n} < x \leq 1
\end{cases}
\end{align*}
\]

Then,

\[
\|u_n - u_{n,I}\|_{L^2(0,1)} \to \left(\int_0^1 (1 - x)^2 \, dx\right)^{\frac{1}{2}} = \frac{1}{\sqrt{3}}
\]

while

\[
\|x^\alpha u_n'\|_{L^2(0,1)}^2 = \int_0^{\frac{1}{n}} n^2 x^{2\alpha} \, dx = \frac{1}{2\alpha + 1} n^{1-2\alpha} \to 0
\]

for \(\alpha > \frac{1}{2}\)
Using these estimates and the weighted a priori estimate

\[ \varepsilon^{2\beta} \| x^\alpha u' \|_{L^2}^2 \leq C \]

if \( \alpha \geq 0, \beta \geq 0, \alpha + \beta = \frac{1}{2} \)

Choosing,

\[ \beta = \frac{1}{2} - \alpha = \frac{1}{\log \frac{1}{\varepsilon}} \]

**ERROR ESTIMATE IN ENERGY NORM**

\[ \| v \|_{\varepsilon}^2 = \| v \|_{L^2}^2 + \varepsilon^2 \| v' \|_{L^2}^2 \]

\[ \| u - u_I \|_{\varepsilon} \leq C \log \frac{1}{\varepsilon} \frac{\log N}{N} \]
THE 2D AND 3D CASES

Classic theory uses “regularity assumption”:

\[
\frac{h_T}{\rho_T} \leq C
\]

\(h_T\) exterior diameter, \(\rho_T\) interior diameter. For both Lagrange and Average Interpolants.

BUT: IT’S KNOWN THAT IT IS NOT NEEDED! First works: Babuska-Aziz, Jamet (1976).

Other references: Krizek, Al Shenk, Dobrowolski, Apel, Nicaise, Formaggia, Perotto, Acosta, D., etc..
FOR EXAMPLE: RECTANGULAR ELEMENTS

$K$ reference element

Given $u \in H^2(K)$, let $p \in P_1$ be such that

$$\left\| \frac{\partial}{\partial x} (u - p) \right\|_{L^2(K)} \leq C \left\| \nabla \frac{\partial u}{\partial x} \right\|_{L^2(K)}$$

For example: $p_1$ the averaged Taylor polynomial of degree 1.

Let $u_I \in Q_1$ be the Lagrange interpolation.
\[ \left\| \frac{\partial}{\partial x} (u - u_I) \right\|_{L^2(K)} \leq \left\| \frac{\partial}{\partial x} (u - p) \right\|_{L^2(K)} + \left\| \frac{\partial}{\partial x} (p - u_I) \right\|_{L^2(K)} \]

So, it is enough to estimate \( \left\| \frac{\partial}{\partial x} (p - u_I) \right\|_{L^2(K)} \)

We use: for \( v = p - u_I \in Q_1(K) \)

\[ \left\| \frac{\partial v}{\partial x} \right\|_{L^2(K)}^2 \sim \left| v(B) - v(A) \right|^2 + \left| v(D) - v(C) \right|^2 \]
\[|v(B) - v(A)| = |(p(B) - u(B)) - (p(A) - u(A))|\]

\[= \left| \int_s \frac{\partial}{\partial x} (p-u) \right| \leq C \left\{ \left\| \frac{\partial}{\partial x} (p-u) \right\|_{L^2(K)} + \left\| \nabla \frac{\partial u}{\partial x} \right\|_{L^2(K)} \right\}\]

where we have used a trace theorem.

Analogously we bound \( |v(D) - v(C)| \) and so we obtain:

\[\left\| \frac{\partial}{\partial x} (u - u_I) \right\|_{L^2(K)} \leq C \left\| \nabla \frac{\partial u}{\partial x} \right\|_{L^2(K)}\]

\( \frac{\partial^2 u}{\partial y^2} \) DOES NOT APPEAR!
Therefore, changing variables we obtain for a rectangle $R$:

\[ \left\| \frac{\partial}{\partial x} (u - u_I) \right\|_{L^2(R)} \leq C \left\{ h_1 \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L^2(R)} + h_2 \left\| \frac{\partial^2 u}{\partial x \partial y} \right\|_{L^2(R)} \right\} \]

THE CONSTANT $C$ IS INDEPENDENT OF THE RELATION BETWEEN $h_1$ and $h_2$ !

A SIMILAR ESTIMATE IN 3D IS NOT TRUE !!

WHAT FAILS IN 3D IN THE ARGUMENT GIVEN ABOVE?
THE TRACE THEOREM:

\[ \|u\|_{L^2(s)} \leq C\|u\|_{H^1(R)} , \]

WHERE \( s \) IS AN EDGE OF \( R \), IS NOT TRUE!

Counterexamples for the interpolation error estimate were given by:


They showed that the constant in the estimate

\[ \|u - u_I\|_{H^1(R_\varepsilon)} \leq C_\varepsilon h\|u\|_{H^2(R_\varepsilon)} \]

goes to \( \infty \) when \( \varepsilon \to 0 \)
THIS IS ONE REASON TO WORK WITH AVERAGE INTERPOLANTS.

The other reason is the classic one: to approximate non-smooth functions.
GENERALIZED HARDY INEQUALITY:

$D \subset \mathbb{R}^n$ convex domain, $u \in H^1_0(D)$

d(x) distance to the boundary

$$\left\| \frac{u}{d} \right\|_{L^2(D)} \leq 2\left\| \nabla u \right\|_{L^2(D)}$$

ANISOTROPIC VERSION

$R = \prod_{i=1}^{n} (a_i, b_i)$ \quad $h_i = b_i - a_i$

$u \in H^1_0(R)$, $\delta_R$ is a “normalized distance”:

$$\delta_R(x) = \min \left\{ \frac{x_i - a_i}{h_{R,i}}, \frac{b_i - x_i}{h_{R,i}} : 1 \leq i \leq n \right\}$$
\[
\left\| \frac{u}{\delta R} \right\|_{L^2(R)} \leq 2 \sum_{i=1}^{n} h_i \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2(R)}.
\]

**DUAL INEQUALITY**

\( \frac{1}{\delta} \leq h_i \leq \delta, \quad \psi \in C_0(R), \quad \int_R \psi = 1. \)

\( u \in H^1(R) \) such that \( \int_R u \psi = 0 \)

\( \|u\|_{L^2(R)} \leq C \|d\nabla u\|_{L^2(R)} \)

with \( C \) depending only on \( \delta \) and \( \psi \).

**Proof :** REPEAT THE ARGUMENT GIVEN IN 1D:

\( v := u - (\int_R u)\psi \) has vanishing mean value.
So, there exists $F \in H^1_0(R)^2$ such that

$$-\text{div } F = v$$

and

$$\|F\|_{H^1_0(R)^2} \leq C\|v\|_{L^2(R)}$$

$C$ DEPENDS ONLY ON $\delta$: It follows from the explicit bound given in DM.

Since $\int_R u\psi = 0$, then

$$\|u\|_{L^2(R)}^2 = \int_R uv = -\int_R u \text{div } F$$

and the proof finish as in the 1D case.
AN AVERAGE INTERPOLANT

Our construction is a modification of that in D. (Math. Comp. 1999).

DIFFERENCE: We do not use reference elements for the definition!

In this way we can relax the regularity assumptions on the mesh.

ASSUMPTION: local regularity in each direction

\( h_{R,i} \leq \sigma \leq h_{S,i} \leq n. \)
OUR ERROR ESTIMATES DEPEND ONLY ON $\sigma$.

$\mathcal{N}_{in}$ set of interior nodes. For $z \in \mathcal{N}_{in}$:

$$h_{z,i} = \min\{h_{R,i} : z \text{ is a vertex of } R\}, \quad 1 \leq i \leq n$$

$$\tilde{R} = \bigcup\{S \in \mathcal{T} : S \text{ is a neighboring element of } R\}.$$ 

and

$$R_v = \bigcup\{S \in \mathcal{T} : v \text{ is a vertex of } S\}.$$
Taylor polynomial of \( u \) of degree 1 at the point \( x \):

\[
Tu(x, y) = u(x) + \nabla u(x) \cdot (y - x)
\]

**AVERAGED TAYLOR POLYNOMIAL**

\[\psi \in C^\infty(\mathbb{R}^n) \quad \int \psi = 1\]

\[\text{supp} \psi \subset B(0, r) \quad r \leq 1/\sigma\]

For \( z \in \mathcal{N}_{in} \) define:

\[
\psi_z(x) = \frac{1}{h_{z,1} h_{z,2} h_{z,3}} \psi \left( \frac{z_1 - x_1}{h_{z,1}}, \frac{z_2 - x_2}{h_{z,2}}, \frac{z_3 - x_3}{h_{z,3}} \right)
\]

We introduce the averaged Taylor polynomial of order 1 of \( u \) at \( z \in \mathcal{N}_{in} \):

\[
T_{1,z}(u)(y) = \int Tu(x, y) \psi_z(x) \, dx
\]
Analogously, we introduce the average of $u$ at $z \in \mathcal{N}_{in}$:

$$T_{0,z}(u) = \int u(x)\psi_z(x)dx$$

**INTERPOLANT:**

For $u \in H_0^1(\Omega)$ define $\Pi u$ as the unique piecewise $Q_1$ function such that, for $z \in \mathcal{N}_{in}$,

$$\Pi u(z) = T_{1,z}(u)(z)$$

$\Pi u(z) = 0$ for boundary nodes $z$. 
\[ \Pi u(x) = \sum_{z \in \mathcal{N}_{in}} T_{1,z}(u)(z)\lambda_z(x) \]

\( \lambda_z: \) standard basis functions

**ERROR ESTIMATES IN WEIGHTED NORMS**

**\( L^2 \)- ERROR ESTIMATES:**

**THEOREM:** If \( R \) is an interior element,

\[
\|u - \Pi u\|_{L^2(R)} \leq C \sum_{i=1}^{n} h_{R,i} \left\| \delta_{\tilde{R}} \frac{\partial u}{\partial x_i} \right\|_{L^2(\tilde{R})} 
\]

\[
C = C(\sigma, \psi)
\]
Proof: First we prove stability:

\[ \| \Pi u \|_{L^2(R)} \leq C \| u \|_{L^2(\bar{R})} \]

then, for \( z_1 \) a vertex of \( R \),

\[
\| u - \Pi u \|_{L^2(R)} \leq \| u - T_{0,z_1}(u) \|_{L^2(R)} \\
+ \| \Pi (T_{0,z_1}(u) - u) \|_{L^2(R)} \\
\leq C \| u - T_{0,z_1}(u) \|_{L^2(R)}
\]

and use the dual Hardy inequality. \( \square \)

\( H^1 \)- ERROR ESTIMATES:

**Theorem**: If \( R \) is an interior element,

\[
\left\| \frac{\partial}{\partial x_j} (u - \Pi u) \right\|_{L^2(R)} \leq C \sum_{i=1}^{n} h_{R,i} \left\| \delta \bar{R} \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L^2(\bar{R})}
\]
Proof: VERY TECHNICAL!

IDEA: Decompose the error in two parts (as in the proof of the estimate for Lagrange interpolation)

\[ u - \Pi u = (u - T_{1,z_1}(u)) + (T_{1,z_1}(u) - \Pi u) \]

First term:

\[ \left\| \frac{\partial (u - T_{1,z_1}(u))}{\partial x_1} \right\|_{L^2(R)} \]

and use the dual Hardy inequality.

Second term: \( w := T_{1,z_1}(u) - \Pi u \in Q_1 \) then

\[ \frac{\partial w}{\partial x_1} = \sum_{i=1}^{4} \left( w(z_i) - w(z_i+4) \right) \frac{\partial \lambda_{z_i}}{\partial x_1} \]
So,

$$\left\| \frac{\partial w}{\partial x_1} \right\|_{L^2(R)} \leq \sum_{i=1}^{4} |w(z_i) - w(z_i + 4)| \left\| \frac{\partial \lambda z_i}{\partial x_1} \right\|_{L^2(R)}$$

But,

$$\left\| \frac{\partial \lambda z_i}{\partial x_1} \right\|_{L^2(R)} \leq C \left( \frac{h_{z_i,2} h_{z_i,3}}{h_{z_i,1}} \right)^{1/2}$$
So, we have to estimate $|w(z_i) - w(z_{i+4})|$

For example,

$$w(z_1) - w(z_5) = T_{1,z_5}(u)(z_5) - T_{1,z_1}(u)(z_5)$$

$$= \int Tu(x, z_5)\psi_{z_5}(x)dx - \int Tu(x, z_5)\psi_{z_1}(x)dx$$

Which, after long technical details! can be bounded by

$$C \frac{1}{h_{z_1,2} h_{z_1,3}} \sum_{i=1}^{3} h_{z_1,i} \int \left| \frac{\partial^2 u}{\partial x_1 \partial x_i} (\bar{x}) \right| \psi(\bar{x}) d\bar{x}$$

where the function $\psi(\bar{x})$ has support in $\tilde{R}$. Then, use the Cauchy-Schwarz inequality and the Hardy inequality for $\psi(\bar{x})$.  \[\square\]
APPLICATIONS

REACTION-DIFFUSION EQUATION:

\[-\varepsilon^2 \Delta u + u = f \quad \text{in} \quad \Omega = (0, 1)^2\]
\[u = 0 \quad \text{on} \quad \partial\Omega\]

\[\|v\|_\varepsilon^2 = \|v\|_{L^2}^2 + \varepsilon^2 \|
abla v\|_{L^2}^2\]

\[\|u - u_h\|_\varepsilon \leq \|u - \Pi u\|_\varepsilon\]

**THEOREM:** With graded mesh:

\[\|u - u_h\|_\varepsilon \leq C \frac{\log N}{\sqrt{N}}\]
CONVECTION-DIFFUSION EQUATION:

\[-\varepsilon \Delta u - u_x - u_y = 1 \quad \text{in } \Omega = (0, 1)^2\]

\[u = 0 \quad \text{on } \partial \Omega\]

For graded meshes it follows from our weighted error estimates:

\[\| u - \Pi u \|_\varepsilon \leq C \log \frac{1}{\varepsilon} \frac{\log N}{\sqrt{N}}\]

However, in this case we don’t have:

\[\| u - u_h \|_\varepsilon \leq C \| u - \Pi u \|_\varepsilon\]

So

\[\| u - u_h \|_\varepsilon \quad ? \quad \text{WE DON'T KNOW!}\]

PRELIMINARY NUMERICAL EXPERIMENTS SHOW GOOD RESULTS!
Numerical Solution with graded mesh

$\varepsilon = 0.01$
Order of convergence with graded mesh and $Q_1$ elements:

\[
\begin{align*}
\varepsilon & = 0.01 \\
\end{align*}
\]

Order of convergence with Shishkin mesh and $Q_1$ elements:

\[
\begin{align*}
\varepsilon & = 0.01 \\
\end{align*}
\]
Numerical Solution with graded mesh

$\varepsilon = 0.0001$
Order of convergence with graded mesh and $Q_1$ elements:

\[ \varepsilon = 0.0001 \]

Order of convergence with Shishkin mesh and $Q_1$ elements:

\[ \varepsilon = 0.0001 \]
### ORDER OF CONVERGENCE

**Graded Meshes, $\varepsilon = 0.01$**

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**Shishkin Meshes, $\varepsilon = 0.01$**

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ORDER OF CONVERGENCE

Graded Meshes, $\varepsilon = 0.0001$

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Shishkin Meshes, $\varepsilon = 0.0001$

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FOURTH ORDER MODEL EQUATION

\[-\varepsilon^2 \Delta^2 u + \Delta u = 1 \quad \text{in } \Omega\]
\[u = 0 \quad \text{on } \partial \Omega\]
\[\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega\]

Numerical solution with Adini’s element:

\[\varepsilon = 0.001\]
Order of convergence with uniform and graded meshes:

\[ \varepsilon = 0.001 \]
For uniform meshes:

\[
\|u - u_h\|_\varepsilon \leq \frac{C}{4\sqrt{N}}
\]

this can be proved (but not yet written!).

Expected order for graded meshes:

\[
\|u - u_h\|_\varepsilon \leq \frac{C}{\sqrt{N}}
\]

BUT: NO THEORY FOR ANISOTROPIC ELEMENTS!

WE ARE WORKING ON THAT!