# WEIGHTED A PRIORI ESTIMATES FOR POISSON EQUATION

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Abstract. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^2$ and let $u$ be a solution of the classical Poisson problem in $\Omega$; i.e.,

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $f \in L^p_\omega(\Omega)$ and $\omega$ is a weight in $A_p$.

The main goal of this paper is to prove the following a priori estimate

$$\|u\|_{W^{2,p}_\omega(\Omega)} \leq C\,\|f\|_{L^p_\omega(\Omega)},$$

and to give some applications for weights given by powers of the distance to the boundary.

## 1. Introduction

We will use the standard notation for Sobolev spaces and for derivatives, namely, if $\alpha$ is a multi-index, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n_+$ we denote $|\alpha| = \sum_{j=1}^n \alpha_j$, $D^\alpha = \partial^{\alpha_1}_{x_1} \ldots \partial^{\alpha_n}_{x_n}$ and

$$W^{k,p}(\Omega) = \{v \in L^p(\Omega) \,:\, D^\alpha v \in L^p(\Omega) \quad \forall |\alpha| \leq k\}.$$

Let $\Gamma$ be the standard fundamental solution of the Laplacian operator, namely,

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi} \log |x|^{-1} & n = 2 \\ \frac{1}{n(n-2)\,w_n} |x|^{2-n} & n \geq 3 \end{cases}$$

with $w_n$ the area of the unit sphere in $\mathbb{R}^n$.

Given a function $f \in C^\infty_0(\mathbb{R}^n)$ it is a classic result that the potential $u$ given by

$$u(x) = \int \Gamma(x-y) f(y)\, dy$$

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is a solution of $-\Delta u = f$ in $\mathbb{R}^n$ and satisfies the estimate

$$\|u\|_{W^{2,p}(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)} \tag{1.1}$$

for $1 < p < \infty$. Indeed, this estimate is a consequence of the Calderón-Zygmund theory of singular integrals (see for example [11]).

Since the work by Muckenhoupt [9], many results on weighted estimates for maximal functions and singular integral operators have been obtained. In particular, generalizations of (1.1) to weighted norms are known to hold for weights in the class $A_p$ (see for example [12]).

On the other hand, a priori estimates like (1.1) for solutions of the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

on smooth bounded domains $\Omega$ are also well known (see for example the classic paper by Agmon, Douglis and Nirenberg [3] where a priori estimates for general elliptic problems are proved).

Therefore, it is a natural question whether weighted a priori estimates are valid also for the solution of the Dirichlet problem (1.2). In this paper we give a positive answer to this question, namely, we prove that

$$\|u\|_{W^{2,p}_\omega(\Omega)} \le C\,\|f\|_{L^p_\omega(\Omega)},$$

for $\omega \in A_p$, where the constant $C$ depends only on $\Omega$ and on the weight $\omega$.

The main ideas for the proof of these estimates were explained to the first author around twenty years ago by Professor Eugene B. Fabes, but several technical details needed to be developed and this is the goal of this paper.

As an application we obtain a priori estimates for weights given by powers of the distance to $\partial\Omega$. Estimates of this type are of interest in the analysis of some non-linear problems and were derived using different arguments (see [13]).

Using our general results, together with some weighted Sobolev imbedding theorems, we are able to prove some of the estimates given in [13] as well as a new one corresponding to a border case of the results of that paper.

## 2. Weighted a priori estimates

We consider the Dirichlet problem (1.2) in bounded domains $\Omega$. From now on we will assume that $\partial\Omega$ is of class $C^2$. The solution of this problem is given by

$$u(x) = \int_\Omega G(x,y)\, f(y)\, dy \tag{2.1}$$

where $G(x,y)$ is the Green function which can be written as

$$G(x,y) = \Gamma(x-y) + h(x,y) \tag{2.2}$$

with $h(x,y)$ satisfying, for each fixed $y \in \Omega$,

$$\begin{cases} \Delta_x h(x,y) = 0 & x \in \Omega \\ h(x,y) = -\Gamma(x-y) & x \in \partial\Omega. \end{cases}$$

If $P(y,Q)$ is the Poisson kernel $h(x,y)$ is given by

$$h(x,y) = -\frac{1}{(n-2)\, w_n} \int_{\partial\Omega} \frac{1}{|x-Q|^{n-2}}\, P(y,Q)\, dS(Q) \tag{2.3}$$

where $dS$ denotes the surface measure on $\partial\Omega$.

In what follows the letter $C$ will denote a generic constant not necessarily the same at each occurrence. It is known that the Green function satisfies the following estimates (see [14]),

$$|G(x,y)| \le \begin{cases} C \log|x-y| & \text{if}\, n = 2 \\ C|x-y|^{2-n} & \text{if}\, n \ge 3 \end{cases} \tag{2.4}$$

and

$$|D_{x_i} G(x,y)| \le C|x-y|^{1-n}. \tag{2.5}$$

Therefore,

$$D_{x_i}u(x) = \int_\Omega D_{x_i}G(x,y)\, f(y)\, dy \tag{2.6}$$

To obtain the second derivatives of $u$ from the representation (2.1) we will use the following Lemma. We denote with $d(x)$ the distance to the boundary, namely, $d(x) = \inf_{Q\in\partial\Omega} |x-Q|$.

**Lemma 2.1.** *Given $\alpha \in \mathbb{Z}_+^n$ ($|\alpha| > 0$ if $n = 2$) there exists a constant $C$ depending only on $n$ and $\alpha$ such that*

$$|D^\alpha h(x,y)| \le C\, d(x)^{2-n-|\alpha|} \tag{2.7}$$

**Proof:** To simplify notation we assume that $n \ge 3$ (but the argument applies also in the case $n = 2$). Using (2.3) and $P(y,Q) \ge 0$, $\forall Q \in \partial\Omega$ we have

$$\begin{aligned} |D^\alpha h(x,y)| &= \left| \frac{1}{(n-2)\, w_n} \int_{\partial\Omega} D^\alpha |x-Q|^{2-n}\, P(y,Q)\ dS(Q) \right| \\ &\le C \int_{\partial\Omega} |x-Q|^{2-n-|\alpha|}\, P(y,Q)\, dS(Q) \end{aligned}$$

and then (2.7) follows by using that $\int_{\partial\Omega} P(y,Q)\, dS(Q) = 1$. $\square$

It follows from this Lemma that for each $x \in \Omega$, $D_{x_ix_j}h(x,y)$ is bounded uniformly in a neighborhood of $x$ and so

$$D_{x_ix_j} \int_\Omega h(x,y)\, f(y)\, dy = \int_\Omega D_{x_ix_j}h(x,y)\, f(y)\, dy \tag{2.8}$$

On the other hand, since $|D_{x_j}\Gamma(x)| \le C|x|^{1-n}$ we have

$$D_{x_j} \int_\Omega \Gamma(x-y)\, f(y)\, dy = \int_\Omega D_{x_j}\Gamma(x-y)\, f(y)\, dy$$

However, $D_{x_ix_j}\Gamma$ is not an integrable function and we can not interchange the order between second derivatives and integration. A known standard argument shows that

$$D_{x_i} \int_\Omega D_{x_j}\Gamma(x-y)\, f(y)\, dy = Kf(x) + c(x)f(x) \tag{2.9}$$

where $c$ is a bounded function and

$$Kf(x) = \lim_{\epsilon\to 0} \int_{|x-y|>\epsilon} D_{x_i x_j}\Gamma(x-y)\, f(y)\, dy. \tag{2.10}$$

Here and in what follows we consider $f$ defined in $\mathbb{R}^n$ extending the original $f$ by zero.

The operator $K$ is a Calderón-Zygmund singular integral operator. Indeed, since $D_{x_j}\Gamma \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and it is a homogeneous function of degree $1-n$ it follows that $D_{x_i x_j}\Gamma(x-y)$ is homogeneous of degree $-n$ and has vanishing average on the unit sphere (see Lemma 11.1 in [2, page 152]). Then, it follows from the general theory given in [5] that $K$ is a bounded operator in $L^p$ for $1 < p < \infty$.

Moreover, the maximal operator

$$\widetilde{K}f(x) = \sup_{\epsilon>0} \left| \int_{|x-y|>\epsilon} D_{x_i x_j}\Gamma(x-y)\, f(y)\, dy \right|$$

is also bounded in $L^p$ for $1 < p < \infty$.

We will need the following estimate for the Green function. This estimate has been proved by A. Dall'Acqua and G. Sweers in [6], however they assume that the domain is more regular than $C^2$. Therefore, we will give a different proof valid for $C^2$ domains following the arguments given in [14].

**Lemma 2.2.** *Let $\Omega$ be a bounded $C^2$ domain and $G(x,y)$ be the Green function of problem (1.2) in $\Omega$. There exists a constant $C$ depending only on $n$ and $\Omega$ such that for $(x,y) \in \Omega \times \Omega$*

$$|D_{x_i x_j} G(x,y)| \le C \frac{d(x)}{|x-y|^{n+1}} \tag{2.11}$$

Since the proof of this Lemma is very technical it will be given in Appendix 1.

Our main result follows from the following Lemma.

**Lemma 2.3.** *There exists a constant $C$ depending only on $n$ and $\Omega$ such that, for any $x \in \Omega$,*

$$|u(x)| + |D_{x_i} u(x)| \leq C\, Mf(x)$$

$$|D_{x_i x_j} u(x)| \leq C \left\{ \widetilde{K} f(x) + Mf(x) + |f(x)| \right\}$$

*where $Mf(x)$ is the usual Hardy-Littlewood maximal function of $f$.*

**Proof:** Calling $\delta$ the diameter of $\Omega$ and using (2.5) and (2.6) we have

$$|D_{x_i} u(x)| \leq C \int_{|x-y|\leq\delta} \frac{|f(y)|}{|x-y|^{n-1}}\, dy = \sum_{k=0}^{\infty} \int_{\{2^{-(k+1)}\delta \leq |x-y| \leq 2^{-k}\delta\}} \frac{|f(y)|}{|x-y|^{n-1}}\, dy$$

and then, it follows easily that

$$|D_{x_i} u(x)| \leq C\, Mf(x). \tag{2.12}$$

(see Lemma 2.8.3 in [16, page 85] for details).

Analogously we obtain

$$|u(x)| \leq C\, Mf(x)$$

using now (2.1) and (2.4).

Therefore, the most interesting and difficult part of the Lemma is the estimate of the second derivatives. Using the representation given by (2.1) and (2.2), (2.8), (2.9) and (2.10) we have

$$D_{x_i x_j} u(x) = \lim_{\epsilon\to 0} \int_{\epsilon < |x-y| \leq d(x)} D_{x_i x_j} \Gamma(x-y)\, f(y)\, dy + \int_{|x-y|>d(x)} D_{x_i x_j} \Gamma(x-y)\, f(y)\, dy$$
$$+ c(x) f(x) + \int_{|x-y|\leq d(x)} D_{x_i x_j} h(x,y)\, f(y)\, dy + \int_{|x-y|>d(x)} D_{x_i x_j} h(x,y)\, f(y)\, dy$$

and then,

$$\begin{aligned} D_{x_i x_j} u(x) &= \lim_{\epsilon\to 0} \int_{\epsilon < |x-y| \leq d(x)} D_{x_i x_j} \Gamma(x-y)\, f(y)\, dy + c(x) f(x) \\ &+ \int_{|x-y|\leq d(x)} D_{x_i x_j} h(x,y)\, f(y)\, dy + \int_{|x-y|>d(x)} D_{x_i x_j} G(x,y)\, f(y)\, dy \\ &=: I + II + III + IV \end{aligned} \tag{2.13}$$

Now we have

$$I = \lim_{\epsilon \to 0} \int_{\epsilon < |x-y|} D_{x_i x_j}\Gamma(x-y)\, f(y)\, dy - \int_{|x-y|>d(x)} D_{x_i x_j}\Gamma(x-y)\, f(y)\, dy$$

but,

$$\left| \int_{|x-y|>d(x)} D_{x_i x_j}\Gamma(x-y)\, f(y)\, dy \right| \le \sup_{\epsilon>0} \left| \int_{|x-y|>\epsilon} D_{x_i x_j}\Gamma(x-y)\, f(y)\, dy \right| = \widetilde{K} f(x)$$

and therefore

$$|I| \le |Kf(x)| + \widetilde{K} f(x) \le 2\widetilde{K} f(x).$$

Since $c$ is a bounded function we have $|II| \le C|f(x)|$. Therefore, it only remains to estimate the last two terms in (2.13).

By (2.7) we have

$$|III| = \frac{C}{d(x)^n} \int_{|x-y|\le d(x)} |f(y)|\, dy \le CMf(x).$$

Finally, from (2.11) we obtain

$$|IV| \le C \int_{|x-y|>d(x)} \frac{d(x)}{|x-y|^{n+1}} f(y)\, dy$$

and therefore, by the same arguments used to prove (2.12) we conclude that

$$|IV| \le CMf(x)$$

and the Lemma is proved. $\square$

We can now state and prove our main result. First we recall the definition of the $A_p$ class for $1 < p < \infty$. A non-negative locally integrable function $\omega$ belongs to $A_p$ if there exists a constant $C$ such that

$$\left( \frac{1}{|Q|} \int_Q \omega(x)\ dx \right) \left( \frac{1}{|Q|} \int_Q \omega(x)^{-1/(p-1)}\, dx \right)^{p-1} \le C$$

for all cube $Q \subset \mathbb{R}^n$.

For any weight $\omega$, $L^p_\omega(\Omega)$ is the space of measurable functions $f$ defined in $\Omega$ such that

$$\|f\|_{L^p_\omega(\Omega)} = \left( \int_\Omega |f(x)|^p \, \omega(x) \, dx \right)^{1/p} < \infty$$

and $W^{k,p}_\omega(\Omega)$ is the space of functions such that

$$\|f\|_{W^{k,p}_\omega(\Omega)} = \left( \sum_{|\alpha| \le k} \|D^\alpha f\|^p_{L^p_\omega(\Omega)} \right)^{1/p} < \infty.$$

**Theorem 2.4.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^2$ domain. If $\omega \in A_p$, $f \in L^p_\omega(\Omega)$ and $u$ is the solution of problem (1.2), then there exists a constant $C$ depending only on $n$, $\omega$ and $\Omega$ such that*

$$\|u\|_{W^{2,p}_\omega(\Omega)} \le C \, \|f\|_{L^p_\omega(\Omega)}. \tag{2.14}$$

**Proof:** Since $M$ and $\widetilde{K}$ are bounded operators in $L^p_\omega$ (see [12, Chapter V]) (2.14) follows immediately from Lemma 2.3. $\square$

## 3. Application to weights of the form $d(x)^\beta$

In this section we show how the weighted estimate proved in the previous section can be used to obtain some of the a priori estimates given in [13]. Moreover, our arguments allows us to prove a new estimate which was not contained in the results in [13].

We will also make use of some imbedding Theorems for weighted Sobolev spaces which, as we will show, can be proved in a simple way by using an argument of Buckley and Koskela [4].

First of all we need to see which powers of the distance to the boundary belong to the class $A_p$. For the particular case of $\Omega$ being a ball it was shown in [8] that $d(x)^\gamma \in A_p$ for $-1 < \gamma < p-1$. We were not able to find in the literature an analogous result for general smooth domains. In the following Theorem we give a proof based on Whitney decomposition.

**Theorem 3.1.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^2$ domain and $d(x)$ the distance from $x$ to $\partial\Omega$. Then, $d(x)^\beta \in A_p$ for $-1 < \beta < p-1$.*

**Proof:** We have to prove that

$$\left(\frac{1}{|Q|}\int_Q d(x)^\beta \, dx\right)\left(\frac{1}{|Q|}\int_Q d(x)^{-\beta/(p-1)}\, dx\right)^{p-1} \leq C \tag{3.1}$$

for all cube $Q \subset \mathbb{R}^n$.

We will consider the following cases:

(1) $Q \cap \partial\Omega \neq \emptyset$

(2) $Q \cap \partial\Omega = \emptyset$

1. Let $\ell$ be the side of $Q$. We consider first $\beta \geq 0$.

It is easy to see that for $x \in Q$, $d(x) \leq diam(Q)$ and then

$$\frac{1}{|Q|}\int_Q d(x)^\beta \, dx \leq \frac{1}{|Q|}\int_Q diam(Q)^\beta \, dx = n^{\beta/2}\,\ell^\beta$$

In order to estimate the second integral in (3.1) we define $D_1 = Q \cap \Omega$ and $D_2 = Q \cap \Omega^c$. We consider a Whitney decomposition of $D_1$, i.e, a family $\{Q_j^k\}$ of closed dyadic cubes, whose interiors are pairwise disjoint, and which satisfy

- $D_1 = \bigcup_{k=k_0}^{\infty} \bigcup_{j=1}^{N_k} Q_j^k$
- $diam(Q_j^k) \leq dist(Q_j^k, \partial D_1) \leq 4\, diam(Q_j^k)$
- $|Q_j^k| = (\ell\, 2^{-k})^n$

If $x \in Q_j^k$, we have that $d(x) \geq dist(x, \partial D_1) \geq dist(Q_j^k, \partial D_1) \geq diam(Q_j^k)$.

Then

$$\begin{aligned}
\int_{D_1} d(x)^{-\beta/(p-1)}\, dx &= \sum_{k=k_0}^{\infty} \sum_{j=1}^{N_k} \int_{Q_j^k} d(x)^{-\beta/(p-1)}\, dx \\
&\leq \sum_{k=k_0}^{\infty} \sum_{j=1}^{N_k} (\ell\, 2^{-k})^{-\beta/(p-1)}\, |Q_j^k| \\
&\leq \sum_{k=k_0}^{\infty} \sum_{j=1}^{N_k} (\ell\, 2^{-k})^{-\beta/(p-1)}\, (\ell\, 2^{-k})^{n} \\
&= \ell^{n-\beta/(p-1)} \sum_{k=k_0}^{\infty} N_k\, 2^{k\beta/(p-1)}\, 2^{-kn} \\
&\leq C\, \ell^{n-\beta/(p-1)} \sum_{k=k_0}^{\infty} 2^{-k+k\beta/(p-1)}
\end{aligned}$$

where we have used that $N_k \leq C\, 2^{(n-1)k}$ for $k \geq k_0$ since $\Omega$ is smooth enough (indeed $D_1$ is an $n-1$-set and then we can apply the results given in [15]).

Applying an analogous argument for $D_2$ we can see that

$$\int_{D_2} d(x)^{-\beta/(p-1)}\, dx \leq C\, \ell^{n-\beta/(p-1)} \sum_{k=k_0}^{\infty} 2^{-k+k\beta/(p-1)}.$$

Therefore, for $\beta \geq 0$ we have

$$\begin{aligned}
\left\{ \frac{1}{|Q|} \int_Q d(x)^{-\beta/(p-1)}\, dx \right\}^{p-1} &= \left\{ \frac{1}{\ell^n} \left( \int_{D_1} d(x)^{-\beta/(p-1)}\, dx + \int_{D_2} d(x)^{-\beta/(p-1)}\, dx \right) \right\}^{p-1} \\
&\leq C \left\{ \frac{1}{\ell^n}\, \ell^{-\beta/(p-1)}\, \ell^n \sum_{k=k_0}^{\infty} 2^{-k+k\beta/(p-1)} \right\}^{p-1} \\
&= C\, \ell^{-\beta} \left\{ \sum_{k=k_0}^{\infty} 2^{-k+k\beta/(p-1)} \right\}^{p-1}
\end{aligned}$$

and finally

$$\left( \frac{1}{|Q|} \int_Q d(x)^{\beta}\, dx \right) \left( \frac{1}{|Q|} \int_Q d(x)^{-\beta/(p-1)}\, dx \right)^{p-1} \leq C \left\{ \sum_{k=k_0}^{\infty} 2^{-k+k\beta/(p-1)} \right\}^{p-1}$$

which is finite whenever $-k + k\beta/(p-1) < 0$, i.e, $0 \leq \beta < p-1$.

Now, we consider $\beta < 0$.

By the same arguments that in the case $\beta \geq 0$, we have that

$$\left(\frac{1}{|Q|}\int_Q d(x)^\beta \, dx\right)\left(\frac{1}{|Q|}\int_Q d(x)^{-\beta/(p-1)} \, dx\right)^{p-1} \leq C \sum_{k=k_0}^{\infty} 2^{-k(\beta+1)}$$

that it is finite whenever $-k(\beta+1) < 0$, i.e, $-1 < \beta < 0$.

2. Let $Q \subset \mathbb{R}^n$ be a cube with side $\ell$ such that $dist(Q, \partial\Omega) > 0$. If $diam(Q) \leq dist(Q, \partial\Omega)$ we have, for $x \in Q$,

$$dist(Q, \partial\Omega) \leq d(x) \leq diam(Q) + dist(Q, \partial\Omega) \leq 2\, dist(Q, \partial\Omega)$$

and then there exists a constant $C$ not depending on $Q$ such that

$$\left(\frac{1}{|Q|}\int_Q d(x)^\beta\right)\left(\frac{1}{|Q|}\int_Q d(x)^{-\beta/(p-1)}\right)^{p-1} \leq C$$

for any $\beta$.

On the other hand, if $diam(Q) \geq dist(Q, \partial\Omega)$, we consider a Whitney decomposition of $Q$ given by $\{Q_j^k\}$. Then for each $x \in Q_j^k$ we have

$$d(x) \leq diam(Q_j^k) + dist(Q_j^k, \partial Q) + dist(Q, \partial\Omega) \leq 6\, diam(Q)$$

Consequently, if $\beta \geq 0$ we have

$$\begin{aligned}
\int_Q d(x)^\beta \, dx &= \sum_{k=k_0}^{\infty} \sum_{j=1}^{N_k} \int_{Q_j^k} d(x)^\beta \, dx \\
&\leq C \sum_{k=k_0}^{\infty} \sum_{j=1}^{N_k} \int_{Q_j^k} diam(Q)^\beta \, dx \\
&\leq C \, \ell^\beta \sum_{k=k_0}^{\infty} N_k \, |Q_j^k| \\
&\leq C \, \ell^\beta \, \ell^n \sum_{k=k_0}^{\infty} 2^{-k}.
\end{aligned}$$

Using now, that for $x \in Q_j^k$,

$$d(x) \geq dist(Q_j^k, \partial Q) \geq diam(Q_j^k)$$

we obtain

$$\begin{aligned}
\int_Q d(x)^{-\beta/(p-1)} \, dx &= \sum_{k=k_0}^{\infty} \sum_{j=1}^{N_k} \int_{Q_j^k} d(x)^{-\beta/(p-1)} \, dx \\
&\leq \sum_{k=k_0}^{\infty} \sum_{j=1}^{N_k} \int_{Q_j^k} diam(Q_j^k)^{-\beta/(p-1)} \, dx \\
&\leq \sum_{k=k_0}^{\infty} \sum_{j=1}^{N_k} \int_{Q_j^k} (\ell \, 2^{-k})^{-\beta/(p-1)} \, dx \\
&\leq C \, \ell^{-\beta/(p-1)} \sum_{k=k_0}^{\infty} 2^{-k+k\beta/(p-1)}.
\end{aligned}$$

Therefore,

$$\left( \frac{1}{|Q|} \int_Q d(x)^\beta \, dx \right) \left( \frac{1}{|Q|} \int_Q d(x)^{-\beta/(p-1)} \, dx \right)^{p-1} \leq C \left\{ \sum_{k=k_0}^{\infty} 2^{-k+k\beta/(p-1)} \right\}^{p-1}$$

which is finite whenever $-k + k\beta/(p-1) < 0$, i.e, $0 \leq \beta < p-1$.

The result for the case $-1 < \beta < 0$ can be proved in an analogous way. Then the Theorem is proved. $\square$

The following theorem is an immediate consequence of Theorems 3.1 and 2.4.

**Theorem 3.2.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^2$ domain, $f \in L^p_{d^\gamma}(\Omega)$ and $u$ be the solution of problem (1.2). If $-1 < \gamma < p-1$, then there exists a constant $C$ depending only on $\gamma$, $p$, $n$ and $\Omega$ such that*

$$\|u\|_{W^{2,p}_{d^\gamma}(\Omega)} \le C\,\|f\|_{L_{d^\gamma}(\Omega)}.$$

Our next goal is to obtain some estimates of the type given in [13]. To do that we prove first some weighted imbedding Theorems for weights being powers of the distance to the boundary. We will use the following classical Theorem (see for instance [10]) and arguments introduced in [4].

**Theorem 3.3.** *Let $D \subset \mathbb{R}^{n+k}$ be a bounded Lipschitz domain and $u \in W^{2,p}(D)$.*

(1) *If $1 \le p < \dfrac{n+k}{2}$ and $\dfrac{1}{p} - \dfrac{1}{q} \le \dfrac{2}{n+k}$ there exists a constant $C$ not depending on $u$ such that*

$$\|u\|_{L^q(D)} \le C\,\|u\|_{W^{2,p}(D)}.$$

(2) *If $p = \dfrac{n+k}{2}$ and $1 \le q < \infty$ or $p > \dfrac{n+k}{2}$ and $1 \le q \le \infty$ there exists a constant $C$ not depending on $u$ such that,*

$$\|u\|_{L^q(D)} \le C\,\|u\|_{W^{2,p}(D)}.$$

**Theorem 3.4.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Suppose $u \in W^{2,p}_{d^\gamma}(\Omega)$ with $\gamma = k\beta$, where $k \in \mathbb{N}$ and $0 \le \beta \le 1$. Then,*

(1) *If $1 \le p < \dfrac{n+k}{2}$ and $\dfrac{1}{p} - \dfrac{1}{q} \le \dfrac{2}{n+k}$ there exists a constant $C$ not depending on $u$ such that*

$$\|u\|_{L^q_{d^\gamma}(\Omega)} \le C\,\|u\|_{W^{2,p}_{d^\gamma}(\Omega)}$$

(2) *If* $p = \dfrac{n+k}{2}$ *and* $1 \leq q < \infty$ *or* $p > \dfrac{n+k}{2}$ *and* $1 \leq q \leq \infty$, *there exists a constant* $C$ *not depending on* $u$ *such that*

$$\|u\|_{L^q_{d^\gamma}(\Omega)} \leq C\,\|u\|_{W^{2,p}_{d^\gamma}(\Omega)}.$$

**Proof:** 1. As in [4], we introduce the domains

$$\Omega_{k,\beta} = \{(x,y) \in \Omega \times \mathbb{R}^k \text{ such that } |y| < d(x)^\beta\}$$

It can be proved that if $\Omega$ is a Lipschitz domain, $\Omega_{k,\beta}$ is also a Lipschitz domain (see [1]).

For any $v \in L^p_{d^\gamma}(\Omega)$ let $V : \Omega_{k,\beta} \longrightarrow \mathbb{R}$ given by $V(x,y) = v(x)$. Then

(3.2)
$$\int_{\Omega_{k,\beta}} |V(x,y)|^p\,dx\,dy = \int_\Omega \int_{\{|y|<d(x)^\beta\}} |v(x)|^p\,dy\,dx = c_k \int_\Omega |v(x)|^p\,d(x)^{k\beta}\,dx$$

where $c_k$ denotes the measure of the unit ball in $\mathbb{R}^k$. Therefore, since $u \in W^{2,p}_{d^{k\beta}}(\Omega)$, the function $U(x,y) := u(x)$ belongs to $W^{2,p}(\Omega_{k,\beta})$. Then, by 1 of Theorem 3.3 we have

$$\|U\|_{L^q(\Omega_{k,\beta})} \leq C\,\|U\|_{W^{2,p}(\Omega_{k,\beta})}.$$

Therefore, applying again (3.2) we conclude the proof of the first part of the Theorem.

The proof of 2 is analogous using, now 2 of Theroem 3.3. □

We can now give the main result of this section.

**Theorem 3.5.** *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded* $C^2$ *domain,* $f \in L^p_{d^\gamma}(\Omega)$, *with* $\gamma = k\beta$, *where* $k \in \mathbb{N}$ *and* $0 \leq \beta \leq 1$. *If* $u$ *be the solution of problem (1.2),* $0 \leq \gamma < p-1$ *and* $\dfrac{1}{p} - \dfrac{1}{q} \leq \dfrac{2}{n+k}$ *( with* $q < \infty$ *when* $2p = n+k$*), then there exists a constant* $C$ *depending only on* $\gamma$ *,* $p$*,* $q$*,* $n$ *and* $\Omega$ *such that*

$$\|u\|_{L^q_{d^\gamma}(\Omega)} \leq C\, \|f\|_{L^p_{d^\gamma}(\Omega)} \tag{3.3}$$

**Proof:** The result follows immediately from Theorems 3.2 and 3.4. □

*Remark* 3.6. A particular case of the Theorem given above is

$$\|u\|_{L^q_{d}(\Omega)} \leq C\, \|f\|_{L^p_{d}(\Omega)}$$

for $p > 2$ and $\frac{1}{p} - \frac{1}{q} \leq \frac{2}{n+1}$ ( with $q < \infty$ when $2p = n+1$). This result is proved in [13] using different arguments for the case $\frac{1}{p} - \frac{1}{q} < \frac{2}{n+1}$. In that paper the author also shows that the estimate does not hold for $\frac{1}{p} - \frac{1}{q} > \frac{2}{n+1}$. Our results shows that, at least in the case $p > 2$, the estimate remains valid when $\frac{1}{p} - \frac{1}{q} = \frac{2}{n+1}$.

## 4. Appendix

**Proof of Lemma 2.2** If $|x-y| < 2d(x)$ the estimate (2.11) follows easily from

$$|D_{x_ix_j}\Gamma(x-y)| \leq C|x-y|^{-n}$$

and (2.7).

Therefore, the difficult part is to prove the estimate for

$$y \in \Omega_2 := \{y \in \Omega : |x-y| \geq 2d(x)\}$$

First of all, we write $\Omega \times \Omega_2 = D_1 \cup D_2$, where

$$D_1 = \{(x,y) \in \Omega \times \Omega_2 : d(y) \leq 2d(x)\} \quad \textit{and} \quad D_2 = \{(x,y) \in \Omega \times \Omega_2 : d(y) > 2d(x)\}.$$

For $(x,y) \in D_1$ we will prove later that

$$\left|D_{x_ix_j}G(x,y)\right| \leq C\,|x-y|^{-n} \tag{4.1}$$

$$\left|D_{x_i\,x_j} G(x,y)\right| \longrightarrow 0 \text{ as } d(y)\to 0 \tag{4.2}$$

where $C$ is a constant depending only on $n$ and $\Omega$.

Once we have proved (4.1) and (4.2), the proof of estimate (2.11) for $(x,y)\in D_1$ follows in the same way that the proof of Theorem 2.3 i) in [14].

On the other hand, for $(x,y)\in D_2$, we have $d(y) > 2\,d(x)$. Then it is easy to check that $d(y) < 2\,|x-y|$. Therefore, we can prove, using the same arguments as for $D_1$ that

$$\left|D_{y_i\,y_j}\, G(x,y)\right| \le C\,\frac{d(y)}{|x-y|^{n+1}} \tag{4.3}$$

and so, using the symmetry of $G(x,y)$ it is easy to conclude that,

$$\left|D_{x_i\,x_j}\, G(x,y)\right| \le C\,\frac{d(x)}{|x-y|^{n+1}} \tag{4.4}$$

for $(x,y)\in D_2$.

**Proof of (4.1)**

For $(x_0,y)\in D_1$ let $v$ given by

$$\begin{cases} -\Delta v = 0 & \text{in } B(x_0,\tfrac{1}{2}\,d(x_0)) \\ v = G(\cdot,y) & \text{on } \partial B(x_0,\tfrac{1}{2}\,d(x_0)) \end{cases} \tag{4.5}$$

Using the representation formula

$$v(x) \quad = \quad \int_{|z-x_0|=r} \frac{r^2-|x-x_0|^2}{r\,n\,w_n}\,\frac{v(z)}{|x-z|^n}\,dS(z) \tag{4.6}$$

with $r=\frac{1}{2}\,d(x_0)$, we have

$$\begin{aligned}\left|D_{x_i x_j} v(x_0)\right| &\leq \frac{(n+2)}{w_n} \int_{|z-x_0|=r} r^{-n-1}\, |v(z)|\, dS(z) \\ &= \frac{(n+2)}{w_n}\, r^{-n-1} \int_{|z-x_0|=r} |G(z,y)|\, dS(z).\end{aligned}$$

From Theorem 3.3 iii) in [7] we know that $|G(z,y)| \leq \dfrac{d(y)\, d(z)}{|z-y|^n}$ and then

$$\left|D_{x_i x_j} v(x_0)\right| \leq \frac{(n+2)}{w_n}\, r^{-n-1}\, d(y) \int_{|z-x_0|=r} \frac{d(z)}{|z-y|^n}\, dS(z). \tag{4.7}$$

Now, taking into account that for $(x_0, y) \in D_1$ and $z \in \partial B(x_0, \frac{1}{2}\, d(x_0))$, we have that $d(y) \leq 4r,\ d(z) \leq 3r$ and $|z-y| \geq \frac{3}{4}\, |x_0 - y|$, therefore

$$\begin{aligned}\left|D_{x_i x_j} v(x_0)\right| &\leq \frac{(n+2)}{w_n}\, 4\, r^{-n} \int_{|z-x_0|=r} \frac{d(z)}{|z-y|^n}\, dS(z) \\ &\leq \frac{4^{n+1}\,(n+2)}{3^{n-1}\, w_n}\, r^{-n+1}\, |x_0 - y|^{-n} \int_{|z-x_0|=r} dS(z) \\ &\leq \frac{4^{n+1}\,(n+2)}{3^{n-1}}\, |x_0 - y|^{-n}.\end{aligned}$$

Thus, (4.1) follows observing that $v(x) = G(x,y)\ \forall x \in B(x_0, \frac{1}{2}\, d(x_0))$.

**Proof of (4.2)**

For fixed $x \in \Omega$, and $y$ such that $|x-y| = \rho$, we have $G(x,y) \geq C\, |x-y|^{2-n}$ if $\rho$ is small enough (see [7]). Now, let $h \in \mathbb{R}$ with $|h| \leq \frac{1}{2}\, \rho$ and such that, for all $\xi$ in the segment $[x, x + h\, e_j]$

$$d(\xi) < c_1\, |\xi - y| \quad \text{and} \quad d(y) < c_2\, d(\xi)$$

where $c_1$ and $c_2$ are constants.

Then, in the same way that we proved (4.1), we obtain

$$\left|D_{x_i x_j} G(\xi, y)\right| \leq C\, |\xi - y|^{-n}.$$

Therefore,

$$\begin{aligned} \frac{1}{|h|} \left|D_{x_j} G(x + h\, e_i, y) - D_{x_j} G(x, y)\right| &\leq \left|D_{x_i x_j} G(\xi, y)\right| \leq C\, |\xi - y|^{-n} \\ &\leq C\, |x - y|^{-n} \leq C\, \rho^{-2}\, G(x, y). \end{aligned} \tag{4.8}$$

On the other hand, if $y \in \partial\Omega$, estimate (4.8) holds since $G(x, y) = 0$ on $\partial\Omega$.

Then, by the maximum principle for harmonic functions we have (4.8) $\forall y$ with $\rho \leq |x - y|$.

Finally, taking $h \to 0$ we obtain

$$\left|D_{x_i x_j} G(x, y)\right| \leq C\, \rho^{-2}\, G(x, y) \longrightarrow 0 \qquad \text{as } d(y) \to 0$$

as we wanted to show. □

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