ON THE TRACES OF $W^{2,p}(\Omega)$ FOR A LIPSCHITZ DOMAIN

RICARDO G. DURÁN AND MARIA AMELIA MUSCHIETTI

Abstract. We extend to the case $1 < p$ the results obtained by Geymonat and Krasucki for $p = 2$ on the characterization of the traces of $W^{2,p}(\Omega)$ for a bounded Lipschitz domain.

The object of this note is to give a characterization of the traces of the Sobolev space $W^{2,p}(\Omega)$ when $\Omega \subset \mathbb{R}^2$ is a connected bounded Lipschitz domain (possible not simply connected). Denote with γ_0 the trace operator (see its precise definition below) and with ∂_n the normal derivative on the boundary Γ of Ω. When Γ is smooth it is known that the range of the operator $\psi \longmapsto (\gamma_0(\psi), \partial_n\psi)$ is $W^{2-1/p,p}(\Gamma) \times W^{1-1/p,p}$. In the Lipschitz case, Geymonat and Krasucki [4] considered the case $p = 2$ and proved that $(g_0, g_1) \in H^1(\Gamma) \times L^2(\Gamma)$ is in the range of this map if and only if $(\partial_t g_0)\mathbf{n} - g_1\mathbf{t} \in H^{1/2}(\Gamma)$, where $\mathbf{n}$ is the exterior unit normal and $\mathbf{t} = (-n_2, n_1)$ is the tangential unit vector on Γ, which are defined almost everywhere since Ω is Lipschitz. In this work we extend the result of [4] to the case $1 < p$. Our argument is different from that in [4] and is based on the existence of a continuous right inverse of the divergence operator on $W_0^{1,p}(\Omega)$ which is a known non trivial result.

Let Γ_j, $j = 0, \dots, m$ be the connected components of Γ, where Γ_0 is the exterior component. For a scalar function ψ we write $\mathbf{curl}\,\psi = (\frac{\partial\psi}{\partial x_2}, -\frac{\partial\psi}{\partial x_1})$. We recall that the trace operator $\psi \longmapsto \psi|_\Gamma$, defined for regular functions ψ, has a continuous extension γ_0 from $W^{1,p}(\Omega)$ onto $W^{1-1/p,p}(\Gamma)$, where

$$W^{1-1/p,p}(\Gamma) = \{\phi \in L^p(\Gamma) \, : \int_\Gamma \int_\Gamma \frac{|\phi(x) - \phi(y)|^p}{|x - y|^p} \, dx \, dy < \infty\}$$

Moreover, γ_0 has a continuous right inverse that we will call E, i.e., $\gamma_0(E\phi) = \phi$ (see [2, 7]). To simplify notation we will also write $\gamma_0(\mathbf{v}) = (\gamma_0(v_1), \gamma_0(v_2))$ for vector fields $\mathbf{v} \in W^{1,p}(\Omega)^2$.

If Ω is multiply connected,

1991 *Mathematics Subject Classification.* 46E35.

Key words and phrases. Lipschitz domains, Sobolev spaces, traces.

The research of the first author was supported by Universidad de Buenos Aires, under grant TX48, by ANPCyT under grant PICT 03-00000-00137 and by CONICET under grant PIP 0660/98.

The research of the second author was supported by Universidad Nacional de La Plata under grant 11/X228 and by CONICET under grant PIP 4723/96.

Both authors are members of CONICET, Argentina.

$$W^{1-1/p,p}(\Gamma) = \prod_{j=0}^{m} W^{1-1/p,p}(\Gamma_j)$$

For $1 < p < \infty$, the space

$$W^{-1/p,p}(\Gamma) = \prod_{j=0}^{m} W^{-1/p,p}(\Gamma_j)$$

is the dual space of $W^{1/p,q}(\Gamma) = W^{1-1/q,q}(\Gamma)$, where q is the dual exponent of p.

The linear map $\mathbf{v} \longmapsto \mathbf{v}|_{\Gamma} \cdot \mathbf{n}$ defined for smooth vector fields $\mathbf{v}$ admits a continuous extension

$$\gamma_n \, : W^p(\mathrm{div}\,, \Omega) \longrightarrow W^{-1/p,p}(\Gamma)$$

where

$$W^p(\mathrm{div}\,, \Omega) = \{\mathbf{v} \in L^p(\Omega)^2 \,:\, \mathrm{div}\,\mathbf{v} \in L^p(\Omega)\}$$

Indeed, given $\phi \in W^{1-1/q,q}(\Gamma)$ we set

$$\langle \gamma_n(\mathbf{v}), \phi \rangle = \int_\Omega E\phi \, \mathrm{div}\,\mathbf{v} + \int_\Omega \nabla E\phi \cdot \mathbf{v}$$

For $\mathbf{v} \in W^p(\mathrm{div}\,, \Omega)$ given, the right hand side defines a continuous linear operator on $W^{1-1/q,q}(\Gamma)$ and, by density, it is easy to see that it does not depend on the extension use. Therefore $\gamma_n(\mathbf{v})$ is well defined. For the sake of clarity, we will write $\mathbf{v} \cdot \mathbf{n} = \gamma_n(\mathbf{v})$ also for functions in $W^p(\mathrm{div}\,, \Omega)$.

A basic tool for our results is the existence of a stream function for a non smooth divergence free vector field on a Lipschitz domain. This will be proven in Theorem 1 below. The existence of a stream function is easy to prove for a divergence free vector field with compact support defined in $\mathbb{R}^2$. This is shown in the next lemma.

Lemma 1. *Let $1 \le p \le \infty$. If $\mathbf{v} \in L^p(\mathbb{R}^2)^2$, $\mathbf{v}$ has compact support and div $\mathbf{v} = 0$ then, there exists $\psi \in W^{1,p}(\mathbb{R}^2)$ such that*

$$\mathbf{curl}\,\psi = \mathbf{v}$$

Proof. Let G be the fundamental solution of the Laplace operator and define ψ as

$$\psi(x) = \mathbf{curl}\, G * \mathbf{v} = \frac{\partial G}{\partial x_2} * v_1 - \frac{\partial G}{\partial x_1} * v_2$$

Observe that, since the first derivatives of G are locally integrable and $\mathbf{v}$ has compact support, the convolution is well defined. On the other hand, by Young inequality, it follows that $\psi \in L^p(\mathbb{R}^2)$ and so, if $\mathbf{curl}\,\psi = \mathbf{v}$, $\psi \in W^{1,p}(\mathbb{R}^2)$.

In order to check that ψ is the desired function one can proceed by density. Indeed, for regular functions it follows immediately by differentiating the convolution and using that $\operatorname{div}\mathbf{v} = 0$. On the other hand, any $\mathbf{v} \in L^p(\mathbb{R}^2)^2$ such that $\operatorname{div}\mathbf{v} = 0$ can be approximated by regular divergence free vector fields obtained by convolution with a standard approximation of unity. $\square$

The next lemma will allow us to show that a divergence free vector field satisfying appropriate boundary conditions can be extended to R^2 still with vanishing divergence. For $p = 2$ this result is proven in [6] by using a priori estimates for the Neumann problem which do not hold in general for arbitrary p on a Lipschitz domain.

Lemma 2. *Let $1 < p < \infty$ and $\phi^* \in W^{-1/p,p}(\Gamma)$ be such that $\langle \phi^*, 1\rangle_\Gamma = 0$. Then, there exists $\mathbf{v} \in L^p(\Omega)^2$ such that,*

$$\mathit{div}\,\mathbf{v} = 0 \qquad \mathit{in}\ \ \Omega$$

and

$$\mathbf{v} \cdot \mathbf{n} = \phi^* \qquad \mathit{on}\ \ \Gamma$$

Proof. Since the trace operator γ_0 is surjective from $W^{1,q}(\Omega)$ onto $W^{1-1/q,q}$ the statement of the lemma can be equivalently written in the following way. There exists $\mathbf{v} \in L^p(\Omega)^2$ such that

$$\langle \phi^*, \gamma_0(\psi)\rangle_\Gamma = \int_\Omega \mathbf{v} \cdot \nabla\psi \qquad \forall\psi \in W^{1,q}(\Omega) \tag{1}$$

Now, to show the existence of this $\mathbf{v}$, observe that, since $\langle \phi^*, 1\rangle_\Gamma = 0$, the map

$$\nabla\psi \longmapsto \langle \phi^*, \gamma_0(\psi)\rangle_\Gamma \tag{2}$$

is well defined on the subspace of $L^q(\Omega)^2$ defined as

$$G = \{\mathbf{g} \in L^q(\Omega)^2 \ : \ \mathbf{g} = \nabla\psi \ \text{ for some}\, \psi \in W^{1,q}(\Omega)\}$$

Moreover, by using the continuity of the trace operator and the Poincaré inequality, we have

$$\langle \phi^*, \gamma_0(\psi)\rangle_\Gamma \leq C \inf_{k\in\mathbb{R}} \|\psi - k\|_{W^{1,q}(\Omega)} \leq C\|\nabla\psi\|_{L^q(\Omega)}$$

Then, (2) defines a continuous functional on G which can be extended to $L^q(\Omega)^2$. Therefore, the existence of $\mathbf{v} \in L^p(\Omega)^2$ satisfying (1) follows from the Riesz representation theorem for functionals in L^q. $\square$

We can now prove the existence of a stream function for divergence free vector fields $\mathbf{v} \in L^p(\Omega)^2$.

Theorem 1. *Let $1 < p < \infty$, $\mathbf{v} \in L^p(\Omega)^2$, $\mathit{div}\,\mathbf{v} = 0$ in Ω and $\langle \mathbf{v}\cdot\mathbf{n}, 1\rangle_{\Gamma_j} = 0$ for $j = 0, \ldots, m$. Then, there exists $\psi \in W^{1,p}(\Omega)$ such that $\mathbf{curl}\,\psi = \mathbf{v}$.*

Proof. Let us show that the vector field $\mathbf{v}$ can be extended to a field $\tilde{\mathbf{v}} \in L^p(\mathbb{R}^2)^2$ in such a way that $\operatorname{div} \tilde{\mathbf{v}} = 0$ in $\mathbb{R}^2$ and $\operatorname{supp} \tilde{\mathbf{v}} \subset B$ where B is a ball containing $\overline{\Omega}$.

For $j = 1, \ldots, m$, let us call Ω_j the domain with exterior boundary Γ_j. Let $\tilde{\mathbf{v}} \in L^p(\Omega_j)$ be such that $\operatorname{div} \tilde{\mathbf{v}} = 0$ in Ω_j and $\tilde{\mathbf{v}} \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n}$ on Γ_j. Note that such a $\tilde{\mathbf{v}}$ exists in view of Lemma 2 and the hypothesis $\langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Gamma_j} = 0$. In the same way we define $\tilde{\mathbf{v}}$ in the subdomain of B with boundary ∂B and Γ_0 with vanishing divergence and satisfying $\tilde{\mathbf{v}} \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n}$ on Γ_0 and $\tilde{\mathbf{v}} \cdot \mathbf{n} = 0$ on ∂B. Finally, we extend $\tilde{\mathbf{v}}$ by zero outside of B.

In view of the continuity of the normal component across Γ_j, $j = 0, \ldots, m$ and ∂B it follows that $\operatorname{div} \tilde{\mathbf{v}} = 0$ in $\mathbb{R}^2$. Therefore, the existence of the stream function $\psi \in L^p(\Omega)$ follows from Lemma 1. $\square$

Our next lemma is a simple consequence of the following known result which can be found in [8]. See also [5] where a different proof of a dual result is given.

Let $1 < p < \infty$. Given a function $g \in L^p(\Omega)$, with vanishing mean value, there exists $\mathbf{u} \in W_0^{1,p}(\Omega)^2$ such that,

$$\operatorname{div} \mathbf{u} = g \qquad \text{in } \ \Omega$$

It is interesting to remark that this result does not hold in general if the domain is not Lipschitz. If the domain satisfies the segment property, this result is equivalent to the so called Lions lemma stating that if $f \in W^{-1,p}(\Omega)$ and $\nabla f \in W^{-1,p}(\Omega)^2$ then, $f \in L^p(\Omega)$. That this Lemma does not hold in irregular domains is shown in [3] by giving a counterexample. For the case $p = 2$ this can be deduced also from the results of [1].

Lemma 3. *Let $1 < p < \infty$ and $\mathbf{f} \in W^{1-1/p,p}(\Gamma)^2$ be such that $\int_\Gamma \mathbf{f} \cdot \mathbf{n} = 0$. Then, there exists $\mathbf{v} \in W^{1,p}(\Omega)^2$ such that,*

$$\mathit{div}\, \mathbf{v} = 0 \qquad \mathit{in} \ \ \Omega$$

and

$$\gamma_0(\mathbf{v}) = \mathbf{f} \qquad \mathit{on} \ \ \Gamma$$

Proof. Let $\mathbf{w} \in W^{1,p}(\Omega)^2$ be such that $\gamma_0(\mathbf{w}) = \mathbf{f}$. Since $\int_\Gamma \mathbf{f} \cdot \mathbf{n} = 0$ then, $\int_\Omega \operatorname{div} \mathbf{w} = 0$. Therefore, there exists $\mathbf{u} \in W_0^{1,p}(\Omega)^2$ such that $\operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{w}$ and then, $\mathbf{v} = \mathbf{w} - \mathbf{u}$ is the desired function. $\square$

Remark 1. *Clearly we could use Lemma 3 instead of Lemma 2 in the proof of Theorem 1 when the vector field $\mathbf{v}$ is in $W^{1,p}(\Omega)^2$ (which will be the case of interest for our arguments). However, we prefer to include Lemma 2 because its proof is simpler and it provides the existence of a stream function for divergence free vector fields which are only in $L^p(\Omega)^2$, a result that can be of interest in itself.*

Our next theorem gives a characterization of the range of the map

$$\psi \longmapsto (\gamma_0(\psi), \partial_n \psi)$$

which is linear and continuous from $W^{2,p}(\Omega)$ into $W^{1,p}(\Gamma) \times L^p(\Gamma)$. The result generalizes to the case $1 < p < \infty$ that obtained in [4] for $p = 2$. We recall that, when Γ is smooth, the range of this map is $W^{2-1/p,p}(\Gamma) \times W^{1-1/p,p}(\Gamma)$.

Theorem 2. *Let $g_0 \in W^{1,p}(\Gamma)$ and $g_1 \in L^p(\Gamma)$. Then, there exists $\psi \in W^{2,p}(\Omega)$ such that*

$$\gamma_0(\psi) = g_0 \quad \textit{and} \quad \partial_n \psi = g_1$$

if and only if

$$(\partial_t g_0)\mathbf{n} - g_1 \mathbf{t} \in W^{1-1/p,p}(\Gamma)^2$$

Proof. Let us call $\mathbf{f} = (\partial_t g_0)\mathbf{n} - g_1\mathbf{t}$. Given $\psi \in W^{2,p}(\Omega)$ let $g_0 = \gamma_0(\psi)$ and $g_1 = \partial_n \psi$. It is easy to see that $\partial_t g_0 = \mathbf{curl}\,\psi \cdot \mathbf{n}$ and $\partial_n \psi = -\mathbf{curl}\,\psi \cdot \mathbf{t}$. Therefore, $\mathbf{f} = \gamma_0(\mathbf{curl}\,\psi) \in W^{1-1/p,p}(\Gamma)^2$.

Assume now that $\mathbf{f} \in W^{1-1/p,p}(\Gamma)^2$. Since $\mathbf{f} \cdot \mathbf{n} = \partial_t g_0$ we have that $\int_{\Gamma_j} \mathbf{f} \cdot \mathbf{n} = 0$, for $j = 0, \cdots, m$. Then, from Lemma 3 we know that there exists $\mathbf{v} \in W^{1,p}(\Omega)^2$ such that $\operatorname{div} \mathbf{v} = 0$ and $\gamma_0(\mathbf{v}) = \mathbf{f}$. Therefore, it follows from Theorem 1 that there exists $\psi \in W^{1,p}(\Omega)$ such that $\mathbf{curl}\,\psi = \mathbf{v}$. But, since $\mathbf{v} \in W^{1,p}(\Omega)^2$, it follows that $\psi \in W^{2,p}(\Omega)$. Finally, observe that

$$\partial_n \psi = -\mathbf{curl}\,\psi \cdot \mathbf{t} = -\mathbf{f} \cdot \mathbf{t} = g_1$$

and

$$\partial_t \psi = \mathbf{curl}\,\psi \cdot \mathbf{n} = \mathbf{f} \cdot \mathbf{n} = \partial_t g_0$$

Therefore, $\psi - g_0$ is constant on each Γ_j and so, ψ can be modified by adding a smooth function which is constant in a neighborhood of each Γ_j in order to obtain the desired function. $\square$

References

[1] Friedrichs, K. O., On certain inequalities and characteristic value problems for analytic functions and for functions of two variables, Trans. Amer. Math. Soc. **41**, 321-364, 1937.

[2] Gagliardo, M., *Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili*, Ren. Sem. Mat. Univ. Padova **27**, pp. 284-305, 1957.

[3] Geymonat, G., Gilardi, G., Contre-exemples á l'inégalité de Korn et au Lemme de Lions dans des domaines irréguliers, Equations aux Dérivées Partielles et Applications, Gauthiers-Villars, pp. 541-548, 1998.

[4] Geymonat, G., Krasucki, F., *On the existence of the Airy function in Lipschitz domains. Application to the traces of H^2*, C. R. Acad. Sci. Paris, t. 330, serie 1, pp. 355-360, 2000.

[5] Geymonat, G., Suquet, P., Functional spaces for Norton-Hoff materials, Math. Meth. in the Appl. Sci. **8**, pp. 206-222, 1986.

[6] Girault, V., Raviart, P. A., Finite Element Methods for Navier-Stokes Equations, Theory and Algorithms, Springer Verlag, Berlin, 1986.

[7] Grisvard, P., Elliptic Problems in Nonsmooth Domains, Pitman, Boston, 1985.

[8] Nečás, J., Equations aux Dérivées Partielles, Presses de l'Université de Montréal, 1966.

Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, 1428 Buenos Aires, Argentina.
E-mail address: `rduran@dm.uba.ar`

Departamento de Matemática, Facultad de Ciencias Exactas, Universidad Nacional de La Plata, Casilla de Correo 172, 1900 La Plata, Provincia de Buenos Aires, Argentina.
E-mail address: `mariam@mate.unlp.edu.ar`