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STABILITY OF THE STOKES PROJECTION ON WEIGHTED SPACES AND APPLICATIONS

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Abstract. We show that, on convex polytopes in two or three dimensions, the finite element Stokes projection is stable on weighted spaces $\mathbf{W}_0^{1,p}(\omega,\Omega) \times L^p(\omega,\Omega)$, where the weight belongs to a certain Muckenhoupt class and the integrability index can be different from two. We show how this estimate can be applied to obtain error estimates for approximations of the solution to the Stokes problem with singular sources.

1. Introduction

In this work we shall be interested in the stability and approximation properties of the finite element Stokes projection when measured over weighted norms. To be precise, let $d \in \{2,3\}$ and $\Omega \subset \mathbb{R}^d$ be a convex polytope. Assume that $\mathbb{T} = \{\mathscr{T}_h\}_{h>0}$ is a family of quasiuniform triangulations of $\bar{\Omega}$ parametrized by their mesh size $h > 0$ and $\mathbf{V}_h \times \mathcal{P}_h$ is a pair of finite element spaces constructed over the mesh $\mathscr{T}_h$. To describe the question that we wish to address here let $(\mathsf{u},\pi) \in \mathbf{W}_0^{1,1}(\Omega) \times L^1(\Omega)/\mathbb{R}$, with u solenoidal (see Section 2 for notation), and define $(\mathsf{u}_h,\pi_h) \in \mathbf{V}_h \times \mathcal{P}_h$ to be its Stokes projection, i.e., the pair (u_h,π_h) is such that

$$\begin{cases} \displaystyle\int_\Omega [\nabla \mathsf{u}_h : \nabla \mathbf{v}_h - \pi_h \operatorname{div} \mathbf{v}_h]\,\mathrm{d}x = \int_\Omega [\nabla \mathsf{u} : \nabla \mathbf{v}_h - \pi \operatorname{div} \mathbf{v}_h]\,\mathrm{d}x & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ \displaystyle\int_\Omega q_h \operatorname{div} \mathsf{u}_h \,\mathrm{d}x = 0 & \forall q_h \in \mathcal{P}_h. \end{cases} \tag{1.1}$$

With this notation, the main result in our work is that, for a certain range of integrability indices p and a certain class of Muckenhoupt weights ω, we have

$$\|\nabla \mathsf{u}_h\|_{\mathbf{L}^p(\omega,\Omega)} + \|\pi_h\|_{L^p(\omega,\Omega)} \lesssim \|\nabla \mathsf{u}\|_{\mathbf{L}^p(\omega,\Omega)} + \|\pi\|_{L^p(\omega,\Omega)}. \tag{1.2}$$

Our main motivation for the development of such estimates is the study of the Stokes problem

$$\begin{cases} -\Delta \mathsf{u} + \nabla \pi = \mathbf{f}, & \text{in } \Omega, \\ \operatorname{div} \mathsf{u} = 0, & \text{in } \Omega, \\ \mathsf{u} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

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in the case where the forcing term $\mathbf{f}$ is allowed to be singular. Essentially, by introducing a weight, we can allow for forces such that $\mathbf{f} \notin \mathbf{W}^{-1,2}(\Omega)$. In particular, our theory will allow the following particular examples. For a fixed $\mathbf{F} \in \mathbb{R}^d$ we can set $\mathbf{f} = \mathbf{F}\delta_z$, where δ_z denotes the Dirac delta supported at the interior point $z \in \Omega$. Similarly, if Γ denotes a smooth curve or surface without boundary contained in Ω, we can allow the components of $\mathbf{f}$ to be measures supported in Γ.

While the stability and approximation properties for the Stokes problem in energy type norms has a rich history and is by now well established, the derivation of these properties in non energy norms is more delicate. To our knowledge, the first works that address these questions in a non energy setting are [14, 18]. In these references, the authors establish a L^∞-norm almost stability (up to logarithmic factors) in two dimensions. Later, in view of the weighted a priori estimate for a solution of the divergence operator of [17], the results of [18] were extended to three dimensions; see [17, Section 3] for a discussion. We would also like to mention reference [8] for results on domains with smooth boundaries. Results withouth logarithmic factors where first established in [25], albeit under certain restrictions on the internal angles of the domain. This last assumption was finally removed in [29] and, not so much after and with a different techinque, in [24]. The state of the art is that, simply put, the Stokes projection is stable in $\mathbf{W}^{1,p}_0(\Omega) \times L^p(\Omega)/\mathbb{R}$ for $p \in (1, \infty]$ if the domain Ω is a convex polytope.

We must remark that, in the PDE literature, the idea of introducing weights to handle singular sources is by now well established. There is a vast amount of literature dealing with weighted a priori estimates for solutions of elliptic equations and systems, and for models of incompressible fluids that are even more general than (1.3); see for instance [6]. However, in most of these works, it is usually assumed that the domain is at least C^1, which is not finite element friendly. Two exceptions are [12, 39]. In [12] the well posedness of the Poisson problem in $W^{1,p}_0(\omega, \Omega)$ is established for all $p \in (1, \infty)$ and $\omega \in A_p$, provided Ω is a convex polytope. In addition, the stability of the Ritz projection is obtained for $p \in [2, \infty)$ and $\omega \in A_1$, and for $p = 2$ and $\omega^{-1} \in A_1$. On the other hand, [39] works on general Lipschitz domains, and shows that the Poisson and Stokes problems are well posed, provided p, that depends on the domain, is restricted to a neighborhood of 2 and the weight is regular near the boundary ($\omega \in A_p(\Omega)$ in the notation of that work).

From the discussion given above, it is clear that the stability of the Stokes projection is open and, in light of applications, needed. This is the main contribution of our work.

Our presentation will be organized as follows. We set notation in Section 2, where we also recall the definition of Muckenhoupt weights and introduce the weighted spaces we shall work with. In addition, in Section 2.2, we introduce a saddle point formulation of the Stokes problem (1.3) in weighted spaces and review well-posedness results. In Section 3 we introduce the discrete setting in which we will operate. Section 4 is dedicated to obtaining the stability of the finite element Stokes projection in weighted spaces; this is one of the highlights of our work. As an immediate application, Section 5 studies the development of $\mathbf{L}^p$–error estimates for the error approximation of the velocity field. We also specialize these results and study the approximation of the Stokes problem with a forcing term that is a linear combination of Dirac measures. All the developments of the previous sections rest on a series of assumptions on the finite element velocity–pressure pairs. For this reason in, the final, Section 6 we derive a continuous weighted inf–sup condition and study some suitable finite element pairs that satisfy all the assumptions that our theory rests upon.

2. Notation and preliminaries

We begin by fixing notation and the setting in which we will operate. Throughout this work $d \in \{2,3\}$ and $\Omega \subset \mathbb{R}^d$ is an open, bounded, and convex polytope. If $\mathcal{W}$ and $\mathcal{Z}$ are Banach function spaces, we write $\mathcal{W} \hookrightarrow \mathcal{Z}$ to denote that $\mathcal{W}$ is continuously embedded in $\mathcal{Z}$. We denote by $\mathcal{W}'$ and $\|\cdot\|_{\mathcal{W}}$ the dual and the norm of $\mathcal{W}$, respectively.

For $E \subset \Omega$ open and $f : E \to \mathbb{R}$, we set

$$\fint_E f \,\mathrm{d}x = \frac{1}{|E|}\int_E f \,\mathrm{d}x.$$

For $w \in L^1_{\mathrm{loc}}(\Omega)$, the Hardy–Littlewood maximal operator is defined by

$$\mathcal{M}w(x) = \sup_{Q \ni x} \fint_Q |w(y)| \,\mathrm{d}y, \tag{2.1}$$

where the supremum is taken over all cubes Q containing x.

Given $p \in (1,\infty)$, we denote by p' its Hölder conjugate, i.e., the real number such that $1/p + 1/p' = 1$. By $a \lesssim b$ we will denote that $a \le Cb$, for a constant C that does not depend on a, b nor the discretization parameters. The value of C might change at each occurrence.

2.1. Weights and weighted Sobolev spaces.

By a weight we mean a locally integrable, nonnegative function defined on $\mathbb{R}^d$. If ω is a weight and $E \subset \mathbb{R}^d$ we set

$$\omega(E) = \int_E \omega \,\mathrm{d}x.$$

Of particular interest to us will be the so–called Muckenhoupt A_p weights [13, 36, 44].

Definition 2.1 (Muckenhoupt class A_p)**.** Let $p \in [1,\infty)$ we say that a weight $\omega \in A_p$ if

$$\begin{aligned} [\omega]_{A_p} &:= \sup_B \left(\fint_B \omega \,\mathrm{d}x\right)\left(\fint_B \omega^{1/(1-p)} \,\mathrm{d}x\right)^{p-1} < \infty, \quad p \in (1,\infty), \\ [\omega]_{A_1} &:= \sup_B \left(\fint_B \omega \,\mathrm{d}x\right) \sup_{x \in B} \frac{1}{\omega(x)} < \infty, \quad p = 1, \end{aligned} \tag{2.2}$$

where the supremum is taken over all balls B in $\mathbb{R}^d$. In addition, $A_\infty := \bigcup_{p>1} A_p$. We call $[\omega]_{A_p}$, for $p \in [1,\infty)$, the Muckenhoupt characteristic of ω.

Notice that there is a certain symmetry in the A_p classes with respect to Hölder conjugate exponents. If $\omega \in A_p$, then its conjugate $\omega' := \omega^{1/(1-p)} \in A_{p'}$ and

$$[\omega']_{A_{p'}} = [\omega]_{A_p}^{1/(p-1)}.$$

We comment also that, following [13, Chapter 7.1], an equivalent characterization of $\omega \in A_1$ is that for almost every x,

$$\mathcal{M}\omega(x) \lesssim \omega(x). \tag{2.3}$$

The class of A_p weights was introduced by Muckenhoupt in [36] where he showed that the A_p weights are precisely those weights for which the Hardy-Littlewood maximal operator is bounded on weighted Lebesgue spaces; see [36] and [13, Theorem 7.3].

Distances to lower dimensional objects are prototypical examples of Muckenhoupt weights. In particular, if $\mathcal{K} \subset \Omega$ is a smooth compact submanifold of dimension $k \in [0,d) \cap \mathbb{Z}$ then, owing to [2] and [21, Lemma 2.3(vi)], we have that the function

$$\mathsf{d}_{\mathcal{K}}^{\alpha}(x) = \operatorname{dist}(x,\mathcal{K})^{\alpha} \tag{2.4}$$

belongs to the class A_p provided

$$\alpha \in (-(d-k),(d-k)(p-1)) .$$

This allows us to identify three particular cases:

(i) Let $d > 1$ and $z \in \Omega$, then the weight $\mathsf{d}_z^{\alpha} \in A_2$ if and only if $\alpha \in (-d,d)$.
(ii) Let $d \geq 2$ and $\gamma \subset \Omega$ be a smooth closed curve without self intersections. We have that $\mathsf{d}_{\gamma}^{\alpha} \in A_2$ if and only if $\alpha \in (-(d-1),d-1)$.
(iii) Finally, if $d = 3$ and $\Gamma \subset \Omega$ is a smooth surface without boundary, then $\mathsf{d}_{\Gamma}^{\alpha} \in A_2$ if and only if $\alpha \in (-1,1)$.

It is important to notice, first, that in all the examples shown above we have that either the weight or its inverse, which is the conjugate within the A_2 class, belongs to A_1. Second, since the lower dimensional objects are strictly contained in Ω, there is a neighborhood of $\partial\Omega$ where the weight has no degeneracies or singularities. In fact, it is continuous and strictly positive. This observation motivates us to define a restricted class of Muckenhoupt weights that will be of importance for the analysis that follows. The next definition is inspired by [21, Definition 2.5].

Definition 2.2 (class $A_p(D)$)**.** Let $D \subset \mathbb{R}^d$ be a Lipschitz domain. For $p \in (1,\infty)$ we say that $\omega \in A_p$ belongs to $A_p(D)$ if there is an open set $\mathcal{G} \subset D$, and positive constants $\varepsilon > 0$ and $\omega_l > 0$, such that:
(a) $\{x \in \Omega : \operatorname{dist}(x,\partial D) < \varepsilon\} \subset \mathcal{G}$,
(b) $\omega \in C(\bar{\mathcal{G}})$, and
(c) $\omega_l \leq \omega(x)$ for all $x \in \bar{\mathcal{G}}$.

Notice that the weights described in (i)–(iii) belong to the restricted Muckenhoupt class $A_2(\Omega)$. The latter has been shown to be crucial in the analysis of [39] that guarantees the well–posedness of problem (1.3) in the weighted Sobolev spaces that we define below.

Let $p \in (1,\infty)$, $\omega \in A_p$, and $E \subset \mathbb{R}^d$ be an open set. We define $L^p(\omega,E)$ as the space of Lebesgue p–integrable functions with respect to the measure $\omega\,\mathrm{d}x$. We also define the weighted Sobolev space $W^{1,p}(\omega,E)$ as the set of functions $v \in L^p(\omega,E)$ with weak derivatives $D^{\alpha}v \in L^p(\omega,E)$ for $|\alpha| \leq 1$. The norm of a function $v \in W^{1,p}(\omega,E)$ is given by

$$\|v\|_{W^{1,p}(\omega,E)} := \left(\|v\|_{L^p(\omega,E)}^p + \|\nabla v\|_{L^p(\omega,E)}^p \right)^{1/p} . \tag{2.5}$$

We also define $W_0^{1,p}(\omega,E)$ as the closure of $C_0^{\infty}(E)$ in $W^{1,p}(\omega,E)$. It is remarkable that most of the properties of classical Sobolev spaces have a weighted counterpart. This is not because of the specific form of the weight but rather due to the fact that the weight ω belongs to the Muckenhoupt class A_p. If $p \in (1,\infty)$ and ω

belongs to A_p, then $L^p(\omega, E)$ and $W^{1,p}(\omega, E)$ are Banach spaces [44, Proposition 2.1.2] and smooth functions are dense [44, Corollary 2.1.6]; see also [28, Theorem 1]. In addition, [20, Theorem 1.3] guarantees a weighted Poincaré inequality which, in turn, implies that over $W_0^{1,p}(\omega, E)$ the seminorm $\|\nabla v\|_{L^p(\omega,E)}$ is an equivalent norm to the one defined in (2.5).

Spaces of vector valued functions will be denoted by boldface, that is

$$\mathbf{W}_0^{1,p}(\omega, E) = [W_0^{1,p}(\omega, E)]^d, \quad \|\nabla \mathbf{v}\|_{\mathbf{L}^p(\omega,E)} := \left(\sum_{i=1}^{d} \|\nabla v^i\|_{\mathbf{L}^p(\omega,E)}^p \right)^{1/p},$$

where $\mathbf{v} = (v^1, \ldots, v^d)^\intercal$.

For future use we recall a particular Sobolev–type embedding theorem between weighted spaces. For the general case we refer to [7, 22, 34] and [38, Section 6].

Proposition 2.3 (embedding in weighted spaces)**.** *Let $p \in (1, \infty)$ and $\omega \in A_p$. Assume that, for all $x \in \Omega$ and $0 < r \leq R$, we have that*

$$\frac{r^{p+d}}{R^{p+d}} \frac{\omega(B(x, R))}{\omega(B(x, r))} \lesssim 1,$$

then $W^{1,p}(\omega, \Omega) \hookrightarrow L^p(\Omega)$ and $W^{1,p'}(\Omega) \hookrightarrow L^{p'}(\omega', \Omega)$.

2.2. **The Stokes problem in weighted spaces.** We begin with a motivation for the use of weights. Let us assume that (1.3) is posed over the whole space $\mathbb{R}^d$ and that $\mathbf{f} = \mathbf{F}\delta_z$ for some $z \in \mathbb{R}^d$. The results of [23, Section IV.2] thus provide the following asymptotic behavior of the solution (u, π) to problem (1.3) near the point z:

$$|\nabla \mathsf{u}(x)| \approx |x - z|^{1-d} \quad \text{and} \quad |\pi(x)| \approx |x - z|^{1-d}, \tag{2.6}$$

so that $|\nabla \mathsf{u}|, \pi \notin L^2(\mathbb{R}^d)$. However, basic computations reveal that, for every ball B,

$$\alpha \in (d - 2, \infty) \implies \int_B \mathsf{d}_z^\alpha |\nabla \mathsf{u}|^2 \, \mathrm{d}x < \infty, \quad \int_B \mathsf{d}_z^\alpha |\pi|^2 \, \mathrm{d}x < \infty.$$

This heuristic suggests to seek solutions to problem (1.3) in weighted Sobolev spaces [6, 39]. In what follows we will make these considerations rigorous.

Let $\omega \in A_p$. Given $\mathbf{f} \in \mathbf{W}^{-1,p}(\omega, \Omega)$, we seek for $(\mathsf{u}, \pi) \in \mathbf{W}_0^{1,p}(\omega, \Omega) \times L^p(\omega, \Omega)/\mathbb{R}$ such that

$$\begin{cases} a(\mathsf{u}, \mathbf{v}) + b(\mathbf{v}, \pi) = \langle \mathbf{f}, \mathbf{v} \rangle & \forall \mathbf{v} \in \mathbf{W}_0^{1,p'}(\omega', \Omega), \\ b(\mathsf{u}, q) = 0 & \forall q \in L^{p'}(\omega', \Omega)/\mathbb{R}, \end{cases} \tag{2.7}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathbf{W}^{-1,p}(\omega, \Omega) := \mathbf{W}_0^{1,p'}(\omega', \Omega)'$ and $\mathbf{W}_0^{1,p'}(\omega', \Omega)$. Finally, to shorten notation, here and in what follows, we set

$$a(\mathbf{v}, \mathbf{w}) = \int_\Omega \nabla \mathbf{v} : \nabla \mathbf{w} \, \mathrm{d}x, \qquad b(\mathbf{v}, q) = -\int_\Omega q \operatorname{div} \mathbf{v} \, \mathrm{d}x.$$

The well–posedness of (2.7) in Lipschitz domains was studied in [39, Theorem 17]. The main result is summarized below.

Proposition 2.4 (well–posedness in weighted spaces)**.** *Let $d \in \{2, 3\}$ and $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. There exists $\epsilon = \epsilon(d, \Omega) \in (0, 1]$ such that if $P = 2 + \epsilon$, $p \in (P', P)$, and $\omega \in A_p(\Omega)$, problem (2.7) is well posed. In other words, for all $\mathbf{f} \in$*

$\mathbf{W}^{-1,p}(\omega,\Omega)$ *problem* (2.7) *has a unique solution* $(\mathsf{u},\pi) \in \mathbf{W}_0^{1,p}(\omega,\Omega) \times L^p(\omega,\Omega)/\mathbb{R}$ *and the following stability estimate holds*

$$\|\nabla \mathsf{u}\|_{\mathbf{L}^p(\omega,\Omega)} + \|\pi\|_{L^p(\omega,\Omega)} \lesssim \|\mathbf{f}\|_{\mathbf{W}^{-1,p}(\omega,\Omega)}. \tag{2.8}$$

Remark 2.5 ($p < 2$)*.* Strictly speaking [39, Theorem 17] only shows well–posedness for $p \geq 2$. However, using the equivalent characterization of well–posedness via inf–sup conditions given in [4, Theorem 2.1], see also [19, Exercise 2.14], one can deduce that (2.7) is also well–posed for $p \in (P', 2)$.

Notice that Proposition 2.4 assumes only that the domain is Lipschitz. Finer results can be obtained provided more information on the domain is available. Since we are working on convex polytopes we have the following result; see [35, Corollary 1.8].

Proposition 2.6 (L^p–regularity)**.** *Let* $d \in \{2,3\}$ *and* $\Omega \subset \mathbb{R}^d$ *be a convex polytope. If* $p \in (1,2]$ *and* $\mathbf{f} \in \mathbf{L}^p(\Omega)$*, then the solution of* (1.3) *is such that*

$$\mathsf{u} \in \mathbf{W}^{2,p}(\Omega) \cap \mathbf{W}_0^{1,p}(\Omega), \qquad \pi \in W^{1,p}(\Omega)/\mathbb{R},$$

with a corresponding estimate.

3. Finite element approximation

We now introduce the discrete setting in which we will operate. We first introduce some terminology and a few basic ingredients and assumptions that will be common to all our methods.

3.1. Triangulation and finite element spaces.

We denote by $\mathscr{T}_h = \{T\}$ a conforming partition, or mesh, of $\bar{\Omega}$ into closed simplices T with size $h_T = \operatorname{diam}(T)$ and define $h = \max_{T \in \mathscr{T}_h} h_T$. We assume that $\mathbb{T} = \{\mathscr{T}_h\}_{h>0}$ is a collection of conforming and quasiuniform meshes [9, 19]. For $T \in \mathscr{T}_h$, we define the *star* or *patch* associated with the element T as

$$\mathcal{S}_T := \bigcup \{T' \in \mathscr{T}_h : T \cap T' \neq \emptyset\}. \tag{3.1}$$

In the literature, several finite element approximations have been proposed and analyzed to approximate the solution to the Stokes problem (2.7) when the forcing term of the momentum equation is not singular; see, for instance, [19, Section 4], [26, Chapter II], and references therein. Initially we shall not be specific about the type of finite element approximation that we are using. We will only state a set of assumptions that our discrete spaces need to satisfy. Given a mesh $\mathscr{T}_h \in \mathbb{T}$, we denote by $\mathbf{V}_h$ and $\mathcal{P}_h$ the finite element spaces that approximate the velocity field and the pressure, respectively, constructed over $\mathscr{T}_h$. We assume that, for every $p \in (1,\infty)$ and $\omega \in A_p$,

$$\mathbf{V}_h \subset \mathbf{W}_0^{1,\infty}(\Omega) \subset \mathbf{W}_0^{1,p}(\omega,\Omega), \qquad \mathcal{P}_h \subset L^\infty(\Omega)/\mathbb{R} \subset L^p(\omega,\Omega)/\mathbb{R}.$$

In addition, we require that functions in $\mathbf{V}_h$ and $\mathcal{P}_h$ are locally polynomials of degree at least one and zero, respectively. Moreover, we need to assume that these spaces are compatible, in the sense that they satisfy weighted versions of the classical LBB condition [19, Proposition 4.13]. Namely, we assume that if $\omega \in A_p$

then, there exists a positive constant $\beta = \beta([\omega]_{A_p})$ such that, for all $\mathscr{T}_h \in \mathbb{T}$,

$$
\begin{cases} \inf_{q_h \in \mathcal{P}_h} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \dfrac{b(\mathbf{v}_h, q_h)}{\|\nabla \mathbf{v}_h\|_{\mathbf{L}^{p'}(\omega',\Omega)} \|q_h\|_{L^p(\omega,\Omega)}} \geq \beta, \\ \inf_{q_h \in \mathcal{P}_h} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \dfrac{b(\mathbf{v}_h, q_h)}{\|\nabla \mathbf{v}_h\|_{\mathbf{L}^{p}(\omega,\Omega)} \|q_h\|_{L^{p'}(\omega',\Omega)}} \geq \beta. \end{cases} \tag{3.2}
$$

3.2. **A quasi–interpolation operator.** Since our interest is to approximate rough functions the classical Lagrange interpolation operator cannot be applied. Instead, we can consider a variant of the quasi–interpolation operator analyzed in [38]. Its construction is inspired in the ideas developed by Clément [10], Scott and Zhang [42], and Durán and Lombardi [15]: it is built on local averages over stars and is thus well–defined for locally integrable functions; it also exhibits optimal approximation properties.

For $\mathscr{T}_h \in \mathbb{T}$, we let X_h be the space of piecewise linear, continuous, functions over the mesh $\mathscr{T}_h$. For $w \in L^1(\Omega)$, we define $\Pi_{X_h} w \in X_h$ to be the interpolation operator of [38] onto piecewise linears. Define $\mathbf{X}_h = [X_h \cap H^1_0(\Omega)]^d$. For $\mathbf{v} \in \mathbf{W}^{1,1}_0(\Omega)$, we set $\Pi_{\mathbf{V}_h}\mathbf{v} \in \mathbf{X}_h \subset \mathbf{V}_h$ to be the operator Π_{X_h} applied component–wise and accordingly modified to preserve boundary conditions.

To define an interpolant onto the pressure space $\mathcal{P}_h$ we distinguish two cases. If $\mathcal{P}_h$ contains piecewise constants, then, for $q \in L^1(\Omega)/\mathbb{R}$, we simply define $\Pi_{\mathcal{P}_h} q \in \mathcal{P}_h$ to be the local average of q. On the other hand, if $\mathcal{P}_h$ contains piecewise linears $\Pi_{\mathcal{P}_h} q = \Pi_{X_h} q + c_q$, where $c_q \in \mathbb{R}$ is chosen so that $\Pi_{\mathcal{P}_h} q \in \mathcal{P}_h$.

To alleviate notation, if there is no source of confusion, we shall use Π_h to denote indistinctely $\Pi_{\mathbf{V}_h}$ or $\Pi_{\mathcal{P}_h}$. The properties of Π_h are summarized below. For a proof we refer the reader to [38, Section 5].

Proposition 3.1 (stability and interpolation estimates). *Let $p \in (1,\infty)$, $\omega \in A_p$, and $T \in \mathscr{T}_h$. Then, for every $v \in W^{1,p}(\omega, \mathcal{S}_T)$, we have the local stability bound*

$$
\|\nabla \Pi_h v\|_{\mathbf{L}^p(\omega,T)} \lesssim \|\nabla v\|_{\mathbf{L}^p(\omega,\mathcal{S}_T)} \tag{3.3}
$$

and the interpolation error estimate

$$
\|v - \Pi_h v\|_{L^p(\omega,T)} \lesssim h_T \|\nabla v\|_{\mathbf{L}^p(\omega,\mathcal{S}_T)}. \tag{3.4}
$$

The hidden constants, in (3.3) *and* (3.4)*, are independent of v, T, and h.*

This operator also enjoys the following approximation property [38, Section 6].

Proposition 3.2 (interpolation in different metrics). *Assume that $\omega \in A_p$ is such that Proposition 2.3 holds. Then, for every $v_p \in W^{1,p}(\omega, \mathcal{S}_T)$, we have that*

$$
\|v_p - \Pi_h v_p\|_{L^p(T)} \lesssim h_T^{1+d/p} \omega(\mathcal{S}_T)^{-1/p} \|\nabla v_p\|_{\mathbf{L}^p(\omega,\mathcal{S}_T)}.
$$

Similarly, for $v_{p'} \in W^{1,p'}(\mathcal{S}_T)$, we have

$$
\|v_{p'} - \Pi_h v_{p'}\|_{L^{p'}(\omega',T)} \lesssim h_T^{1-d/p'} \omega'(\mathcal{S}_T)^{1/p'} \|\nabla v_{p'}\|_{\mathbf{L}^{p'}(\mathcal{S}_T)}.
$$

The hidden constants in the previous inequalities are independent of the functions being iterpolated, the cell T, and h.

Remark 3.3 (higher order elements). We comment that the construction of [38] allows for polynomial degrees of any order, with the corresponding analogue of (3.3) and (3.4) being true. Since, as mentioned in the introduction, our main motivation for the introduction of weights is to handle problems with singular data we do not expect the solution to possess much regularity. For this reason we only consider interpolations into piecewise linears for the velocity and either constants or linears for the pressure, respectively.

3.3. **Approximate Green's function.** Let $z \in \Omega$ be such that $z \in \mathring{T}_z$ for some $T_z \in \mathscr{T}_h$. Let $\tilde{\delta}_z$ be a regularized Dirac delta satisfying the following properties:

1. $\tilde{\delta}_z \in C_0^\infty(T_z)$;
2. $\int_\Omega \tilde{\delta}_z \,\mathrm{d}x = 1$;
3. $\|\tilde{\delta}_z\|_{L^\infty(T_z)} \lesssim h_{T_z}^{-d}$;
4. $\int_\Omega \tilde{\delta}_z \mathbf{v}_h \,\mathrm{d}x = \mathbf{v}_h(z)$ for all $\mathbf{v}_h \in \mathbf{V}_h$.

We refer to [43] and [5, Exercise 8.1] for a construction of such a function. Notice that, if $\mathbf{v}_h = (v_h^1, \dots, v_h^d)^\intercal \in \mathbf{V}_h$ and $j \in \{1, \dots, d\}$, we have

$$\partial_{x_i} \mathbf{v}_h^j(z) = \int_\Omega \partial_{x_i} \mathbf{v}_h^j \tilde{\delta}_z \,\mathrm{d}x = -\int_\Omega \mathbf{v}_h^j \partial_{x_i} \tilde{\delta}_z \,\mathrm{d}x, \quad i \in \{1, \dots, d\}.$$

With these ingredients at hand, we define a regularized Green's function $(\mathbf{G}, Q)$ as the solution to the following problem: Find $(\mathbf{G}, Q) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ such that

$$\begin{cases} a(\mathbf{G}, \mathbf{v}) + b(\mathbf{v}, Q) = \int_\Omega \tilde{\delta}_z \partial_{x_i} \mathbf{v}^j \,\mathrm{d}x & \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ b(\mathbf{G}, q) = 0 & \forall q \in L^2(\Omega)/\mathbb{R}, \end{cases} \tag{3.5}$$

where $i, j \in \{1, \cdots, d\}$. Notice that the functions $\mathbf{G}$ and Q depend on z and the indices i and j. However, to alleviate notation we will omit this dependence.

We also define $(\mathbf{G}_h, Q_h)$, the Stokes projection of $(\mathbf{G}, Q)$, as the solution to the discrete problem: Find $(\mathbf{G}_h, Q_h) \in \mathbf{V}_h \times \mathcal{P}_h$ such that

$$\begin{cases} a(\mathbf{G}_h, \mathbf{v}_h) + b(\mathbf{v}_h, Q_h) = \int_\Omega \tilde{\delta}_z \partial_{x_i} \mathbf{v}_h^j \,\mathrm{d}x & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{G}_h, q_h) = 0 & \forall q_h \in \mathcal{P}_h. \end{cases} \tag{3.6}$$

Let R be a fixed positive number such that for any $x \in \bar{\Omega}$ the ball $B(x, R)$ contains Ω. For $y \in \Omega$, we define the weight function σ_y, introduced by Natterer [37], as

$$\sigma_y(x) = \left(|x - y|^2 + (\kappa h)^2 \right)^{1/2}, \tag{3.7}$$

where $\kappa > 1$ is a parameter independent of h but such that $\kappa h \le R$; see [24, Section 1.7]. We recall that this weight verifies [25, inequality (0.18)]

$$\int_\Omega \sigma_y^{-d-\lambda} \,\mathrm{d}x \lesssim h^{-\lambda}, \quad \lambda \in (0, 1). \tag{3.8}$$

We shall assume that if $\nu \in (0, 1/2)$, $0 < \lambda < \nu/2$, $\mu = d + \lambda$, and $\mathscr{T}_h$ is quasiuniform, then there exists $\kappa_1 > 1$ such that for all $\kappa \ge \kappa_1$ and for all meshsizes $h > 0$ such that $\kappa h \le R$, we have

$$\sup_{y \in \Omega} \left\| \sigma_y^{\frac{\mu}{2}} \nabla(\mathbf{G} - \mathbf{G}_h) \right\|_{\mathbf{L}^2(\Omega)} \lesssim h^{\lambda/2}. \tag{3.9}$$

Examples of spaces that satisfy this assumption will be presented below.

4. Discrete stability estimates in weighted spaces

Let $p \in (1,\infty)$, $\omega \in A_p$, $(\mathsf{u},\pi) \in \mathbf{W}_0^{1,p}(\omega,\Omega) \times L^p(\omega,\Omega)/\mathbb{R}$ with u solenoidal, and the pair $(\mathsf{u}_h, p_h) \in \mathbf{V}_h \times \mathcal{P}_h$ be the finite element approximation of (u,π). Our goal in this section is to, on the basis of the weighted compatibility conditions (3.2), derive the weighted stability estimate (1.2). To do so, we must place some restrictions on the range of the integrability p and the weight ω. We codify these in the following assumption

$$\text{(S)} \qquad \begin{cases} p \in (2,\infty) & \implies \omega \in A_1, \\ p = 2 & \implies \omega \in A_1, \text{ or } \omega^{-1} \in A_2(\Omega) \cap A_1, \\ p \in (P',2] & \implies \omega' \in A_{p'}(\Omega) \cap A_1, \end{cases}$$

where P is as in Proposition 2.4 and P' is its Hölder conjugate.

Theorem 4.1 (weighted stability estimate)**.** *Let $d \in \{2,3\}$ and $\Omega \subset \mathbb{R}^d$ be an open convex polytope. Assume that* (S) *holds and that $(\mathsf{u},\pi) \in \mathbf{W}_0^{1,p}(\omega,\Omega) \times L^p(\omega,\Omega)/\mathbb{R}$ with u solenoidal. Let $(\mathsf{u}_h,\pi_h) \in \mathbf{V}_h \times \mathcal{P}_h$ be its finite element Stokes projection. If the spaces $(\mathbf{V}_h,\mathcal{P}_h)$ satisfy* (3.2) *and* (3.9)*, then estimate* (1.2) *holds. The hidden constant in this estimate is independent of (u,π), (u_h,π_h), and h.*

Proof. We begin by noticing that, by density, it suffices to show the estimate assuming that u and π are smooth.

We split the proof in several steps.

1. Assume that we have already shown that

$$\|\nabla \mathsf{u}_h\|_{\mathbf{L}^p(\omega,\Omega)} \lesssim \|\nabla \mathsf{u}\|_{\mathbf{L}^p(\omega,\Omega)} + \|\pi\|_{L^p(\omega,\Omega)}. \tag{4.1}$$

Utilizing the first discrete inf–sup condition of (3.2) and that (u_h,π_h) solves (1.1), we arrive at

$$\|\pi_h\|_{L^p(\omega,\Omega)} \lesssim \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h,\pi_h)}{\|\nabla \mathbf{v}_h\|_{\mathbf{L}^{p'}(\omega',\Omega)}} = \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{a(\mathsf{u},\mathbf{v}_h) - a(\mathsf{u}_h,\mathbf{v}_h) + b(\mathbf{v}_h,\pi)}{\|\nabla \mathbf{v}_h\|_{\mathbf{L}^{p'}(\omega',\Omega)}},$$

which immediately yields

$$\|\pi_h\|_{L^p(\omega,\Omega)} \lesssim \|\nabla \mathsf{u}\|_{\mathbf{L}^p(\omega,\Omega)} + \|\pi\|_{L^p(\omega,\Omega)} + \|\nabla \mathsf{u}_h\|_{\mathbf{L}^p(\omega,\Omega)}.$$

This, in view of (4.1), implies the desired bound for $\|\pi_h\|_{L^p(\omega,\Omega)}$.

2. Assume that $p \geq 2$ and $\omega \in A_1$. Set $\mathbf{v}_h = \mathsf{u}_h$ in (3.6) to arrive at

$$a(\mathbf{G}_h,\mathsf{u}_h) = \int_\Omega \tilde{\delta}_z \partial_{x_i} \mathsf{u}_h^j \,\mathrm{d}x = \partial_{x_i} \mathsf{u}_h^j(z).$$

Set now $\mathbf{v}_h = \mathbf{G}_h$ in (1.1) and use that $b(\mathbf{G}_h,q_h) = 0$ for all $q_h \in \mathcal{P}_h$ to obtain

$$a(\mathsf{u}_h,\mathbf{G}_h) = a(\mathsf{u},\mathbf{G}_h) + b(\mathbf{G}_h,\pi). \tag{4.2}$$

Using that $b(\mathbf{G},\pi) = 0$, we can thus conclude the identity

$$a(\mathsf{u}_h,\mathbf{G}_h) = a(\mathsf{u},\mathbf{G}_h) + b(\mathbf{G}_h,\pi) = a(\mathsf{u},\mathbf{G}_h - \mathbf{G}) + b(\mathbf{G}_h - \mathbf{G},\pi) + a(\mathsf{u},\mathbf{G}).$$

Since the bilinear form a is symmetric, we have

$$\begin{aligned} \partial_{x_i} \mathsf{u}_h^j(z) &= a(\mathsf{u},\mathbf{G}_h - \mathbf{G}) + b(\mathbf{G}_h - \mathbf{G},\pi) + a(\mathsf{u},\mathbf{G}) \\ &= a(\mathsf{u},\mathbf{G}_h - \mathbf{G}) + b(\mathbf{G}_h - \mathbf{G},\pi) + \int_\Omega \tilde{\delta}_z \partial_{x_i} \mathsf{u}^j \,\mathrm{d}x. \end{aligned}$$

Notice that here we used the smoothness assumption on u to be able to assert that this is an admissible test function in (3.5).

Let now $\mathbf{E} = \mathbf{G} - \mathbf{G}_h$. The previous equality implies that

$$\int_\Omega \omega |\partial_{x_i} \mathsf{u}_h^j|^p \,\mathrm{d}z \lesssim \int_\Omega \omega \left[\int_\Omega \nabla \mathsf{u} : \nabla \mathbf{E} \,\mathrm{d}x \right]^p \mathrm{d}z + \int_\Omega \omega \left[\int_\Omega \pi \operatorname{div} \mathbf{E} \,\mathrm{d}x \right]^p \mathrm{d}z + \int_\Omega \omega \left[\fint_{T_z} |\nabla \mathsf{u}| \,\mathrm{d}x \right]^p \mathrm{d}z =: \mathrm{I} + \mathrm{II} + \mathrm{III},$$

where we have used that $\tilde{\delta}_z$ is supported on T_z and that $\|\tilde{\delta}_z\|_{L^\infty(\Omega)} \lesssim h^{-d}$.

We estimate the terms I, II, and III with the help of (3.9), similar arguments to those developed in the proof of [12, Theorem 3.1], and modifications inspired by [40]. We begin by controlling the term III. Since the weight $\omega \in A_1 \subset A_p$, we utilize that the Hardy–Littlewood maximal operator $\mathcal{M}$ is continuous from $L^p(\omega, \mathbb{R}^d)$ to $L^p(\omega, \mathbb{R}^d)$ to arrive at

$$\mathrm{III} = \int_\Omega \omega \left[\fint_{T_z} |\nabla \mathsf{u}| \,\mathrm{d}x \right]^p \mathrm{d}z \lesssim \int_\Omega \omega \mathcal{M}(|\nabla \mathsf{u}|)^p \,\mathrm{d}z \lesssim \int_\Omega \omega |\nabla \mathsf{u}|^p \,\mathrm{d}z.$$

We now control I and II. Using the weight σ_z, defined in (3.7), and its property (3.8) we have that for any $\lambda \in (0,1)$

$$\int_\Omega \nabla \mathsf{u} : \nabla \mathbf{E} \,\mathrm{d}x \lesssim h^{-\lambda(p-2)/(2p)} \left(\int_\Omega \sigma_z^{-d-\lambda} |\nabla \mathsf{u}|^p \,\mathrm{d}x \right)^{1/p} \left(\int_\Omega \sigma_z^{d+\lambda} |\nabla \mathbf{E}|^2 \,\mathrm{d}x \right)^{1/2}$$

and

$$\int_\Omega \pi \operatorname{div} \mathbf{E} \,\mathrm{d}x \lesssim h^{-\lambda(p-2)/(2p)} \left(\int_\Omega \sigma_z^{-d-\lambda} |\pi|^p \,\mathrm{d}x \right)^{1/p} \left(\int_\Omega \sigma_z^{d+\lambda} |\operatorname{div} \mathbf{E}|^2 \,\mathrm{d}x \right)^{1/2}.$$

Thus, we have that

$$\mathrm{I} + \mathrm{II} \lesssim h^{-\lambda(p-2)/2} \int_\Omega \omega \left(\int_\Omega \sigma_z^{d+\lambda} |\nabla \mathbf{E}|^2 \,\mathrm{d}x \right)^{p/2} \left(\int_\Omega \frac{|\nabla \mathsf{u}|^p + |\pi|^p}{\sigma_z^{d+\lambda}} \,\mathrm{d}x \right) \mathrm{d}z.$$

Assume now that $0 < \lambda < \nu/2$ with $\nu \in (0, 1/2)$. In this case estimate (3.9) immediately yields

$$h^{-\lambda(p-2)/2} \left(\int_\Omega \sigma_z^{d+\lambda} |\nabla \mathbf{E}|^2 \,\mathrm{d}x \right)^{p/2} \lesssim h^\lambda.$$

In addition, the arguments developed in the proof of [12, Theorem 3.1] yield

$$\int_\Omega \frac{h^\lambda \omega(z)}{(|x-z|^2 + (\kappa h)^2)^{(d+\lambda)/2}} \,\mathrm{d}z \lesssim \mathcal{M}\omega(x) \lesssim \omega(x), \tag{4.3}$$

where, in the last step, we used (2.3). For completness, we have provided a detailed proof of this estimate in Appendix A. In conclusion, we obtained that

$$\mathrm{I} + \mathrm{II} \lesssim \int_\Omega \int_\Omega \frac{\omega(z) h^\lambda}{(|x-z|^2 + (\kappa h)^2)^{(d+\lambda)/2}} \,\mathrm{d}z \left(|\nabla \mathsf{u}(x)|^p + |\pi(x)|^p \right) \mathrm{d}x \lesssim \int_\Omega \omega(x) \left(|\nabla \mathsf{u}(x)|^p + |\pi(x)|^p \right) \mathrm{d}x.$$

A collection of the estimates for the terms I, II, and III yield (4.1) when $p \geq 2$.

3. It remains to consider the case $p \in (P',2]$ with $\omega' \in A_{p'}(\Omega) \cap A_1$. Notice that $p' = p/(p-1) \geq 2$ so that, as in [12, Corollary 3.3], we will reduce our considerations to the previous case. Since $p' \in [2,P)$ and $\omega' \in A_{p'}(\Omega)$ then, as Proposition 2.4 shows, for every $\mathbf{g} \in \mathbf{W}^{-1,p'}(\omega',\Omega)$ we conclude that the Stokes problem

$$\begin{cases} -\Delta \boldsymbol{\varphi}_{\mathbf{g}} + \nabla \psi_{\mathbf{g}} = \mathbf{g}, & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{\varphi}_{\mathbf{g}} = 0, & \text{in } \Omega, \\ \boldsymbol{\varphi}_{\mathbf{g}} = 0, & \text{on } \partial\Omega, \end{cases}$$

is well-posed in $\mathbf{W}_0^{1,p'}(\omega',\Omega) \times L^{p'}(\omega',\Omega)/\mathbb{R}$. So that we have the estimate

$$\|\nabla \boldsymbol{\varphi}_{\mathbf{g}}\|_{\mathbf{L}^{p'}(\omega',\Omega)} + \|\psi_{\mathbf{g}}\|_{L^{p'}(\omega',\Omega)} \lesssim \|\mathbf{g}\|_{\mathbf{W}^{-1,p'}(\omega',\Omega)}.$$

Let $(\boldsymbol{\varphi}_{\mathbf{g},h}, \psi_{\mathbf{g},h}) \in \mathbf{V}_h \times \mathcal{P}_h$ be the Stokes projection of $(\boldsymbol{\varphi}_{\mathbf{g}}, \psi_{\mathbf{g}})$ we have

$$\begin{aligned} \|\nabla \mathsf{u}_h\|_{\mathbf{L}^p(\omega,\Omega)} &= \sup_{\mathbf{g}\in\mathbf{W}^{-1,p'}(\omega',\Omega)} \frac{\langle \mathbf{g}, \mathsf{u}_h\rangle}{\|\mathbf{g}\|_{\mathbf{W}^{-1,p'}(\omega',\Omega)}} \\ &= \sup_{\mathbf{g}\in\mathbf{W}^{-1,p'}(\omega',\Omega)} \frac{a(\mathsf{u}_h, \boldsymbol{\varphi}_{\mathbf{g}}) + b(\mathsf{u}_h, \psi_{\mathbf{g}})}{\|\mathbf{g}\|_{\mathbf{W}^{-1,p'}(\omega',\Omega)}} \\ &= \sup_{\mathbf{g}\in\mathbf{W}^{-1,p'}(\omega',\Omega)} \frac{a(\boldsymbol{\varphi}_{\mathbf{g},h}, \mathsf{u}_h) + b(\mathsf{u}_h, \psi_{\mathbf{g},h})}{\|\mathbf{g}\|_{\mathbf{W}^{-1,p'}(\omega',\Omega)}} \\ &= \sup_{\mathbf{g}\in\mathbf{W}^{-1,p'}(\omega',\Omega)} \frac{a(\mathsf{u}, \boldsymbol{\varphi}_{\mathbf{g},h}) + b(\boldsymbol{\varphi}_{\mathbf{g},h}, \pi)}{\|\mathbf{g}\|_{\mathbf{W}^{-1,p'}(\omega',\Omega)}}, \end{aligned}$$

where we used that both u_h and $\boldsymbol{\varphi}_{\mathbf{g},h}$ are discretely solenoidal. The stability of the Stokes projection in $\mathbf{W}^{1,p'}(\omega',\Omega) \times L^{p'}(\omega',\Omega)$ and the bound on $(\boldsymbol{\varphi}_{\mathbf{g}}, \psi_{\mathbf{g}})$ yield

$$\|\nabla \mathsf{u}_h\|_{\mathbf{L}^p(\omega,\Omega)} \lesssim \|\nabla \mathsf{u}\|_{\mathbf{L}^p(\omega,\Omega)} + \|\pi\|_{L^p(\omega,\Omega)}.$$

The proof is thus complete. □

As usual, the a priori estimate (1.2) implies a best approximation result *à la* Céa.

Corollary 4.2 (best approximation)**.** *In the setting of Theorem 4.1, assume, in addition, that $p \in (P',P)$, and $\omega \in A_p(\Omega)$. Then we have that*

$$\|\nabla(\mathsf{u} - \mathsf{u}_h)\|_{\mathbf{L}^p(\omega,\Omega)} + \|\pi - \pi_h\|_{L^p(\omega,\Omega)} \lesssim \inf_{\mathbf{w}_h\in\mathbf{V}_h} \|\nabla(\mathsf{u} - \mathbf{w}_h)\|_{\mathbf{L}^p(\omega,\Omega)} + \inf_{r_h\in\mathcal{P}_h} \|\pi - r_h\|_{L^p(\omega,\Omega)},$$

where the hidden constant is independent of (u,π), (u_h, p_h), and h.

Proof. The proof is rather standard but we reproduce it here for the sake of completeness. Notice that, if $\mathbf{w}_h \in \mathbf{V}_h$ and $r_h \in \mathcal{P}_h$ are arbitrary, by linearity of (1.1) we obtain that, for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times \mathcal{P}_h$ we have

$$\begin{cases} a(\mathsf{u}_h - \mathbf{w}_h, \mathbf{v}_h) + b(\mathbf{v}_h, \pi_h - r_h) = a(\mathsf{u} - \mathbf{w}_h, \mathbf{v}_h) + b(\mathbf{v}_h, \pi - r_h), \\ b(\mathsf{u}_h - \mathbf{w}_h, q_h) = b(\mathsf{u} - \mathbf{w}_h, q_h). \end{cases}$$

Let now $(\boldsymbol{\varphi}, \psi) \in \mathbf{W}_0^{1,p}(\omega,\Omega) \times L^p(\omega,\Omega)/\mathbb{R}$ be the unique solution of

$$\begin{cases} a(\boldsymbol{\varphi},\mathbf{v}) + b(\mathbf{v},\psi) = a(\mathsf{u}-\mathbf{w}_h,\mathbf{v}) + b(\mathbf{v},\pi - r_h), & \forall \mathbf{v} \in \mathbf{W}_0^{1,p'}(\omega',\Omega), \\ b(\boldsymbol{\varphi},q) = b(\mathsf{u}-\mathbf{w}_h,q), & \forall q \in L^{p'}(\omega',\Omega)/\mathbb{R}. \end{cases}$$

As shown in Proposition 2.4, the assumptions on the integrability index and the weight allow us to conclude that this problem is well posed and we have the estimate

$$\|\nabla \boldsymbol{\varphi}\|_{\mathbf{L}^p(\omega,\Omega)} + \|\psi\|_{L^p(\omega,\Omega)} \lesssim \|\nabla(\mathsf{u}-\mathbf{w}_h)\|_{\mathbf{L}^p(\omega,\Omega)} + \|\pi - r_h\|_{L^p(\omega,\Omega)}. \tag{4.4}$$

Notice now that $(\mathsf{u}_h - \mathbf{w}_h, \pi_h - r_h) \in \mathbf{V}_h \times \mathcal{P}_h$ is nothing but the finite element approximation of $(\boldsymbol{\varphi},\psi) \in \mathbf{W}_0^{1,p}(\omega,\Omega) \times L^p(\omega,\Omega)/\mathbb{R}$. This, in conjunction with Theorem 4.1 and (4.4) then yields

$$\|\nabla(\mathsf{u}_h - \mathbf{w}_h)\|_{\mathbf{L}^p(\omega,\Omega)} + \|\pi_h - r_h\|_{L^p(\omega,\Omega)} \lesssim \|\nabla(\mathsf{u}-\mathbf{w}_h)\|_{\mathbf{L}^p(\omega,\Omega)} + \|\pi - r_h\|_{L^p(\omega,\Omega)}.$$

Conclude with the triangle inequality. □

5. Error estimates

We now provide a $\mathbf{L}^p(\Omega)$–error estimate for the error approximation of the velocity field. For that, obviously, one needs to assume that Proposition 2.3 holds, so that $\mathsf{u} \in \mathbf{L}^p(\Omega)$.

In what follows, for a weight ω, we denote by $\omega(h) = \sup_{T\in\mathscr{T}_h} \omega(T)$. The main error estimate is provided below.

Theorem 5.1 (error estimate)**.** *Let $p \in [2,P)$ and $\omega \in A_p(\Omega)$ be such that condition* (S) *holds. Assume, in addition, that the compatibility condition required for Proposition 2.3 to be valid holds. Let $(\mathsf{u},\pi) \in \mathbf{W}_0^{1,p}(\omega,\Omega) \times L^p(\omega,\Omega)/\mathbb{R}$ with u solenoidal, and let $(\mathsf{u}_h,\pi_h) \in \mathbf{V}_h \times \mathcal{P}_h$ be its Stokes projection, defined as the solution of* (1.1)*. In this setting, we have that*

$$\|\mathsf{u}-\mathsf{u}_h\|_{\mathbf{L}^p(\Omega)} \lesssim h^{1+d/p}\omega(h)^{-1/p}\left(\|\nabla \mathsf{u}\|_{\mathbf{L}^p(\omega,\Omega)} + \|\pi\|_{L^p(\omega,\Omega)}\right), \tag{5.1}$$

where the hidden constant is independent of (u,π), (u_h,π_h), and h.

Proof. We proceed in several steps on the basis of a duality argument.

1. We begin by recalling that, owing to Proposition 2.6, for every $t \in (1,2]$ we have that, if $\mathbf{g} \in \mathbf{L}^t(\Omega)$, the Stokes problem: find $(\boldsymbol{\varphi}_{\mathbf{g}},\psi_{\mathbf{g}}) \in \mathbf{W}_0^{1,t}(\Omega) \times L^t(\Omega)/\mathbb{R}$

$$\begin{cases} a(\boldsymbol{\varphi}_{\mathbf{g}},\mathbf{v}) + b(\mathbf{v},\psi_{\mathbf{g}}) = \displaystyle\int_\Omega \mathbf{g}\cdot\mathbf{v}\,\mathrm{d}x & \forall \mathbf{v} \in \mathbf{W}_0^{1,t'}(\Omega), \\ b(\boldsymbol{\varphi}_{\mathbf{g}},q) = 0 & \forall q \in L^{t'}(\Omega)/\mathbb{R}, \end{cases} \tag{5.2}$$

   is well–posed, $(\boldsymbol{\varphi}_{\mathbf{g}},\psi_{\mathbf{g}}) \in \mathbf{W}^{2,t}(\Omega) \times W^{1,t}(\Omega)$, and

$$\|\boldsymbol{\varphi}_{\mathbf{g}}\|_{\mathbf{W}^{2,t}(\Omega)} + \|\psi_{\mathbf{g}}\|_{W^{1,t}(\Omega)} \lesssim \|\mathbf{g}\|_{\mathbf{L}^t(\Omega)}. \tag{5.3}$$

2. Since $p \geq 2$ and $\omega \in A_p(\Omega)$ satisfies the compatibility condition of Proposition 2.3 we can use the results of the previous step with $t = p'$ and the embedding results of Proposition 2.3 to conclude that

$$(\boldsymbol{\varphi}_{\mathbf{g}},\psi_{\mathbf{g}}) \in \mathbf{W}^{2,p'}(\Omega) \cap \mathbf{W}_0^{1,p'}(\omega',\Omega) \times W^{1,p'}(\Omega) \cap L^{p'}(\omega',\Omega)$$

   with an estimate.

3. Let $\mathbf{g} = |\mathsf{u} - \mathsf{u}_h|^{p-2}(\mathsf{u} - \mathsf{u}_h)$ and note that $\|\mathbf{g}\|_{L^{p'}(\Omega)} = \|\mathsf{u} - \mathsf{u}_h\|_{\mathbf{L}^p(\Omega)}^{p-1}$, which is finite given the assumption on ω and the embedding results of Proposition 2.3.
4. With this choice of $\mathbf{g}$ fixed, we would like to set $\mathbf{v} = \mathsf{u} - \mathsf{u}_h$ in (5.2) to obtain

$$\|\mathsf{u} - \mathsf{u}_h\|_{\mathbf{L}^p(\Omega)}^p = a(\mathsf{u} - \mathsf{u}_h, \boldsymbol{\varphi}_\mathbf{g}) + b(\mathsf{u} - \mathsf{u}_h, \psi_\mathbf{g}). \tag{5.4}$$

However, since $p \geq 2$, $\mathsf{u} - \mathsf{u}_h \notin \mathbf{W}_0^{1,p}(\Omega)$ so that (5.4) must be justified by a density argument. Namely, let $\mathbf{w}_n \in \mathbf{C}_0^\infty(\Omega)$ be such that $\mathbf{w}_n \to \mathsf{u} - \mathsf{u}_h$ in $\mathbf{W}_0^{1,p}(\omega, \Omega)$. Since $\mathbf{w}_n \in \mathbf{C}_0^\infty(\Omega) \subset \mathbf{W}_0^{1,p}(\Omega)$, we set $\mathbf{v} = \mathbf{w}_n$ in (5.2) and arrive at

$$a(\mathbf{w}_n, \boldsymbol{\varphi}_\mathbf{g}) + b(\mathbf{w}_n, \psi_\mathbf{g}) = \int_\Omega |\mathsf{u} - \mathsf{u}_h|^{p-2}(\mathsf{u} - \mathsf{u}_h) \cdot \mathbf{w}_n \,\mathrm{d}x. \tag{5.5}$$

Now, since $\boldsymbol{\varphi}_\mathbf{g} \in \mathbf{W}_0^{1,p'}(\omega', \Omega)$,

$$|a(\mathsf{u} - \mathsf{u}_h, \boldsymbol{\varphi}_\mathbf{g}) - a(\mathbf{w}_n, \boldsymbol{\varphi}_\mathbf{g})| \leq \|\nabla \boldsymbol{\varphi}_\mathbf{g}\|_{\mathbf{L}^{p'}(\omega', \Omega)} \|\nabla(\mathsf{u} - \mathsf{u}_h - \mathbf{w}_n)\|_{\mathbf{L}^p(\omega, \Omega)} \to 0$$

as $n \uparrow \infty$. Similar arguments reveal that $|b(\mathsf{u} - \mathsf{u}_h, \psi_\mathbf{g}) - b(\mathbf{w}_n, \psi_\mathbf{g})| \to 0$ as $n \uparrow \infty$. Finally, in view of the continuous embedding $\mathbf{W}_0^{1,p}(\omega, \Omega) \hookrightarrow \mathbf{L}^p(\Omega)$, the right hand side of (5.5) converges to $\|\mathsf{u} - \mathsf{u}_h\|_{\mathbf{L}^p(\Omega)}^p$. These arguments yield (5.4).
5. From (5.4) and (1.1) we have, for an arbitrary pair $(\mathbf{w}_h, r_h) \in \mathbf{V}_h \times \mathcal{P}_h$,

$$\|\mathsf{u} - \mathsf{u}_h\|_{\mathbf{L}^p(\Omega)}^p = a(\mathsf{u} - \mathsf{u}_h, \boldsymbol{\varphi}_\mathbf{g} - \mathbf{w}_h) - b(\mathsf{u}_h, \psi_\mathbf{g} - r_h) - b(\mathbf{w}_h, \pi - \pi_h),$$

where we also used that u is solenoidal. Set now $\mathbf{w}_h = \boldsymbol{\varphi}_{\mathbf{g},h}$ and $r_h = \psi_{\mathbf{g},h}$, i.e., the Stokes projection of $(\boldsymbol{\varphi}_\mathbf{g}, \psi_\mathbf{g})$. Galerkin orthogonality once again yields

$$\|\mathsf{u} - \mathsf{u}_h\|_{\mathbf{L}^p(\Omega)}^p = a(\mathsf{u}, \boldsymbol{\varphi}_\mathbf{g} - \boldsymbol{\varphi}_{\mathbf{g},h}) + b(\boldsymbol{\varphi}_\mathbf{g} - \boldsymbol{\varphi}_{\mathbf{g},h}, \pi).$$

Consequently

$$\|\mathsf{u} - \mathsf{u}_h\|_{\mathbf{L}^p(\Omega)}^p \lesssim \|\nabla(\boldsymbol{\varphi}_\mathbf{g} - \boldsymbol{\varphi}_{\mathbf{g},h})\|_{\mathbf{L}^{p'}(\omega', \Omega)} \left(\|\nabla \mathsf{u}\|_{\mathbf{L}^p(\omega, \Omega)} + \|\pi\|_{L^p(\omega, \Omega)} \right).$$

6. As a final step we must bound the first term on the right hand side of the previous estimate. Notice that, with $t = p' < 2$, and $\varrho := \omega'$ what we are trying to estimate is the error in the velocity component of the Stokes projection in $\mathbf{W}_0^{1,t}(\varrho, \Omega)$. This means that, since $t < 2$, we can apply Corollary 4.2 provided condition (S) holds, that is

$$\rho' \in A_{t'}(\Omega) \Leftrightarrow (\omega')^{-t'/t} \in A_{p''}(\Omega) \Leftrightarrow (\omega^{-p'/p})^{-p/p'} \in A_p(\Omega) \Leftrightarrow \omega \in A_p(\Omega),$$

and

$$\varrho' \in A_1 \Leftrightarrow (\omega')^{-t'/t} \in A_1 \Leftrightarrow \left(\omega^{-p'/p}\right)^{-p/p'} \in A_1 \Leftrightarrow \omega \in A_1,$$

which is true by assumption. The best approximation result of Corollary 4.2, the interpolation estimates of Proposition 3.2, and the regularity estimate given in (5.3) then yield

$$\|\nabla(\boldsymbol{\varphi}_\mathbf{g} - \boldsymbol{\varphi}_{\mathbf{g},h})\|_{\mathbf{L}^{p'}(\omega', \Omega)} \lesssim h^{1-d/p'} \omega'(h)^{1/p'} \|\mathsf{u} - \mathsf{u}_h\|_{\mathbf{L}^p(\Omega)}^{p-1}.$$

Conclude by observing that, since $\omega \in A_p$, for we have that

$$\omega'(T)^{1/p'} \lesssim h^d \omega(T)^{-1/p}, \quad \forall T \in \mathscr{T}_h.$$

This concludes the proof □

5.1. **Application: The Stokes problem with delta sources.** Let us now, as an application, show how Theorem 5.1 can be applied to the case of singular forces described in item i of Section 2.1. Assume that $\mathcal{Z} \subset \Omega$ with $\#\mathcal{Z} < \infty$, i.e., it is a finite collection of points. We now define

$$\mathbf{f}_{\mathcal{Z}} = \sum_{z \in \mathcal{Z}} \mathbf{F}_z \delta_z, \tag{5.6}$$

with $\mathbf{F}_z \in \mathbb{R}^d$. We begin by establishing the suitable functional framework.

Proposition 5.2 ($\mathbf{f}_{\mathcal{Z}} \in \mathbf{H}^{-1}(\mathsf{d}_{\mathcal{Z}}^{\alpha}, \Omega)$)**.** *Assume that* $\alpha \in (d-2, d)$*, then* $\mathsf{d}_{\mathcal{Z}}^{\alpha} \in A_2(\Omega)$*,* $\mathsf{d}_{\mathcal{Z}}^{-\alpha} \in A_2(\Omega) \cap A_1$*, and* $\mathbf{f}_{\mathcal{Z}} \in \mathbf{H}^{-1}(\mathsf{d}_{\mathcal{Z}}^{\alpha}, \Omega)$*.*

Proof. The bounds on α guarantee that $\mathsf{d}_{\mathcal{Z}}^{\alpha} \in A_2(\Omega)$ and $\mathsf{d}_{\mathcal{Z}}^{-\alpha} \in A_2(\Omega)$. In addition, since $d - 2 \geq 0$, we have that $\mathsf{d}_{\mathcal{Z}}^{-\alpha} \in A_1$.

Now, owing to [31, Remark 21.19], a compactly supported Radon measure ν belongs to the dual of $H_0^1(\omega, \Omega)$ if

$$\int_{\Omega} \int_0^r \frac{t^2 \nu(B(x,t))}{\omega(B(x,t))} \frac{\mathrm{d}t}{t} \, \mathrm{d}\nu(x) < \infty$$

for some $r > 0$. Setting $\nu = \sum_{z \in \mathcal{Z}} \delta_z$ and $\omega = \mathsf{d}_{\mathcal{Z}}^{-\alpha}$ we get

$$\int_{\Omega} \int_0^r \frac{t^2 \nu(B(x,t))}{\omega(B(x,t))} \frac{\mathrm{d}t}{t} \, \mathrm{d}\nu(x) \lesssim \sum_{z \in \mathcal{Z}} \int_0^r \frac{t}{t^{d-\alpha}} \, \mathrm{d}t,$$

which is finite provided $d - 2 < \alpha$. ☐

The previous result shows that, if $\mathbf{f} = \mathbf{f}_{\mathcal{Z}}$ in (1.3), then this problem has a unique solution $(\mathsf{u}, \pi) \in \mathbf{H}_0^1(\mathsf{d}_{\mathcal{Z}}^{\alpha}, \Omega) \times L^2(\mathsf{d}_{\mathcal{Z}}^{\alpha}, \Omega)/\mathbb{R}$. The following result is the missing ingredient to obtain error estimates via Theorem 5.1.

Proposition 5.3 (embedding)**.** *If* $\alpha \in (d-2, 2)$*, then* $\mathbf{H}_0^1(\mathsf{d}_z^{\alpha}, \Omega) \hookrightarrow \mathbf{L}^2(\Omega)$*.*

Proof. We only need to verify the condition of Proposition 2.3. In this case, we have

$$\frac{r^{2+d}}{R^{2+d}} \frac{\mathsf{d}_z^{\alpha}(B(x,R))}{\mathsf{d}_z^{\alpha}(B(x,r))} \approx \frac{r^{2+d}}{R^{2+d}} \frac{R^{d+\alpha}}{r^{d+\alpha}} = \left(\frac{r}{R}\right)^{2-\alpha}.$$

The provided bounds on α guarantee that this ratio is uniformly bounded. ☐

We can now obtain an error estimate. Notice that since $\mathsf{d}_{\mathcal{Z}}^{-\alpha} \in A_2(\Omega) \cap A_1$, the results of Theorem 4.1 and Corollary 4.2 apply.

Corollary 5.4 (error estimate)**.** *Let* $\alpha \in (d-2, 2)$ *and* $(\mathsf{u}, \pi) \in \mathbf{H}_0^1(\mathsf{d}_{\mathcal{Z}}^{\alpha}, \Omega) \times L^2(\mathsf{d}_{\mathcal{Z}}^{\alpha}, \Omega)/\mathbb{R}$ *solve* (1.3) *with* $\mathbf{f} = \mathbf{f}_{\mathcal{Z}}$*. Let* (u_h, π_h) *be the finite element approximation of* (u, π)*. In the setting of Theorem 5.1 we have, for every* $\varepsilon > 0$*,*

$$\|\mathsf{u} - \mathsf{u}_h\|_{\mathbf{L}^2(\Omega)} \lesssim h^{2-d/2-\varepsilon} \left(\|\nabla \mathsf{u}\|_{\mathbf{L}^2(\mathsf{d}_{\mathcal{Z}}^{\alpha}, \Omega)} + \|\pi\|_{L^2(\mathsf{d}_{\mathcal{Z}}^{\alpha}, \Omega)} \right),$$

where the hidden constant does not depend on u*,* π*, nor* h*, but blows up as* $\varepsilon \downarrow 0$*.*

Proof. Proposition 5.2 guarantees that there is a unique pair $(\mathsf{u}, \pi) \in \mathbf{H}_0^1(\mathsf{d}_{\mathcal{Z}}^{\alpha}, \Omega) \times L^2(\mathsf{d}_{\mathcal{Z}}^{\alpha}, \Omega)/\mathbb{R}$ that solves (1.3). In addition, Proposition 5.3 guarantees that $\mathsf{u} \in \mathbf{L}^2(\Omega)$. The rest is just an application of Theorem 5.1. In this case, we have that

$$h^{1+d/2} \omega(h)^{-1/2} = h^{1+d/2} h^{-d/2-\alpha/2} = h^{1-\alpha/2}$$

and

$$\alpha \in (d-2,2) \implies 1 - \frac{\alpha}{2} \in \left(0, 2 - \frac{d}{2}\right).$$

The blowup of the constants is due to the fact that in the limiting case the embedding $\mathbf{H}^1_0(\mathsf{d}_z^\alpha, \Omega) \hookrightarrow \mathbf{L}^2(\Omega)$ does no longer hold. □

We conclude by commenting that via similar techniques we can consider the cases described in items ii and iii of Section 2.1.

6. Examples of suitable pairs

To conclude our analyisis, we study some pairs that satisfy assumptions (3.2), (3.9) so that the theory we have presented above applies.

We begin with a continuous weighted inf–sup condition that immediately follows from the existence of a right inverse of the divergence.

Lemma 6.1 (continuous weighted inf–sup)**.** *Let $p \in (1,\infty)$ and $\omega \in A_p$. For all $q \in L^{p'}(\omega', \Omega)/\mathbb{R}$ we have that*

$$\|q\|_{L^{p'}(\omega',\Omega)} \lesssim \sup_{\mathbf{v} \in \mathbf{W}^{1,p}_0(\omega,\Omega)} \frac{b(\mathbf{v}, q)}{\|\nabla \mathbf{v}\|_{\mathbf{L}^p(\omega,\Omega)}}, \tag{6.1}$$

where the hidden constant depends only on Ω and $[\omega]_{A_p}$, but not on q.

Proof. Let $q \in L^{p'}(\omega', \Omega)/\mathbb{R}$ and we define $\tilde r = \omega' |q|^{p'/p} \operatorname{sign}(q)$. Notice that

$$\|\tilde r\|^p_{L^p(\omega,\Omega)} = \int_\Omega \omega |\tilde r|^p \,\mathrm{d}x = \int_\Omega \omega^{1-p'} |q|^{p'} \,\mathrm{d}x = \|q\|^{p'}_{L^{p'}(\omega',\Omega)},$$

so that $\tilde r \in L^p(\omega, \Omega)$ and, since Ω is bounded $\tilde r \in L^1(\Omega)$. Consequently, we can set $r = \tilde r - \fint_\Omega \tilde r \,\mathrm{d}x$ and we conclude that $r \in L^p(\omega, \Omega)/\mathbb{R}$ with

$$\|r\|_{L^p(\omega,\Omega)} \lesssim \|q\|^{p'-1}_{L^{p'}(\omega',\Omega)}.$$

Our final initial observation is that, since q has zero mean,

$$\int_\Omega q r \,\mathrm{d}x = \int_\Omega q \tilde r \,\mathrm{d}x = \int_\Omega |q|^{p'} \omega' \,\mathrm{d}x = \|q\|^{p'}_{L^{p'}(\omega',\Omega)}.$$

Recall now that there is $\mathbf{w} \in \mathbf{W}^{1,p}_0(\omega, \Omega)$ such that

$$\operatorname{div} \mathbf{w} = r, \qquad \|\nabla \mathbf{w}\|_{\mathbf{L}^p(\omega,\Omega)} \lesssim \|r\|_{L^p(\omega,\Omega)},$$

where the constant in the estimate is independent of r; see [16, Theorem 3.1], [41, Theorem 1], [11, Theorem 5.2], or [1, Theorem 2.8] for a proof. As a consequence, we have

$$\begin{aligned}\sup_{\mathbf{v} \in \mathbf{W}^{1,p}_0(\omega,\Omega)} \frac{b(\mathbf{v}, q)}{\|\nabla \mathbf{v}\|_{\mathbf{L}^p(\omega,\Omega)}} &\geq \frac{b(\mathbf{w}, q)}{\|\nabla \mathbf{w}\|_{\mathbf{L}^p(\omega,\Omega)}} = \frac{\|q\|^{p'}_{L^{p'}(\omega',\Omega)}}{\|\nabla \mathbf{w}\|_{\mathbf{L}^p(\omega,\Omega)}} \\ &\gtrsim \frac{\|q\|^{p'}_{L^{p'}(\omega',\Omega)}}{\|r\|_{L^p(\omega,\Omega)}} \gtrsim \|q\|_{L^{p'}(\omega',\Omega)}.\end{aligned}$$

As we intended to show. □

6.1. **The mini element.** This pair is considered in [3], [19, Section 4.2.4] for the unweighted case and it is defined by:

$$\mathbf{V}_h = \left\{ \mathbf{v}_h \in \mathbf{C}(\bar\Omega) : \ \forall T \in \mathscr{T}_h, \mathbf{v}_h|_T \in [\mathbb{P}_1(T) \oplus \mathbb{B}(T)]^d \right\} \cap \mathbf{H}^1_0(\Omega), \tag{6.2}$$

$$\mathcal{P}_h = \left\{ q_h \in L^2(\Omega)/\mathbb{R} \cap C(\bar\Omega) : \ \forall T \in \mathscr{T}_h, q_h|_T \in \mathbb{P}_1(T) \right\}, \tag{6.3}$$

where $\mathbb{B}(T)$ denotes the space spanned by local bubble functions.

We must immediately note that, for $d \in \{2,3\}$, assumption (3.9) is proved in [24, Theorem 12] and [25, Theorem 8.1]. Thus, we focus on the weighted LBB condition (3.2). This will be obtained with the aid of the, auxiliary, continuous inf–sup condition (6.1).

Theorem 6.2 (discrete inf–sup condition)**.** *Let $p \in (1,\infty)$ and $\omega \in A_p$. If $\mathbf{V}_h$ and $\mathcal{P}_h$ are defined by* (6.2) *and* (6.3)*, respectively, then we have that*

$$\beta \|q_h\|_{L^{p'}(\omega',\Omega)} \leq \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h, q_h)}{\|\nabla \mathbf{v}_h\|_{\mathbf{L}^p(\omega,\Omega)}} \quad \forall q_h \in \mathcal{P}_h, \tag{6.4}$$

where the hidden constant is independent of $\mathscr{T}_h$.

Proof. Our argument will be based on (6.1) and the construction of a so–called Fortin operator [19, Lemma 4.19]. Given $\mathbf{v} \in \mathbf{W}^{1,p}_0(\omega,\Omega)$, we will construct $\mathcal{F}_h \mathbf{v} \in \mathbf{V}_h$ such that

$$b(\mathbf{v}, q_h) = b(\mathcal{F}_h \mathbf{v}, q_h) \quad \forall q_h \in \mathcal{P}_h, \quad \|\nabla \mathcal{F}_h \mathbf{v}\|_{\mathbf{L}^p(\omega,\Omega)} \lesssim \|\nabla \mathbf{v}\|_{\mathbf{L}^p(\omega,\Omega)}, \tag{6.5}$$

with a hidden constant independent of h. To accomplish this task, we first notice that, if $q_h \in \mathcal{P}_h$ then, for all $T \in \mathscr{T}_h$, $\nabla q_{h|T} \in \mathbb{R}^d$. Consequently, an integration by parts argument reveals that $\mathcal{F}_h \mathbf{v}$ must be such that

$$\int_T \mathbf{v} \, \mathrm{d}x = \int_T \mathcal{F}_h \mathbf{v} \, \mathrm{d}x \quad \forall T \in \mathscr{T}_h. \tag{6.6}$$

Let Π_h denote the quasi–interpolation operator introduced in Section 3.2. We define

$$\mathcal{F}_h \mathbf{v} = \Pi_h \mathbf{v} + \sum_{T \in \mathscr{T}_h} \sum_{i=1}^d \gamma^i_T \mathbf{e}_i b_T.$$

Here, $\{\mathbf{e}_1, \cdots, \mathbf{e}_d\}$ denotes the canonical basis of $\mathbb{R}^d$, $\gamma^i_T \in \mathbb{R}$; $i \in \{1, \ldots, d\}$, and b_T is the bubble function associated with T. We thus have that the discrete function $\mathcal{F}_h \mathbf{v}$ satisfies (6.6) if

$$\gamma^i_T = \frac{\int_T (\mathbf{v} - \Pi_h \mathbf{v}) \, \mathrm{d}x}{\int_T b_T \, \mathrm{d}x}, \quad i \in \{1, \cdots, d\}, \quad T \in \mathscr{T}_h.$$

It thus remains to prove the stability bound $\|\nabla \mathcal{F}_h \mathbf{v}\|_{\mathbf{L}^p(\omega,\Omega)} \lesssim \|\nabla \mathbf{v}\|_{\mathbf{L}^p(\omega,\Omega)}$. Write

$$\|\nabla \mathcal{F}_h \mathbf{v}\|_{\mathbf{L}^p(\omega,\Omega)} \leq \|\nabla \Pi_h \mathbf{v}\|_{\mathbf{L}^p(\omega,\Omega)} + \left\| \nabla \left(\sum_{T \in \mathscr{T}_h} \sum_{i=1}^d \gamma^i_T \mathbf{e}_i b_T \right) \right\|_{\mathbf{L}^p(\omega,\Omega)} = \mathrm{I} + \mathrm{II},$$

and notice that the local stability estimate (3.3) and the finite overlapping property of stars yield

$$\mathrm{I} = \|\nabla \Pi_h \mathbf{v}\|_{\mathbf{L}^p(\omega,\Omega)} \lesssim \|\nabla \mathbf{v}\|_{\mathbf{L}^p(\omega,\Omega)}.$$

To bound II we use the interpolation estimate (3.4) and properties of the bubble function to obtain

$$|\gamma_T^i| \lesssim |T|^{-1} h_T \|\nabla \mathbf{v}\|_{\mathbf{L}^p(\omega,\mathcal{S}_T)} \left(\int_T \omega' \,\mathrm{d}x \right)^{\frac{1}{p'}}$$
$$\lesssim h_T^{1-d+d/p'} \|\nabla \mathbf{v}\|_{\mathbf{L}^p(\omega,\mathcal{S}_T)} \left(\fint_T \omega' \,\mathrm{d}x \right)^{\frac{1}{p'}} .$$

Consequently,

$$\mathrm{II} \lesssim \sum_{T\in\mathscr{T}_h} \sum_{i=1}^{d} |\gamma_T^i| \|\nabla b_T\|_{\mathbf{L}^p(\omega,T)}$$
$$\lesssim \sum_{T\in\mathscr{T}_h} h_T^{1-d+d/p'} \|\nabla \mathbf{v}\|_{\mathbf{L}^p(\omega,\mathcal{S}_T)} \left(\fint_T \omega' \,\mathrm{d}x \right)^{\frac{1}{p'}} h_T^{\frac{d}{p}-1} \left(\fint_T \omega \,\mathrm{d}x \right)^{\frac{1}{p}} .$$

Since $(1-d+d/p') + d/p - 1 = 0$ shape regularity allows us to conclude that

$$\mathrm{II} \lesssim \sum_{T\in\mathscr{T}_h} \|\nabla \mathbf{v}\|_{\mathbf{L}^p(\omega,\mathcal{S}_T)} \left[\left(\fint_T \omega \,\mathrm{d}x \right) \left(\fint_T \omega' \,\mathrm{d}x \right)^{p-1} \right]^{\frac{1}{p}} \lesssim [\omega]_{A_p}^{\frac{1}{p}} \|\nabla \mathbf{v}\|_{\mathbf{L}^p(\omega,\Omega)},$$

where we have used (2.2) and the finite overlapping property of stars. The collection of the derived estimates for I and II yield

$$\|\nabla \mathcal{F}_h \mathbf{v}\|_{\mathbf{L}^p(\omega,\Omega)} \lesssim (1 + [\omega]_{A_p}^{\frac{1}{p}}) \|\nabla \mathbf{v}\|_{\mathbf{L}^p(\omega,\Omega)}.$$

The Fortin operator is thus constructed and this concludes the proof. □

6.2. **The lowest order Taylor Hood pair.** The lowest order Taylor Hood element [32], [46], [19, Section 4.2.5] is defined by

$$\mathbf{V}_h = \left\{ \mathbf{v}_h \in \mathbf{C}(\bar{\Omega}) : \ \forall T \in \mathscr{T}_h, \mathbf{v}_h|_T \in \mathbb{P}_2(T)^d \right\} \cap \mathbf{H}_0^1(\Omega), \tag{6.7}$$

$$\mathcal{P}_h = \left\{ q_h \in L^2(\Omega)/\mathbb{R} \cap C(\bar{\Omega}) : \ \forall T \in \mathscr{T}_h, q_h|_T \in \mathbb{P}_1(T) \right\} . \tag{6.8}$$

In two dimensions, estimate (3.9) for this pair is also obtained in [24, Theorem 12] and [25, Theorem 8.1]. In three dimensions, these references only show this result for certain classes of meshes. As a consequence, we will focus on (3.2). Notice that, as in the unweighted case, the technique of proof must differ from that used in Section 6.1. We will follow the ideas of [45, Section 3]; see also [19, Section 4.2.5].

We begin with a preparatory step.

Lemma 6.3 (perturbation). *Let $p \in (1,\infty)$ and $\omega \in A_p$. Assume that all $\{\mathscr{T}_h\}_{h>0}$ are such that every $T \in \mathscr{T}_h$ has at least d edges in Ω, and that $\mathbf{V}_h$ and $\mathcal{P}_h$ are defined as in* (6.7) *and* (6.8)*, respectively. Then we have that*

$$h\|\nabla q_h\|_{L^{p'}(\omega',\Omega)} \lesssim \sup_{\mathbf{v}_h\in\mathbf{V}_h} \frac{b(\mathbf{v}_h, q_h)}{\|\nabla \mathbf{v}_h\|_{\mathbf{L}^p(\omega,\Omega)}} \quad \forall q_h \in \mathcal{P}_h,$$

where the hidden constant does not depend on h.

Proof. We denote by $\mathscr{E}_h, \mathscr{V}_h$, and $\mathscr{M}_h$ be the sets of interior edges, interior vertices, and interior edge midpoints, respectively, of $\mathscr{T}_h$. Let $\mathsf{e} \in \mathscr{E}_h$ and we set $\boldsymbol{\tau}_\mathsf{e}$ to be a unit vector in the direction of e. Notice that there is a bijection between $\mathscr{E}_h$ and $\mathscr{M}_h$.

For $q_h \in \mathcal{P}_h$ we define $\mathbf{w}_h \in \mathbf{V}_h$ as

$$\mathbf{w}_h(\mathtt{v}) = 0, \quad \forall \mathtt{v} \in \mathscr{V}_h,$$

and

$$\mathbf{w}_h(\mathtt{m}) = -|\mathtt{e}|^{p'} \boldsymbol{\tau}_{\mathtt{e}} \operatorname{sign}(\partial_{\boldsymbol{\tau}_{\mathtt{e}}} q_h) |\partial_{\boldsymbol{\tau}_{\mathtt{e}}} q_h|^{p'-1} \frac{\omega'(T)}{|T|}, \quad \forall \mathtt{m} \in \mathscr{M}_h.$$

Let $\{\phi_{\mathtt{m}}\}_{\mathtt{m}\in\mathscr{M}_h} \cup \{\phi_{\mathtt{v}}\}_{\mathtt{v}\in\mathscr{V}_h}$ be the Lagrange nodal basis for piecewise quadratics over $\mathscr{T}_h$. Upon expanding $\mathbf{w}_h$ on this basis we realize that

$$\begin{aligned}
\|\nabla \mathbf{w}_h\|^p_{\mathbf{L}^p(\omega,\Omega)} &= \sum_{T\in\mathscr{T}_h} \int_T \omega \left| \sum_{\mathtt{m}\in\mathscr{M}_h:\mathtt{m}\in T} \mathbf{w}_h(\mathtt{m}) \nabla \phi_{\mathtt{m}} \right|^p \mathrm{d}x \\
&\lesssim h^{p'} \sum_{T\in\mathscr{T}_h} \frac{\omega(T)\,[\omega'(T)]^p}{|T|^p} \sum_{\mathtt{m}\in\mathscr{M}_h:\mathtt{m}\in T} |\partial_{\boldsymbol{\tau}_{\mathtt{e}}} q_h|^{p'} \\
&\lesssim h^{p'} [\omega]_{A_p} \sum_{T\in\mathscr{T}_h} \omega'(T) |\nabla q_h|^{p'} \lesssim h^{p'} \|\nabla q_h\|^{p'}_{\mathbf{L}^{p'}(\omega',\Omega)}.
\end{aligned}$$

Recall now (see [19, Tables 8.2 and 8.3]) that for $d \in \{2,3\}$ there is a quadrature formula on the unit simplex which is exact for quadratics, it is supported on the vertices and edge midpoints of the simplex, and has positive weights on the midpoints. Let $\{\varrho_{\mathtt{m}}\}$ be the weights of this formula, then we have that

$$\begin{aligned}
b(\mathbf{w}_h, q_h) &= -\sum_{T\in\mathscr{T}_h} \nabla q_h \cdot \int_T \mathbf{w}_h \,\mathrm{d}x \\
&= \sum_{T\in\mathscr{T}_h} \omega'(T) \nabla q_h \cdot \sum_{\mathtt{m}\in\mathscr{M}_h:\mathtt{m}\in T} \varrho_m \boldsymbol{\tau}_{\mathtt{e}} |e|^{p'} \operatorname{sign}(\partial_{\boldsymbol{\tau}_{\mathtt{e}}} q_h) |\partial_{\boldsymbol{\tau}_{\mathtt{e}}} q_h|^{p'-1} \\
&\gtrsim h^{p'} \sum_{T\in\mathscr{T}_h} \omega'(T) \sum_{\mathtt{e}\in\mathscr{E}_h:\mathtt{e}\subset T} |\partial_{\boldsymbol{\tau}_{\mathtt{e}}} q_h|^{p'} \gtrsim h^{p'} \sum_{T\in\mathscr{T}_h} \omega'(T) |\nabla q_h|^{p'},
\end{aligned}$$

where, in the last step, we used that the mesh assumption implies that for any element T the collection $\{\boldsymbol{\tau}_{\mathtt{e}}\}_{\mathtt{e}\in\mathscr{E}_h:\mathtt{e}\subset T}$ spans $\mathbb{R}^d$. Conclude by recalling that ∇q_h is constant over T. ☐

With this result at hand we now prove (3.2) for the Taylor Hood pair.

Theorem 6.4 (discrete inf–sup condition)**.** *In the setting of Lemma 6.3, we have*

$$\|q_h\|_{L^{p'}(\omega',\Omega)} \lesssim \sup_{\mathbf{v}_h\in\mathbf{V}_h} \frac{b(\mathbf{v}_h, q_h)}{\|\nabla \mathbf{v}_h\|_{\mathbf{L}^p(\omega,\Omega)}} \quad \forall q_h \in \mathcal{P}_h, \tag{6.9}$$

where the hidden constant is independent of h.

Proof. Given $q_h \in \mathcal{P}_h \subset L^{p'}(\omega',\Omega)/\mathbb{R}$, let $\mathbf{w}_{q_h} \in \mathbf{W}^{1,p}_0(\omega,\Omega)$ be the function constructed in the course of the proof of (6.1) and Π_h the interpolant, onto $\mathbf{V}_h$, described in Section 3.2. The properties of Π_h and arguing as in the proof of (6.1) show that

$$\begin{aligned}
\sup_{\mathbf{v}_h\in\mathbf{V}_h} \frac{b(\mathbf{v}_h, q_h)}{\|\nabla \mathbf{v}_h\|_{\mathbf{L}^p(\omega,\Omega)}} &\geq \frac{b(\Pi_h \mathbf{w}_{q_h}, q_h)}{\|\nabla \Pi_h \mathbf{w}_{q_h}\|_{\mathbf{L}^p(\omega,\Omega)}} \\
&\gtrsim \|q_h\|_{L^{p'}(\omega',\Omega)} + \frac{b\left(\Pi_h \mathbf{w}_{q_h} - \mathbf{w}_{q_h}, q_h\right)}{\|\nabla \mathbf{w}_{q_h}\|_{\mathbf{L}^p(\omega,\Omega)}}.
\end{aligned}$$

Integration by parts, and the properties of Π_h show that

$$\frac{b\left(\Pi_h \mathbf{w}_{q_h} - \mathbf{w}_{q_h}, q_h\right)}{\|\nabla \mathbf{w}_{q_h}\|_{\mathbf{L}^p(\omega,\Omega)}} \geq -\frac{\|\nabla q_h\|_{L^{p'}(\omega',\Omega)}\|\mathbf{w}_{q_h} - \Pi_h \mathbf{w}_{q_h}\|_{\mathbf{L}^p(\omega,\Omega)}}{\|\nabla \mathbf{w}_{q_h}\|_{\mathbf{L}^p(\omega,\Omega)}} \gtrsim -h\|\nabla q_h\|_{L^{p'}(\omega',\Omega)}.$$

Lemma 6.3 allows us to conclude. □

6.3. **Elements with a quasilocal Fortin operator.** We will say that the pair $(\mathbf{V}_h, \mathcal{P}_h)$ has a quasilocal Fortin operator if there is a map $\mathcal{F}_h : \mathbf{W}_0^{1,1}(\Omega) \to \mathbf{V}_h$ such that

$$b(\mathbf{v} - \mathcal{F}_h \mathbf{v}, q_h) = 0 \quad \forall q_h \in \mathcal{P}_h,$$

and, for every $T \in \mathscr{T}_h$,

$$\fint_T \left(|\mathcal{F}_h \mathbf{v}| + h_T |\nabla \mathcal{F}_h \mathbf{v}|\right) \mathrm{d}x \lesssim \fint_{\mathcal{S}_T} \left(|\mathbf{v}| + h_T|\nabla \mathbf{v}|\right) \mathrm{d}x \quad \forall \mathbf{v} \in \mathbf{W}_0^{1,1}(\Omega),$$

where $\mathcal{S}_T$ is defined in (3.1). Our purpose here will be to show that, whenever there is a quasilocal Fortin operator, (3.2) holds.

We begin with the following result for the unweighted case.

Lemma 6.5 (stability)**.** *A quasilocal Fortin operator satisfies*

$$\fint_T |\nabla \mathcal{F}_h \mathbf{v}| \,\mathrm{d}x \lesssim \fint_{\mathcal{S}_T} |\nabla \mathbf{v}| \,\mathrm{d}x \quad \forall \mathbf{v} \in \mathbf{W}_0^{1,1}(\Omega),$$

with a hidden constant that is independent of T, $\mathbf{v}$, and h.

Proof. This is shown, for instance, in [33, formula (3.2)]. □

We now show stability on weighted spaces.

Proposition 6.6 (weighted stability)**.** *Let $p \in (1,\infty)$ and $\omega \in A_p$. A quasilocal Fortin operator is stable in $\mathbf{W}_0^{1,p}(\omega,\Omega)$.*

Proof. The proof is, essentially, a combination of a scaling argument and the definition of the class A_p. Let $\mathbf{v} \in \mathbf{W}_0^{1,p}(\omega,\Omega)$. Since $\nabla \mathcal{F}_h \mathbf{v}_{|T}$ is a polynomial

$$|\nabla \mathcal{F}_h \mathbf{v}_{|T}(z)| \lesssim \fint_T |\nabla \mathcal{F}_h \mathbf{v}| \,\mathrm{d}x \quad \forall z \in T.$$

Therefore,

$$\|\nabla \mathcal{F}_h \mathbf{v}\|^p_{\mathbf{L}^p(\omega,\Omega)} = \sum_{T \in \mathscr{T}_h} \int_T |\nabla \mathcal{F}_h \mathbf{v}(z)|^p \omega \,\mathrm{d}z \lesssim \sum_{T \in \mathscr{T}_h} \int_T \left(\fint_T |\nabla \mathcal{F}_h \mathbf{v}| \,\mathrm{d}x \right)^p \omega \,\mathrm{d}z.$$

The local stability of Lemma 6.5 yields

$$\|\nabla \mathcal{F}_h \mathbf{v}\|^p_{\mathbf{L}^p(\omega,\Omega)} \lesssim \sum_{T \in \mathscr{T}_h} \int_T \left(\fint_{\mathcal{S}_T} |\nabla \mathbf{v}| \,\mathrm{d}x \right)^p \omega \,\mathrm{d}z.$$

Now

$$\left(\fint_{\mathcal{S}_T} |\nabla \mathbf{v}| \,\mathrm{d}x \right)^p \leq \frac{1}{|\mathcal{S}_T|^p} \omega'(\mathcal{S}_T)^{p-1} \int_{\mathcal{S}_T} |\nabla \mathbf{v}|^p \omega \,\mathrm{d}x,$$

which then implies

$$\|\nabla \mathcal{F}_h \mathbf{v}\|^p_{\mathbf{L}^p(\omega,\Omega)} \lesssim \sum_{T \in \mathscr{T}_h} \frac{\omega(T)\omega'(\mathcal{S}_T)^{p-1}}{|\mathcal{S}_T|^p} \int_{\mathcal{S}_T} |\nabla \mathbf{v}|^p \omega \,\mathrm{d}x \lesssim [\omega]_{A_p} \|\nabla \mathbf{v}\|^p_{\mathbf{L}^p(\omega,\Omega)},$$

where, in the last step, we used the shape regularity of $\mathscr{T}_h$, the fact that $\omega \in A_p$, and the finite overlapping property of stars. $\square$

We conclude by recalling a standard result, in this context known as the *Fortin criterion* [19, Lemma 4.19]: If (6.1) holds and there is a stable Fortin operator, then (3.2) holds uniformly in h. Notice that in view of the results of Proposition 6.6, a quasilocal Fortin operator is stable in weighted spaces. This result allows for a rich variety of examples, provided we contempt ourselves to deal with sufficiently high polynomial degree. For instance, in [27, Section 3], such a quasilocal Fortin operator is constructed for:

- Any order Taylor Hood pair if $d = 2$. Section 6.2 had already treated the lowest order case in dimensions $d = 2$ and $d = 3$.
- Taylor Hood pairs with at least cubic velocities for $d = 3$. The lowest order case, in three dimensions, was already discussed in Section 6.2.
- The two dimensional conforming Crouzeix–Raviart pair; each component of the velocity, locally, belongs to $\mathbb{P}_2(T) \oplus \mathbb{B}(T)$, while the pressure is discontinuous and locally in $\mathbb{P}_1(T)$.

In addition, we can also consider spaces such that the velocity, locally, belongs to $\mathbb{P}_2(T)^d$, while the pressure consists of piecewise constants; see [33, Remark 3.4].

We must remark, however, that our main interest in considering weighted spaces is to be able to handle singular data in the Stokes problem. This, in turn, implies that we do not expect the solution to possess much regularity. The approximation power of higher order elements then is lost.

Appendix A. Proof of (4.3)

Although not original, for the sake of readability, here we present a proof of the first estimate in (4.3). We will follow [12, Theorem 3.1] and [30] to show that, for any $x \in \Omega$,

$$\int_\Omega \frac{h^\lambda \omega(z)}{(|x-z|^2 + (\kappa h)^2)^{(d+\lambda)/2}} \, \mathrm{d}z \lesssim \mathcal{M}\omega(x).$$

We begin, for a fixed $x \in \Omega$, by partitioning the integration into points "near" and "far" from it:

$$\int_\Omega \frac{h^\lambda \omega(z)}{(|x-z|^2 + (\kappa h)^2)^{(d+\lambda)/2}} \, \mathrm{d}z = N + F$$

where

$$N = \int_{|x-z|\leq h} \frac{h^\lambda \omega(z)}{(|x-z|^2 + (\kappa h)^2)^{(d+\lambda)/2}} \, \mathrm{d}z \lesssim \frac{1}{h^d} \int_{|x-z|\leq h} \omega(z) \, \mathrm{d}z \lesssim \mathcal{M}\omega(x),$$

and

$$F = \int_{|x-z|>h} \frac{h^\lambda \omega(z)}{(|x-z|^2 + (\kappa h)^2)^{(d+\lambda)/2}} \, \mathrm{d}z \lesssim \int_{|x-z|>h} \frac{h^\lambda \omega(z)}{|x-z|^{d+\lambda}} \, \mathrm{d}z. \tag{A.1}$$

Now we follow [30, Lemma (b)] and [13, Lemma 7.9] and introduce a dyadic decomposition of the last integral in (A.1). We write

$$\begin{aligned}\int_{|x-z|>h} \frac{\omega(z)}{|x-z|^{d+\lambda}}\,\mathrm{d}z &= \sum_{k=0}^{\infty} \int_{h2^k<|x-z|\le h2^{k+1}} \frac{\omega(z)}{|x-z|^{d+\lambda}}\,\mathrm{d}z \\ &\le 2^{\lambda+d} h^{-\lambda} \sum_{k=0}^{\infty} \frac{2^{-\lambda(k+1)}}{(h2^{k+1})^d} \int_{h2^k<|x-z|\le h2^{k+1}} \omega(z)\,\mathrm{d}z \\ &\le 2^{\lambda+d} h^{-\lambda} \sum_{k=0}^{\infty} \frac{2^{-\lambda(k+1)}}{(h2^{k+1})^d} \int_{|x-z|\le h2^{k+1}} \omega(z)\,\mathrm{d}z \\ &\lesssim h^{-\lambda}\mathcal{M}\omega(x) \sum_{k=0}^{\infty} 2^{-\lambda(k+1)} \lesssim h^{-\lambda}\mathcal{M}\omega(x),\end{aligned}$$

where, in the last step, we used that since $\lambda > 0$ the series converges.

A combination of the estimates obtained for N and F is (4.3).

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