AN EXPLICIT RIGHT INVERSE OF THE DIVERGENCE OPERATOR WHICH IS CONTINUOUS IN WEIGHTED NORMS

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Dedicated to the memory of our great teacher Alberto P. Calderón who showed us the beauty and power of Real Analysis

Abstract. The existence of a continuous right inverse of the divergence operator in $W_0^{1,p}(\Omega)^n$, $1 < p < \infty$, is a well known result which is basic in the analysis of the Stokes equations.

The object of this paper is to show that the continuity holds also for some weighted norms. Our results are valid for $\Omega \subset \mathbb{R}^n$ a bounded domain which is star-shaped with respect to a ball $B \subset \Omega$.

The continuity results are obtained by using an explicit solution of the divergence and the classical theory of singular integrals of Calderón and Zygmund together with general results on weighted estimates proven by Stein.

The weights considered here are of interest in the analysis of finite element methods. In particular, our result allows to extend to the three dimensional case the general results on uniform convergence of finite element approximations of the Stokes equations.

1. Introduction

A basic result for the theoretical and numerical analysis of the Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^n$ is the existence of a continuous right inverse of the divergence as an operator from the Sobolev space $H_0^1(\Omega)^n$ into the space $L_0^2(\Omega)$ of functions in $L^2(\Omega)$ with vanishing mean value. In other words, given a function $f \in L_0^2(\Omega)$, the problem is to find a solution $\mathbf{u} \in H_0^1(\Omega)^n$ of the equation

$$\operatorname{div} \mathbf{u} = f \qquad \text{in} \quad \Omega \tag{1.1}$$

such that

$$\|\mathbf{u}\|_{H_0^1(\Omega)^n} \le C \|f\|_{L^2(\Omega)} \tag{1.2}$$

where, here and throughout the paper, the letter C denotes a generic constant.

Several arguments have been given to prove this result. For example, if the domain has a smooth boundary or if it is a convex polygon then, the existence of $\mathbf{u}$ can be proven by using a

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priori estimates for elliptic equations. Indeed, taking $v \in H^1(\Omega)$ as the solution of the Neumann problem

$$\begin{cases} -\Delta v = f & \text{in } \Omega \\ \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

we have that $\tilde{\mathbf{u}} = \nabla v$ satisfies the equation (1.1) and, from a priori estimates for (1.3) (see [12, 10]), it follows that $\|\tilde{\mathbf{u}}\|_{H^1(\Omega)^n} \leq C\|f\|_{L^2(\Omega)}$. Although $\tilde{\mathbf{u}}$ is not in $H_0^1(\Omega)^n$ it is not difficult to modify it by adding a divergence free vector function in order to impose the homogeneus boundary conditions and to obtain $\mathbf{u}$ satisfying (1.1) and also (1.2) (see [5, 11, 3, 13]). This argument can not be applied for a non smooth domain since the solution of the Neumann problem (1.3) is not in general in $H^2(\Omega)$ and so $\tilde{\mathbf{u}}$ will not be in $H^1(\Omega)$.

If Ω is a non convex polygon, solutions of (1.1) satisfying (1.2) were constructed in [2]. The argument in that paper is based on solving the Poisson equation in a larger smooth domain in order to obtain an $\tilde{\mathbf{u}}$ as before. Then, the modification to impose the boundary conditions requires trace theorems for non convex polygons which were developed in [2].

More generally, the result can be proven for a Lipschitz domain in $\mathbb{R}^n$ using different approaches. One possibility is to look at the dual problem. Indeed, by standard functional analysis arguments it can be seen [17] that the existence of $\mathbf{u}$ satisfying (1.1) and (1.2) is equivalent to the existence of a constant C such that for all $q \in L_0^2(\Omega)$

$$\|q\|_{L^2(\Omega)} \leq C\|\nabla q\|_{H^{-1}(\Omega)^n} \tag{1.4}$$

This inequality can be proven for a Lipschitz domain by using compactness arguments. The first and most technical part of the proof is to show that, for any $q \in L^2(\Omega)$,

$$\|q\|_{L^2(\Omega)} \sim \|q\|_{H^{-1}(\Omega)} + \|\nabla q\|_{H^{-1}(\Omega)^n}$$

then, the existence of C such that (1.4) holds follows from this equivalence of norms arguing by contradiction and using the compactness of the inclusion of $L^2(\Omega)$ in $H^{-1}(\Omega)$ (see [14] for details).

A different argument is to construct an explicit solution of (1.1) satisfying (1.2) by means of an integral operator. This has been done in [4].

The object of this work is to prove that there exist solutions of (1.1) satisfying weighted estimates analogous to (1.2).

Our results hold for domains Ω which are star-shaped with respect to a ball $B \subset \Omega$. As in [4], the solution $\mathbf{u}$ will be defined by means of an integral operator.

We will show that the derivatives of $\mathbf{u}$ can be written in terms of a singular integral operator of the Calderón-Zygmund type acting on the right hand side f and that consequently, the weighted estimates can be derived from general results on the continuity of singular integral operators in weighted norms.

Our proof is valid also for the general case of $L^p(\Omega)$, $1 < p < \infty$. However, for the sake of clarity, we will write the weighted estimates only for $p = 2$.

Weighted a-priori estimates are a well known tool for the analysis of uniform convergence of finite element methods (see for example [8]). In particular, the result obtained here allows to generalize to 3-d the general error analysis given in [9] for finite element approximations of the Stokes equations.

2. Construction of the solution and a-priori estimate

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with diameter d. Take $\omega \in C_0^\infty(\Omega)$ such that $\int_\Omega \omega = 1$ and define $G = (G_1, \cdots, G_n)$ as

$$G(x,y) = \int_0^1 \frac{1}{s^{n+1}} (x-y)\, \omega(y + \frac{x-y}{s})\, ds \tag{2.1}$$

The following lemma gives a bound for $G(x,y)$ that will be fundamental in our subsequent arguments.

Lemma 2.1. *For $y \in \Omega$ we have*

$$|G(x,y)| \le \|\omega\|_{L^\infty(\Omega)} \frac{d^n}{(n-1)|x-y|^{n-1}} \tag{2.2}$$

Proof. Since $\omega \in C_0^\infty(\Omega)$ it follows that the integrand in (2.1) vanishes whenever $z = y + (x - y)/s \notin \Omega$. Therefore, since $y \in \Omega$, we can restrict the integral defining $G(x,y)$ to those values of s such that $|z-y| \le d$, that is, $|x-y|/d \le s$ and (2.2) follows easily.

In the next lemma and its corollary we introduce the explicit right inverse of the divergence.

Lemma 2.2. *For any $\phi \in C_0^\infty(\Omega)$ we define $\overline{\phi} = \int_\Omega \phi\,\omega$. Then, for $y \in \Omega$ we have*

$$(\phi - \overline{\phi})(y) = -\int_\Omega G(x,y) \cdot \nabla\phi(x)\, dx$$

Proof. For $y \in \Omega$ we have

$$(\phi - \overline{\phi})(y) = \int_\Omega \int_0^1 (y-z) \cdot \nabla\phi(y + s(z-y))\, \omega(z)\, ds\, dz$$

and interchanging the order of integration and making the change of variable $x = y + s(z-y)$ we obtain

$$(\phi - \overline{\phi})(y) = \int_0^1 \int_\Omega \frac{1}{s^{n+1}} (y-x) \cdot \nabla\phi(x)\, \omega(y + \frac{x-y}{s})\, dx\, ds$$

and the proof concludes by observing that we can interchange again the order of integration. Indeed, using the bound given in (2.2) for G, it is easy to see that the integral of the absolute value of the integrand is finite.

Corollary 2.1. *Given $f \in L^1(\Omega)$ such that $\int_\Omega f = 0$ define*

$$\mathbf{u}(x) = \int_\Omega G(x,y)\, f(y)\, dy \tag{2.3}$$

then,

$$\operatorname{div} \mathbf{u} = f \qquad \text{in } \Omega$$

Proof. For $\phi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega f(y)\phi(y)\,dy = \int_\Omega f(y)(\phi - \overline{\phi})(y)\,dy = -\int_\Omega\int_\Omega f(y)G(x,y)\cdot\nabla\phi(x)\,dx\,dy$$

and interchanging the order of integration, which can be done in view of (2.2), we obtain

$$\int_\Omega f(y)\phi(y)\,dy = -\int_\Omega \mathbf{u}(x)\cdot\nabla\phi(x)\,dx$$

which concludes the proof.

Up to this point, we have not imposed any condition on the domain Ω other than boundedness. Assume now that $\Omega \subset \mathbb{R}^n$ is star-shaped with respect to a ball $B \subset \Omega$ (i.e., for any $z \in B$ and any $x \in \Omega$, the segment joining z and x is contained in Ω). The following lemma shows that in this case the function $\mathbf{u}$ defined in (2.3) vanishes on $\partial\Omega$.

Lemma 2.3. *If Ω is star-shaped with respect to a ball B and $\omega \in C_0^\infty(B)$ then, $G(x,y) = 0$ for all $x \in \partial\Omega$ and all $y \in \Omega$. In particular, $\mathbf{u}$ defined as in (2.3) vanishes on $\partial\Omega$.*

Proof. For $x \in \partial\Omega$, $y \in \Omega$ and any $s \in [0,1]$ we have that $z = y + (x-y)/s \notin B$. Otherwise, since Ω is star-shaped with respect to B, $x = (1-s)y + sz$ would be in Ω. Therefore, the result follows from the definition of $G(x,y)$ recalling that $\omega \in C_0^\infty(B)$.

We want to see that $\frac{\partial u_j}{\partial x_i} \in L^p(\Omega)$ whenever $f \in L^p(\Omega)$, $1 < p < \infty$, and moreover, that there exists a constant C depending only on p and Ω such that $\|\frac{\partial u_j}{\partial x_i}\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}$.

For our subsequent arguments it is convenient to introduce the characteristic function $\mathcal{X}_\Omega$ of Ω. In this way, we will be able to work with operators defined on $L^p(\mathbb{R}^n)$. A function $f \in L^p(\Omega)$ will be extended by zero outside of Ω.

In the next lemma we give an expression for $\frac{\partial u_j}{\partial x_i}$ in terms of f. In order to do that we introduce the following singular integral operator

$$T_{ij}g(y) = \lim_{\varepsilon\to 0}\int_{|y-x|>\varepsilon} \mathcal{X}_\Omega(y)\frac{\partial G_j}{\partial x_i}(x,y)g(x)\,dx$$

and its adjoint

$$T_{ij}^* f(x) = \lim_{\varepsilon\to 0}\int_{|y-x|>\varepsilon} \mathcal{X}_\Omega(y)\frac{\partial G_j}{\partial x_i}(x,y)f(y)\,dy$$

Afterwards, we will prove that the limit defining T_{ij} exists and defines an operator which is bounded in L^p for $1 < p < \infty$. By duality, the same will be true for T_{ij}^*.

Lemma 2.4. *We have*

$$\frac{\partial u_j}{\partial x_i} = T_{ij}^* f + \omega_{ij} f \qquad in \;\; \Omega$$

where

$$\omega_{ij}(y) = \int_{\mathbb{R}^n} \frac{z_j z_i}{|z|^2}\omega(y+z)\,dz$$

Proof. From the definition of G_j and using again (2.2) to interchange the order of integration we have, for any $\phi \in C_0^\infty(\Omega)$,

$$-\int_\Omega u_j(x)\frac{\partial\phi}{\partial x_i}(x)\,dx = -\int_\Omega\int_\Omega G_j(x,y)f(y)\frac{\partial\phi}{\partial x_i}(x)\,dx\,dy \tag{2.4}$$

Now, denoting with $B(y,\varepsilon)$ the ball with center at y and radius ε, we have

$$\begin{aligned}-\int_\Omega G_j(x,y)\frac{\partial\phi}{\partial x_i}(x)\,dx &= -\lim_{\varepsilon\to0}\int_{\Omega\setminus B(y,\varepsilon)} G_j(x,y)\frac{\partial\phi}{\partial x_i}(x)\,dx\\ &= \lim_{\varepsilon\to0}\Big\{\int_{|y-x|>\varepsilon}\frac{\partial G_j}{\partial x_i}(x,y)\phi(x)\,dx\\ &\qquad -\int_{\partial B(y,\varepsilon)} G_j(\zeta,y)\phi(\zeta)\frac{(y_i-\zeta_i)}{|y-\zeta|}\,d\zeta\Big\}\end{aligned} \tag{2.5}$$

Now, we can decompose the integral on $\partial B(y,\varepsilon)$ in two parts in the following way

$$\begin{aligned}\int_{\partial B(y,\varepsilon)} G_j(\zeta,y)\phi(\zeta)\frac{(y_i-\zeta_i)}{|y-\zeta|}\,d\zeta &= \phi(y)\int_{\partial B(y,\varepsilon)} G_j(\zeta,y)\frac{(y_i-\zeta_i)}{|y-\zeta|}\,d\zeta\\ &+\int_{\partial B(y,\varepsilon)} G_j(\zeta,y)(\phi(\zeta)-\phi(y))\frac{(y_i-\zeta_i)}{|y-\zeta|}\,d\zeta := I_\varepsilon + II_\varepsilon\end{aligned}$$

and it easy to see that $II_\varepsilon \to 0$. Indeed, using the bound given in (2.2) for G_j and the fact that ϕ has bounded derivatives we obtain that there exists a constant C depending only on d, n and $\|\phi\|_{W^{1,\infty}(\Omega)}$ such that

$$|II_\varepsilon| \le C\varepsilon$$

On the other hand we have

$$-\lim_{\varepsilon\to0} I_\varepsilon = -\lim_{\varepsilon\to0}\phi(y)\int_{\partial B(y,\varepsilon)}\int_0^1\frac{1}{s^{n+1}}(\zeta_j-y_j)\frac{(y_i-\zeta_i)}{|y-\zeta|}\omega(y+\frac{\zeta-y}{s})\,ds\,d\zeta$$

Then, making the change of variables $r=\varepsilon/s$ and $\sigma=(\zeta-y)/\varepsilon$ and denoting with Σ the unit sphere we obtain

$$\begin{aligned}-\lim_{\varepsilon\to0} I_\varepsilon &= \phi(y)\lim_{\varepsilon\to0}\int_{\partial B(y,\varepsilon)}\int_\varepsilon^\infty(\zeta_j-y_j)\frac{(\zeta_i-y_i)}{|\zeta-y|}\omega(y+r\frac{\zeta-y}{\varepsilon})\frac{r^{n-1}}{\varepsilon^n}\,dr\,d\zeta\\ &= \phi(y)\lim_{\varepsilon\to0}\int_\Sigma\int_\varepsilon^\infty\sigma_j\sigma_i\omega(y+r\sigma)r^{n-1}\,dr\,d\sigma\\ &= \phi(y)\lim_{\varepsilon\to0}\int_\Sigma\int_\varepsilon^\infty\frac{\sigma_j\sigma_i}{|\sigma|^2}\omega(y+r\sigma)r^{n-1}\,dr\,d\sigma\\ &= \phi(y)\lim_{\varepsilon\to0}\int_{\mathbb{R}^n\setminus B(0,\varepsilon)}\frac{z_jz_i}{|z|^2}\omega(y+z)\,dz = \phi(y)\omega_{ij}(y)\end{aligned}$$

and therefore, replacing in (2.5) we obtain that for $y\in\Omega$

$$-\int_\Omega G_j(x,y)\frac{\partial\phi}{\partial x_i}(x)\,dx = T_{ij}\phi(y)+\omega_{ij}(y)\phi(y)$$

which together with (2.4) yields the result.

Since ω_{ij} is a bounded function, in order to see the L^p boundedness of $\frac{\partial u_j}{\partial x_i}$ it is enough to show that the operator T_{ij} is continuous in L^p. We will show that T_{ij} is a singular integral operator of the Calderón-Zygmund type and so, it is bounded in L^p for all $1<p<\infty$.

Calling $\eta_j(y,z)=z_j\omega(y+z)$ we obtain from (2.1) that

$$\frac{\partial G_j}{\partial x_i}(x,y) = \int_0^1 \frac{1}{s^{n+1}}\frac{\partial \eta_j}{\partial z_i}(y,\frac{x-y}{s})\,ds$$

Then, the kernel $\mathcal{X}_\Omega(y)\frac{\partial G_j}{\partial x_i}(x,y)$ and so, the operator T_{ij} can be decomposed in two parts as follows

$$\begin{aligned}\mathcal{X}_\Omega(y)\frac{\partial G_j}{\partial x_i}(x,y) &= \int_0^\infty \mathcal{X}_\Omega(y)\frac{1}{s^{n+1}}\frac{\partial \eta_j}{\partial z_i}(y,\frac{x-y}{s})\,ds - \int_1^\infty \mathcal{X}_\Omega(y)\frac{1}{s^{n+1}}\frac{\partial \eta_j}{\partial z_i}(y,\frac{x-y}{s})\,ds\\ &:= K_1(y,x-y)+K_2(y,x-y)\end{aligned}$$

and

$$T_{ij}=T_1+T_2 \tag{2.6}$$

with

$$T_\ell g(y)=\lim_{\varepsilon\to 0}\int_{|y-x|>\varepsilon} K_\ell(y,x-y)g(x)\,dx \qquad \text{for} \qquad \ell=1,2$$

First, we will show that the second part T_2 defines a bounded operator in L^p for $1\le p<\infty$. This will be a consequence of the bound for K_2 given in the next lemma.

Lemma 2.5. *We have*

$$|K_2(y,z)| \le \frac{(1+d)}{n}\|\omega\|_{W^{1,\infty}(\mathbb{R}^n)}\min\Big\{1,\frac{d^n}{|z|^n}\Big\} \tag{2.7}$$

Proof. From the definition of η_j we can see that

$$\Big|\frac{\partial \eta_j}{\partial z_i}(y,\frac{z}{s})\Big| \le \Big(1+\frac{|z|}{s}\Big)\|\omega\|_{W^{1,\infty}(\mathbb{R}^n)} \tag{2.8}$$

Now, since $\operatorname{supp}\omega\subset B\subset\Omega$ it follows that $\mathcal{X}_\Omega(y)\frac{\partial\eta_j}{\partial z_i}(y,z/s)$ vanishes for $|z|/s>d$. In particular, the integral defining K_2 can be restricted to those values of s such that $s\ge|z|/d$ and from (2.8) we obtain

$$\Big|\mathcal{X}_\Omega(y)\frac{\partial \eta_j}{\partial z_i}(y,\frac{z}{s})\Big| \le (1+d)\|\omega\|_{W^{1,\infty}(\mathbb{R}^n)}$$

Therefore,

$$|K_2(y,z)| \leq (1+d)\|\omega\|_{W^{1,\infty}(\mathbb{R}^n)} \int_{\max\{1,|z|/d\}}^{\infty} \frac{1}{s^{n+1}}\, ds$$

which conludes the proof.

Corollary 2.2. *The operator T_2 is bounded in L^p for $1 \leq p < \infty$.*

Proof. Using (2.7) and the Hölder inequality it follows that, for $g \in L^p(\mathbb{R}^n)$, the integral defining T_2 is absolutly convergent and moreover, there exists a constant C depending on d, n, ω and p such that

$$|T_2 g(y)| \leq C\|g\|_{L^p(\mathbb{R}^n)}$$

and the proof concludes by observing that $T_2 g$ has compact support.

In view of the decomposition (2.6) and Corollary 2.2, it remains to analyze the continuity of the operator T_1. With this goal, we will show in the next two lemmas that $|K_1(y,z)| \leq C|z|^{-n}$, with a constant C independent of y, and that, as a function of the second variable, K_1 is homogeneous of degree $-n$ and with vanishing mean value on the unit sphere. According to the classical theory of Calderón and Zygmund [7, 6], these conditions are sufficient for the continuity in L^p, $1 < p < \infty$, of the associated singular integral operator.

Lemma 2.6. *We have*

$$|K_1(y,z)| \leq \frac{(1+d)}{n}\|\omega\|_{W^{1,\infty}(\mathbb{R}^n)} \frac{d^n}{|z|^n}$$

Proof. It follows by the same arguments used in the proof of Lemma 2.5.

Lemma 2.7. *$K_1(y,z)$ is homogeneous of degree $-n$ and with vanishing mean value on the unit sphere Σ, in the second variable.*

Proof. Given $\lambda > 0$, making the change of variable $t = s/\lambda$ in the definition of K_1 we have

$$K_1(y,\lambda z) = \lambda^{-n} \int_0^\infty \mathcal{X}_\Omega(y) \frac{1}{t^{n+1}} \frac{\partial \eta_j}{\partial z_i}(y, \frac{z}{t})\, dt = \lambda^{-n} K_1(y,z)$$

On the other hand, making the change of variable $r = 1/s$ in the integral defining K_1 we have

$$K_1(y,z) = \int_0^\infty \mathcal{X}_\Omega(y) \frac{\partial \eta_j}{\partial z_i}(y, rz) r^{n-1}\, dr$$

and therefore,

$$\int_\Sigma K_1(y,\sigma)\, d\sigma = \int_\Sigma \int_0^\infty \mathcal{X}_\Omega(y) \frac{\partial \eta_j}{\partial z_i}(y, r\sigma) r^{n-1}\, dr\, d\sigma = \int_{\mathbb{R}^n} \mathcal{X}_\Omega(y) \frac{\partial \eta_j}{\partial z_i}(y,z)\, dz = 0$$

which concludes the proof.

Remark 2.1. A different way of proving Lemma 2.7 is the following. Define

$$H(y,z) = \int_0^\infty \mathcal{X}_\Omega(y) \frac{1}{s^{n+1}} z_j \omega(y + \frac{z}{s})$$

then, proceding as in that lemma it is easy to show that $H(y,z)$ is homogeneous of degree $-n+1$ in the second variable and, since $K_1(x,z) = \frac{\partial H}{\partial z_i}$ the desired properties of K_1 follows (see [1, page 152]).

Remark 2.2. We have considered f such that $\int_\Omega f = 0$. However, the operator giving the solution $\mathbf{u}$ is defined for any $f \in L^1(\mathbb{R}^n)$. It is easy to show directly that

$$\operatorname{div} \mathbf{u} = f - (\int_\Omega f)\omega \qquad \text{in } \Omega \tag{2.9}$$

indeed, using the expressions for the derivatives given in Lemma 2.4 and observing that $\sum_{j=1}^n \omega_{jj} = 1$ we have that

$$\operatorname{div} \mathbf{u} = f + \sum_{j=1}^n T^*_{jj} f \qquad \text{in } \Omega$$

and so, we have to check that

$$\sum_{j=1}^n T^*_{jj} f = -(\int_\Omega f)\omega \qquad \text{in } \Omega$$

But, we have

$$\sum_{j=1}^n T^*_{jj} f(x) = \lim_{\varepsilon \to 0} \sum_{j=1}^n \int_{|y-x|>\varepsilon} \mathcal{X}_\Omega(y) \frac{\partial G_j}{\partial x_j}(x,y) f(y)\, dy \tag{2.10}$$

with

$$\frac{\partial G_j}{\partial x_j}(x,y) = \int_0^1 \frac{1}{s^{n+1}} \frac{\partial \eta_j}{\partial z_j}(y, \frac{x-y}{s})\, ds \tag{2.11}$$

Now,

$$\frac{\partial \eta_j}{\partial z_j}(y,z) = \omega(y+z) + z_j \frac{\partial \omega}{\partial z_j}(y+z)$$

and so, making the change of variable $r = 1/s$ in (2.11) we obtain

$$\sum_{j=1}^n \frac{\partial G_j}{\partial x_j}(x,y) = \sum_{j=1}^n \int_1^\infty \frac{\partial \eta_j}{\partial z_j}(y, r(x-y))\, dr = \int_1^\infty \frac{d}{dr}[\omega(y + r(x-y)r^n]\, dr = -\omega(x)$$

which together with (2.10) concludes the proof of (2.9).

Summing up the above results we obtain the following

Theorem 2.1. *Let Ω be bounded and star-shaped with respect to a ball $B \subset \Omega$. If $f \in L^p(\Omega)$, $1 < p < \infty$, and $\int_\Omega f = 0$ then, the function $\mathbf{u}$ defined in (2.3) is in $W_0^{1,p}(\Omega)^n$ and satisfies*

$$\operatorname{div} \mathbf{u} = f \qquad \text{in } \Omega$$

and,

$$\|\mathbf{u}\|_{W_0^{1,p}(\Omega)^n} \le C\|f\|_{L^p(\Omega)}$$

Proof. In view of Lemmas 2.6 and 2.7, it follows from the theory developed in [7] that the limit defining T_1 exists and defines an operator which is continuous in L^p for $1 < p < \infty$. Then, the boundedness of T_{ij} in L^p, for $1 < p < \infty$, follows from the decomposition $T_{ij} = T_1 + T_2$ recalling that T_2 is continuous in L^p. Then, by duality, T_{ij}^* is also bounded in L^p for $1 < p < \infty$ and the proof concludes by using the representation for $\frac{\partial u_j}{\partial x_i}$ given in Lemma 2.4.

3. Weighted a-priori estimate

A well known technique to prove error estimates in the L^∞ norm for finite element approximations is based on the use of weighted norms (see for example [8] and references therein). In particular, weighted a-priori estimates related with the equation being considered are needed when this approach is used.

For finite element methods for the Stokes equations, a general error analysis for the L^∞ norm has been given in [9]. The results obtained there are based on a weighted inf-sup condition or, equivalently, on a weighted a-priori estimate for a solution of the divergence operator. The proof of this estimate given in [9] is restricted to the 2-d case while the rest of the arguments can be extended strightforward to 3-d.

Here we will show that this weighted a-priori estimate can be derived from our result of the previous section together with a weighted estimate for general singular integral operators given by Stein [15]. Our result holds in any dimension. In particular, the general error analysis given in [9] can be extended to the 3-d case.

In order to state our result we need to introduce first some notation.

Let $0 < \theta < 1/2$ be a parameter and $\sigma(x) = (|x - x_0|^2 + \theta^2)^{1/2}$ where x_0 is a fixed point in the domain Ω. We are interested in the following result (see Lemma 2.2 in [9]):

Given $f \in L_0^2(\Omega)$, find $\mathbf{u} \in H_0^1(\Omega)^n$ solution of

$$\operatorname{div} \mathbf{u} = f \qquad \text{in } \ \Omega$$

and such that

$$\int_\Omega |\nabla \mathbf{u}(x)|^2 \sigma^n(x)\, dx \le C|\log\theta|^2 \int_\Omega |f(x)|^2 \sigma^n(x)\, dx$$

with the constant C independent of θ and x_0.

In order to prove this estimate we will use the following general result of Stein [15]. In the statement of his theorem, Stein considered an operator associated with a kernel of the form $K(x,y) = \frac{H(x,x-y)}{|x-y|^n}$, i.e., such as that associated with T_{ij}. Since we are interested in estimates for T_{ij}^* we could apply his result to T_{ij} and proceed by duality. However, it is interesting to remark that his result applies directly to T_{ij}^*. Indeed, his proof only uses that $|K(x,y)| \le \frac{C}{|x-y|^n}$ and so, in the particular case $p = 2$, the main theorem given in [15] can be sated as follows,

Theorem 3.1. *Let*

$$Tf(x) = \lim_{\varepsilon\to 0} \int_{|y-x|>\varepsilon} K(x,y) f(y)\, dy$$

and assume that there exist constants A_2 and A such that

$$\|Tf\|_{L^2(\mathbb{R}^n)} \leq A_2 \|f\|_{L^2(\mathbb{R}^n)}$$

and,

$$|K(x,y)| \leq \frac{A}{|x-y|^n}$$

Then, for $-n < \alpha < n$,

$$\int_{\mathbb{R}^n} |Tf(x)|^2 \sigma^\alpha(x)\, dx \leq C_\alpha^2 \int_{\mathbb{R}^n} |f(x)|^2 \sigma^\alpha(x)\, dx$$

where C_α is a constant independent of x_0 and θ.

Remark 3.1. The theorem given in [15] is for the weight $|x|^\alpha$ instead of σ^α. However, it is easily seen that the arguments apply for the weight $(|x|+\theta)^\alpha$ (see the proof in page 254 of [15]). Indeed, for $\theta = 1$ this was observed by Stein in his book [16, page 49]. On the other hand, by translation, it is clear that the weight can be replaced by $(|x-x_0|+\theta)^\alpha$ (which is equivalent to σ^α), obtaining a constant which is independent of θ and x_0.

In order to make an extrapolation to the limit case $\alpha = n$ we need to know the dependence of the constant C_α on α. Although this dependence is not given explicitly in [15], it is easy to find out from the proof that, for $0 < \alpha < n$,

$$C_\alpha = \frac{C}{n-\alpha} \tag{3.1}$$

with C independent of α. Indeed, the restriction $\alpha < n$ is used in the proof only to bound the integral (see formula (6) in [15, page 252])

$$\int_0^{1/2} |1-\lambda^{-\beta}|\lambda^{n/2-1}\, d\lambda$$

where $\beta = \alpha/2$ and, it can be easily checked that the constant C_α behaves like this integral and therefore, (3.1) holds.

We can now give the main result of this section.

Theorem 3.2. *If Ω is bounded and star-shaped with respect to a ball $B \subset \Omega$ then, for $f \in L_0^2(\Omega)$, there exists a solution $\mathbf{u} \in H_0^1(\Omega)^n$ of $\operatorname{div}\mathbf{u} = f$ (given as in (2.3))such that, for $0 < \theta < 1/2$,*

$$\int_\Omega |\nabla \mathbf{u}(x)|^2 \sigma^n(x)\, dx \leq C|\log\theta|^2 \int_\Omega |f(x)|^2 \sigma^n(x)\, dx$$

with C independent of θ.

Proof. In view of the representation

$$\frac{\partial u_j}{\partial x_i} = T_{ij}^* f + \omega_{ij} f$$

given in Lemma 2.4 and recalling that ω_{ij} is a bounded function, it is enough to show that

$$\int_\Omega |T_{ij}^* f(x)|^2 \sigma^n(x)\, dx \leq C|\log\theta|^2 \int_\Omega |f(x)|^2 \sigma^n(x)\, dx$$

But, from the previous section we know that T^*_{ij} satisfies the hypotheses of Theorem 3.1. Therefore, for any $0<\alpha<n$ we have

$$\int_\Omega |T^*_{ij}f(x)|^2\sigma^\alpha(x)\,dx \le \frac{C}{(n-\alpha)^2}\int_\Omega |f(x)|^2\sigma^\alpha(x)\,dx$$

Actually, we would have the integrals over all $\mathbb{R}^n$ but we recall that the f is extended by zero outside Ω.

Now, using that Ω is bounded and so, $\sigma^{n-\alpha}$ is also bounded we have

$$\int_\Omega |T^*_{ij}f(x)|^2\sigma^n(x)\,dx \le C\int_\Omega |T^*_{ij}f(x)|^2\sigma^\alpha(x)\,dx \le \frac{C}{(n-\alpha)^2}\int_\Omega |f(x)|^2\sigma^\alpha(x)\,dx$$

and observing that $\sigma^{\alpha-n}\le\theta^{\alpha-n}$ we obtain

$$\int_\Omega |T^*_{ij}f(x)|^2\sigma^n(x)\,dx \le \frac{C}{\theta^{n-\alpha}(n-\alpha)^2}\int_\Omega |f(x)|^2\sigma^n(x)\,dx$$

Then, given $0<\theta<1/2$ we can take α such that $0<\alpha<n$ and $n-\alpha=1/\log(1/\theta)$ and we obtain

$$\int_\Omega |T^*_{ij}f(x)|^2\sigma^n(x)\,dx \le C|\log\theta|^2\int_\Omega |f(x)|^2\sigma^n(x)\,dx$$

concluding the proof.

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