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AN OPTIMAL POINCARÉ INEQUALITY IN L^1 FOR CONVEX DOMAINS

GABRIEL ACOSTA AND RICARDO G. DURÁN

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ABSTRACT. For convex domains $\Omega \subset \mathbb{R}^n$ with diameter d we prove

$$\|u\|_{L^1(\omega)} \leq \frac{d}{2}\|\nabla u\|_{L^1(\omega)}$$

for any u with zero mean value on ω. We also show that the constant $1/2$ in this inequality is optimal.

1. Introduction

Given a bounded domain $\Omega \subset \mathbb{R}^n$, the Poincaré inequality

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)} \tag{1.1}$$

for functions with vanishing mean value over Ω is a very well-known result which holds for $1 \leq p \leq \infty$ under very general assumptions on Ω. In (1.1), as in the rest of the paper, $\|\nabla u\|_{L^p(\Omega)}$ is defined as the L^p-norm of the euclidean norm of ∇u.

The usual proof of (1.1) is based on the compact inclusion of $W^{1,p}(\Omega)$ in $L^p(\Omega)$ which is valid, for example, for domains satisfying the so-called "segment property" (see for example [1]).

An interesting problem is to know the dependence of the constant on the geometry of the domain Ω and, in particular, to find the best constant for a given domain or class of domains. For $p = 2$, the best constant is the inverse of the first positive eigenfrequency of a free membrane. For general p, knowledge of this constant provides explicit estimates for the error in polynomial approximation (see [5]).

The proof based on compactness does not provide any information about the constant other than that it is finite. Therefore, different arguments are needed in order to know more about the constant as, for example, its dependence on the geometry of the domain.

In this paper we consider the case of a convex $\Omega \subset \mathbb{R}^n$. In this case, a beautiful proof for $p = 2$ has been given by Payne and Weinberger [4]. By elementary considerations, Payne and Weinberger showed that (1.1) can be deduced from weighted

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Poincaré inequalities in one dimension. Therefore, the problem of finding the best constant is reduced to a one-dimensional one. In this way, they were able to find the best constant. If d is the diameter of Ω they proved that the optimal constant in (1.1), for $p = 2$, is d/π. For general p, an estimate of the constant for convex domains is given in [2] but, for convex sets with a fixed diameter, this estimate goes to infinity when the measure of the set goes to zero.

The goal of this paper is to find the best constant for convex domains in the case $p = 1$. First, we show that the reduction to a one-dimensional problem can be performed in this case. Indeed, as we show in Section 1, the argument given by Payne and Weinberger also applies for this case.

The proof of the one-dimensional weighted estimates given in [4] strongly uses that $p = 2$, and so we have to introduce different arguments to treat the case $p = 1$. We prove that the optimal constant for $p = 1$ and convex domains is $d/2$. As a byproduct, it follows by interpolation that for any value of p the constant for convex domains depends only on the diameter, a result that, as far as we know, was not known.

2. Reduction to a one-dimensional problem

In this section we show how the arguments given in [4], to reduce the problem to one-dimensional weighted estimates, also applies for the case $p = 1$.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain with diameter d and $u \in L^1(\Omega)$ such that $\int_\Omega u = 0$. Then, for any $\delta > 0$ there exists a decomposition of Ω into a finite number of convex domains Ω_i satisfying*

$$\Omega_i \cap \Omega_j = \emptyset \quad for \quad i \neq j\,, \qquad \overline{\Omega} = \bigcup \overline{\Omega}_i\,, \quad and \qquad \int_{\Omega_i} u = 0$$

and each Ω_i is thin in all but one direction, i.e., in an appropriate rectangular coordinate system, $(x, y) = (x, y_1, \cdots, y_{n-1})$, the set Ω_i is contained in

$$\{(x, y) : 0 \leq x \leq d\,, \quad 0 \leq y_j \leq \delta \quad for \quad j = 1, \cdots, n-1\}.$$

Proof. First consider the case $n = 2$. Let $S^1 \subset \mathbb{R}^2$ be the unitary circle. For any $\alpha \in [0, 2\pi]$ let $v(\alpha) = (cos(\alpha), sin(\alpha)) \in S^1$. By elementary continuity arguments one can see that for any α there exists a unique line $\Pi_{v(\alpha)}$ perpendicular to $v(\alpha)$ that divides Ω into two convex sets of equal area (we remark that this line does not necessarily pass through the centroid as claimed in [4]). We denote these sets by $\Omega_+(\alpha)$ and $\Omega_-(\alpha)$ where

$$\begin{aligned} \Omega_+(\alpha) &= \{x \in \Omega \quad : \quad (x-p) \cdot v(\alpha) > 0\}, \\ \Omega_-(\alpha) &= \{x \in \Omega \quad : \quad (x-p) \cdot v(\alpha) < 0\}, \end{aligned}$$

with $p \in \Pi_{v(\alpha)} \cap \Omega$. Now calling $I(v(\alpha)) = \int_{\Omega_+(\alpha)} u$ we have $I(v(\alpha)) = -I(-v(\alpha))$, and since $-v(\alpha) = v(\alpha + \pi)$, it follows by continuity that there exists an α_0 for which $I(\alpha_0) = 0$. Repeating the described procedure we can decompose Ω into convex sets Ω_i of equal area such that the average of u on each of them vanishes. Now, for a given $\delta > 0$, we can iterate enough in order to obtain that each Ω_i is contained between two parallel lines at a distance less than or equal to δ and so the case $n = 2$ is proved.

The same ideas can be applied in higher dimensions. Indeed, calling $S^{n-1} \subset \mathbb{R}^n$ the unitary sphere, let us define $v_1(\alpha) = (0, \cdots, 0, cos(\alpha), sin(\alpha)) \in S^{n-1}$. As before, there exists a unique hyperplane $\Pi_{v_1(\alpha)}$ perpendicular to $v_1(\alpha)$ that divides Ω into two convex sets of equal measure. Repeating the argument given in the two-dimensional case we obtain a decomposition of Ω into convex sets such that the integral of u vanishes on each of them and each of these sets is contained between two parallel hyperplanes with normal of the form $(0, \cdots, 0, cos(\beta), sin(\beta))$ at a distance less than or equal to δ.

Fixing one of these domains, say $\tilde{\Omega}$, we can choose appropriate coordinates such that these hyperplanes have normal $v_1 = (0, \cdots, 0, 1)$. Using these coordinates we define $v_2(\alpha) = (0, \cdots, 0, cos(\alpha), sin(\alpha), 0) \in S^{n-1}$ perpendicular to v_1. Arguing as above, we can see that $\tilde{\Omega}$ can be decomposed into convex sets, on which the integral of u vanishes, and such that each of these sets is contained between two parallel hyperplanes with normal of the form $(0, \cdots, 0, cos(\beta), sin(\beta), 0)$ at a distance less than or equal to δ. Therefore, we have decomposed Ω into convex sets which are thin in two orthogonal directions and such that the integral of u vanishes on each of them. If $n = 3$ the lemma is proved and if $n > 3$ we can repeat the argument until we obtain the required decomposition. □

We will say that ρ is a concave function if its graph is always above a secant line. This is the usual terminology nowadays but it was not several years ago; for example in [4], this kind of function was called "convex".

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain with diameter d. Suppose that there is a constant C_1 such that, for all $L > 0$, all concave functions $\rho \geq 0$ on $[0, L]$, and all $f \in W^{1,1}(0, L)$ satisfying*

$$\int_0^L f(x)\rho(x)dx = 0,$$

the following estimate holds:

$$\int_0^L |f(x)|\rho(x)dx \leq C_1 L \int_0^L |f'(x)|\rho(x)dx. \tag{2.2}$$

Then,

$$\|u\|_{L^1(\Omega)} \leq C_1 d \|\nabla u\|_{L^1(\Omega)} \tag{2.3}$$

for all $u \in W^{1,1}(\Omega)$ such that $\int_\Omega u = 0$.

Proof. By density, we can assume that the function $u \in C^\infty(\overline{\Omega})$. Let M be a bound for u and all its derivatives up to the second order.

Given $\delta > 0$, we decompose Ω into convex sets Ω_i as in Lemma 2.1. So, for each i, we have a rectangular coordinate system $(x, y) = (x, y_1, \cdots, y_{n-1})$ such that the projection of Ω_i into the x-axis is the interval $(0, d_i)$, $d_i \leq d$, and Ω_i is contained in

$$\{(x, y) : 0 \leq x \leq d_i, \quad 0 \leq y_j \leq \delta \quad \text{for} \quad j = 1, \cdots, n-1\}.$$

Let $\rho(x_0)$ be the $n-1$ volume of the intersection of Ω_i with the hyperplane $x = x_0$. Since Ω_i is convex, the function $\rho(x)$ is concave.

Then, it is easy to see that

$$\Big|\int_{\Omega_i} |u(x,y)|dxdy - \int_0^{d_i} |u(x,0)|\rho(x)dx\Big| \le (n-1)M|\Omega_i|\delta, \tag{2.4}$$

$$\Big|\int_{\Omega_i} |\frac{\partial u}{\partial x}(x,y)|dxdy - \int_0^{d_i} |\frac{\partial u}{\partial x}(x,0)|\rho(x)dx\Big| \le (n-1)M|\Omega_i|\delta, \tag{2.5}$$

$$\Big|\int_{\Omega_i} u(x,y)dxdy - \int_0^{d_i} u(x,0)\rho(x)dx\Big| \le (n-1)M|\Omega_i|\delta. \tag{2.6}$$

Now, if g is any function in $W^{1,1}(0,L)$, we can apply (2.2) to

$$f = g - \int_0^L g(x)\rho(x)dx/\int_0^L \rho(x)dx$$

to obtain

$$\int_0^L |g(x)|\rho(x)dx \le C_1 L\int_0^L |g'(x)|\rho(x)dx - \Big|\int_0^L g(x)\rho(x)dx\Big|. \tag{2.7}$$

But, from (2.6) and the fact that $\int_{\Omega_i} u = 0$, it follows that

$$\Big|\int_0^{d_i} u(x,0)\rho(x)dx\Big| \le (n-1)M|\Omega_i|\delta$$

and therefore, applying (2.7) to the function $g(x) = u(x,0)$ with $L = d_i$, we have

$$\int_0^{d_i} |u(x,0)|\rho(x)dx \le C_1 d_i \int_0^{d_i} |\frac{\partial u}{\partial x}(x,0)|\rho(x)dx + (n-1)M|\Omega_i|\delta,$$

and so, using (2.5),

$$\begin{aligned}\int_0^{d_i} |u(x,0)|\rho(x)dx &\le C_1 d_i \int_{\Omega_i} |\frac{\partial u}{\partial x}(x,y)|dxdy + (C_1 d_i + 1)(n-1)M|\Omega_i|\delta \\ &\le C_1 d\int_{\Omega_i} |\nabla u(x,y)|dxdy + (C_1 d + 1)(n-1)M|\Omega_i|\delta.\end{aligned}$$

Now applying (2.4) and summing up in i we obtain

$$\int_\Omega |u(x,y)|dxdy \le C_1 d\int_\Omega |\nabla u(x,y)|dxdy + (C_1 d + 2)(n-1)M|\Omega|\delta$$

and, since $\delta > 0$ is arbitrary, we conclude the proof. □

3. The weighted one-dimensional inequality

The goal of this section is to prove that the inequality (2.2) holds and to find the best possible constant C_1. The key point in our argument is the following lemma which gives an inequality for concave functions.

Lemma 3.1. *Let ρ be a non-negative concave function on $[0,1]$ such that $\int_0^1 \rho(x)dx = 1$. Then,*

$$h(\rho,x) := \frac{\int_0^x \rho(t)dt\int_x^1 \rho(t)dt}{\rho(x)} \le \frac{1}{4} \qquad \forall x \in (0,1). \tag{3.8}$$

We postpone the proof of Lemma 3.1 to first show how the weighted estimate follows from it.

Theorem 3.1. *Let ρ be a non-negative concave function on $[0, L]$ and $f \in W^{1,1}(0, L)$ such that*

$$\int_0^L f(x)\rho(x)dx = 0.$$

Then,

$$\int_0^L |f(x)|\rho(x)dx \leq \frac{1}{2}L \int_0^L |f'(x)|\rho(x)dx. \tag{3.9}$$

Moreover, the constant $1/2$ is optimal.

Proof. By a simple scaling argument it is enough to prove (3.9) for $L = 1$. Moreover, dividing the inequality by $\int_0^1 \rho(x)dx$, we can assume that $\int_0^1 \rho(x)dx = 1$.

Now, since $\int_0^1 f(x)\rho(x)dx = 0$, it follows by integration by parts that

$$f(y) = \int_0^y f'(x)\Big(\int_0^x \rho(t)dt\Big)dx - \int_y^1 f'(x)\Big(\int_x^1 \rho(t)dt\Big)dx$$

and therefore,

$$|f(y)| \leq \int_0^y |f'(x)|\Big(\int_0^x \rho(t)dt\Big)dx + \int_y^1 |f'(x)|\Big(\int_x^1 \rho(t)dt\Big)dx.$$

Hence, multiplying by $\rho(y)$, integrating in the y variable and applying Fubini's theorem, we obtain

$$\int_0^1 |f(y)|\rho(y)dy \leq \int_0^1 |f'(x)|2\Big(\int_0^x \rho(t)dt\Big)\Big(\int_x^1 \rho(t)dt\Big)dx$$

and the proof of (3.9) concludes by using (3.8).

To see that the constant $1/2$ is optimal, take $\rho \equiv 1$, $L = 1$, and $f = f_\varepsilon$ where $\varepsilon > 0$ and f_ε is defined by

$$f_\varepsilon(x) = \begin{cases} -1 & \text{for} \quad x \in [0, \frac{1}{2} - \varepsilon], \\ \frac{1}{\varepsilon}(x - \frac{1}{2}) & \text{for} \quad x \in (\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon), \\ 1 & \text{for} \quad x \in [\frac{1}{2} + \varepsilon, 1]. \end{cases} \tag{3.10}$$

Then, $\int_0^1 f_\varepsilon(x)\rho(x)dx = 0$ and

$$\frac{\int_0^1 |f_\varepsilon(x)|dx}{\int_0^1 |f'_\varepsilon(x)|dx} = \frac{1-\varepsilon}{2} \longrightarrow \frac{1}{2} \qquad \text{when} \quad \varepsilon \longrightarrow 0. \qquad \square$$

Remark 3.1. As in the case $p = 2$ (see [4]) the worst possible constant in inequality (2.2) is attained when $\rho \equiv 1$.

Summing up our results we obtain our main theorem:

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain with diameter d and $u \in W^{1,1}(\Omega)$ such that $\int_\Omega u = 0$. Then,*

$$||u||_{L^1(\Omega)} \leq \frac{1}{2}\, d||\nabla u||_{L^1(\Omega)}.$$

Moreover, the constant $1/2$ is optimal.

Proof. The estimate follows immediately from Theorems 2.1 and 3.1. To see that the constant is optimal, let f_ε be the function defined in (3.10) and consider

$$u_\varepsilon(x_1, \cdots, x_n) = f_\varepsilon(x_1)$$

on $\Omega_\varepsilon = [0,1] \times [0,\varepsilon]^{n-1}$. It is easy to check that

$$\frac{\|u\|_{L^1(\Omega_\varepsilon)}}{\|\nabla u\|_{L^1(\Omega_\varepsilon)}} \longrightarrow \frac{1}{2} \qquad \text{when} \quad \varepsilon \longrightarrow 0$$

while the diameter of $\Omega_\varepsilon \longrightarrow 1$. ☐

Proof of Lemma 3.1. Since $\int_0^1 \rho(x)dx = 1$ we have

$$h(\rho, x) = \frac{\int_0^x \rho(t)dt\Big(1 - \int_0^x \rho(t)dt\Big)}{\rho(x)}. \tag{3.11}$$

Then, for those values of x such that $\rho(x) \geq 1$,

$$h(\rho, x) \leq \int_0^x \rho(t)dt\Big(1 - \int_0^x \rho(t)dt\Big),$$

and so (3.8) follows from the fact that the function $\phi(\alpha) = \alpha(1-\alpha)$ is bounded by $1/4$.

So, the most difficult part of the proof is for the case $\rho(x) < 1$. Observe first that $\rho(1/2) \geq 1$. Indeed, by concavity, the graph of ρ is always below a straight line passing through the point $(1/2, \rho(1/2))$ (the tangent line if ρ is smooth). But, $\rho(1/2)$ equals the area below this line on $[0,1]$, which is greater than or equal to $\int_0^1 \rho(x)dx = 1$.

Therefore, we can assume that $\rho(x) < 1$ and that $x \in (0, 1/2)$ (the same argument applies by symmetry for the other half of the interval).

Fix $x_0 \in (0, 1/2)$ and consider the linear function ℓ_{x_0} defined by the conditions $\ell_{x_0}(x_0) = \rho(x_0)$ and $\ell_{x_0}(1/2) = 1$. Then, since ρ is concave and $\rho(1/2) \geq 1$, it follows that

$$\rho(t) \leq \ell_{x_0}(t) \quad \forall t \in [0, x_0],$$

and so

$$\int_0^{x_0} \rho(t)dt \leq \int_0^{x_0} \ell_{x_0}(t)dt < \int_0^{\frac{1}{2}} \ell_{x_0}(t)dt \leq \frac{1}{2}.$$

Then, using that the function $\phi(\alpha) = \alpha(1-\alpha)$ is monotone increasing for $\alpha \in (0, 1/2)$, it follows from the expression (3.11) of h that

$$h(\rho, x_0) \leq h(\ell_{x_0}, x_0).$$

Therefore, it is enough to prove (3.8) for functions of the form

$$\ell(x) = m(x - 1/2) + 1$$

with $0 \leq m \leq 2$. But, a straightforward computation yields

$$h(\ell, x) = x(1-x)g(m, x)$$

where

$$g(m, x) = \frac{(mx/2 + 1 - m/2)(1 + mx/2)}{m(x - 1/2) + 1}$$

and it is not difficult to check that

$$\frac{\partial g(m, x)}{\partial m} < 0$$

for any $x \in (0,1)$ and $m \in (0,2)$. Hence, for a fixed x, $h(\ell, x)$ reaches its maximum in m when $m = 0$, i.e., for $\ell \equiv 1$. Therefore, the proof concludes by observing that

$$h(1,x) = x(1-x) \leq \frac{1}{4} \qquad \forall x. \qquad \square$$

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Added after posting

For dimension $n > 2$, the correct statement of Theorem 2.1 should say "all functions $\rho \geq 0$ such that $\rho^{1/(n-1)}$ is concave" instead of "all concave functions $\rho \geq 0$". Indeed, the function defined in the proof of that theorem, which for each x_0 gives the $n-1$ volume of the intersection of a convex set with the hyperplane $x = x_0$, is not concave in general, but it can be proved that its $(n-1)$-th root is concave (see, for example, [6, p. 361]).

Therefore, we need to prove that the result stated in Theorem 3.1 is true also for functions ρ such that $\rho^{1/(n-1)}$ is concave. Looking at the proof of that theorem one sees that it is enough to prove the inequality (3.8) for this class of functions, and this is the goal of the following lemma.

Lemma. *Let ρ be a nonnegative function on $[0,1]$ such that $\int_0^1 \rho(x)dx = 1$ and such that, for some $m \in \mathbb{N}$, $\rho^{1/m}$ is concave. Then,*

$$\frac{\int_0^x \rho(t)dt \int_x^1 \rho(t)dt}{\rho(x)} \leq \frac{1}{4} \qquad \forall x \in (0,1). \tag{A.1}$$

Proof. Given a positive concave function f on $(0,1)$ let $c_m = \int_0^1 f^m(t)dt$. Then, (A.1) will follow if we prove that

$$\frac{\int_0^x f^m(t)dt \int_x^1 f^m(t)dt}{c_m f^m(x)} \leq \frac{1}{4} \qquad \forall x \in (0,1) \tag{A.2}$$

for all $m \in \mathbb{N}$.

We proceed by induction on m. For $m = 1$ this is exactly the result of Lemma 3.1 applied to the normalized function f/c_1. Now, for $m \geq 1$ we will show that

$$\frac{\int_0^x f^{m+1}(t)dt \int_x^1 f^{m+1}(t)dt}{c_{m+1} f^{m+1}(x)} \leq \frac{\int_0^x f^m(t)dt \int_x^1 f^m(t)dt}{c_m f^m(x)} \quad \text{for} \quad f(x) \leq c_{m+1}/c_m. \tag{A.3}$$

Indeed, making the change of variable given by $s = \phi(t) := \int_0^t f^m/c_m$ and defining $g := (c_m/c_{m+1}) f \circ \phi^{-1}$ we have

$$\frac{\int_0^x f^{m+1}(t)dt \int_x^1 f^{m+1}dt}{c_{m+1} f^{m+1}(x)} = \frac{c_m \int_0^{\phi(x)} g(s)ds \int_{\phi(x)}^1 g(s)ds}{g(\phi(x))\, f^m(x)}. \tag{A.4}$$

Now, it is not difficult to check that, since f is concave, the function g is also concave. We also have $\int_0^1 g(s)ds = 1$. Then, following the steps of the proof of Lemma 3.1 we conclude that

$$\frac{\int_0^{\phi(x)} g(s)ds \int_{\phi(x)}^1 g(s)ds}{g(\phi(x))} \leq \phi(x)(1-\phi(x))$$

for $g(\phi(x)) \le 1$ or, equivalently, for $f(x) \le c_{m+1}/c_m$. Then, (A.3) follows from this estimate combined with (A.4) and the definition of ϕ.

Now suppose that (A.2) is true for m. Then, if x is such that $f^{m+1}(x) \le c_{m+1}$ we are in the condition under which we have proved (A.3). Indeed, this follows from the inequality $c_{m+1} \le (c_{m+1}/c_m)^{m+1}$, which is an immediate consequence of the Hölder inequality. On the other hand, if $f^{m+1}(x) \ge c_{m+1}$, (A.2) for $m+1$ can be easily shown by writing the numerator as $\int_0^x f^{m+1}(t)dt(c_{m+1} - \int_0^x f^{m+1}(t)dt)$, and so the proof is concluded. □

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Instituto de Ciencias, Universidad Nacional de General Sarmiento, J. M. Gutierrez 1150, Los Polvorines, B1613GSX Provincia de Buenos Aires, Argentina
E-mail address: gacosta@ungs.edu.ar

Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, 1428 Buenos Aires, Argentina
E-mail address: rduran@dm.uba.ar