SOLUTIONS OF THE DIVERGENCE AND ANALYSIS OF THE STOKES EQUATIONS IN PLANAR HÖLDER-α DOMAINS

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ABSTRACT. If $\Omega \subset \mathbb{R}^n$ is a bounded domain, the existence of solutions $\mathbf{u} \in H_0^1(\Omega)^n$ of $\operatorname{div} \mathbf{u} = f$ for $f \in L^2(\Omega)$ with vanishing mean value, is a basic result in the analysis of the Stokes equations. In particular it allows to show the existence of a solution $(\mathbf{u}, p) \in H_0^1(\Omega)^n \times L^2(\Omega)$, where $\mathbf{u}$ is the velocity and p the pressure.

It is known that the above mentioned result holds when Ω is a Lipschitz domain and that it is not valid for arbitrary Hölder-α domains.

In this paper we prove that if Ω is a planar simply connected Hölder-α domain, there exist solutions of $\operatorname{div} \mathbf{u} = f$ in appropriate weighted Sobolev spaces, where the weights are powers of the distance to the boundary. Moreover, we show that the powers of the distance in the results obtained are optimal.

For some particular domains with an external cusp we apply our results to show the well posedness of the Stokes equations in appropriate weighted Sobolev spaces obtaining as a consequence the existence of a solution $(\mathbf{u}, p) \in H_0^1(\Omega)^n \times L^r(\Omega)$ for some $r < 2$ depending on the power of the cusp.

1. INTRODUCTION

Let $\Omega \in \mathbb{R}^n$ be a bounded open domain. We will use standard notations for Sobolev spaces and, for $1 < p < \infty$, $L_0^p(\Omega)$ will denote the subspace of functions in $L^p(\Omega)$ with vanishing mean value.

The Stokes equations modeling the displacement of a viscous incompressible fluid contained in Ω are given by

$$
\begin{cases}
-\Delta \mathbf{u} + \nabla p & = \mathbf{f} \qquad \text{in } \Omega \\
\operatorname{div} \mathbf{u} & = 0 \qquad \text{in } \Omega \\
\mathbf{u} & = 0 \qquad \text{on } \partial\Omega.
\end{cases}
\tag{1.1}
$$

where $\mathbf{u}$ is the velocity and p the pressure.

Variational analysis of this system of equations is fundamental for the development of finite element numerical approximations as well as for the study of their accuracy. Consequently, a lot of work has been done in this direction. However, the standard well known theory is based on the assumption that Ω is a Lipschitz domain. Our goal is to extend the variational analysis of the Stokes equations to a more general class of domains. With this purpose we will work with Hölder-α domains.

For Lipschitz domains it is known that, for $\mathbf{f} \in H^{-1}(\Omega)^n$, there exists a unique solution $(\mathbf{u}, p) \in H_0^1(\Omega)^n \times L_0^2(\Omega)$ of (1.1) and moreover,

$$
\|\mathbf{u}\|_{H_0^1(\Omega)} + \|p\|_{L^2(\Omega)} \le C \|\mathbf{f}\|_{H^{-1}(\Omega)},
$$

see for example [8, 9, 26, 41].

The basic result to prove this existence and uniqueness theorem is the following result on solutions of the divergence: If Ω is a Lipschitz domain, for any $f \in L_0^2(\Omega)$ there exists $\mathbf{u} \in H_0^1(\Omega)^n$ such that

$$\text{div}\,\mathbf{u} = f \qquad \text{in}\ \ \Omega \tag{1.2}$$

and

$$\|\mathbf{u}\|_{H_0^1(\Omega)} \le C\|f\|_{L^2(\Omega)} \tag{1.3}$$

where the constant C depends only on Ω.

The existence of solutions of (1.2) satisfying (1.3) is also closely connected with the Korn inequality which is fundamental in the analysis of the elasticity equations (see [29]).

Because of these applications, this problem, as well as it generalizations to the L^p case, has been widely analyzed and several different arguments have been given to prove it under different assumptions on the domain. We refer the reader for example to [4, 7, 6, 8, 5, 26, 17]. Also, generalizations of this result to the case of fractional order Sobolev spaces have been proved recently in [11, 22].

Moreover, in [3], the existence of solutions of (1.2) satisfying

$$\|\mathbf{u}\|_{W_0^{1,p}(\Omega)} \le C\|f\|_{L^p(\Omega)}, \tag{1.4}$$

for $1 < p < \infty$, was proved for the so called John domains, which form a large class containing properly the Lipschitz domains. For the particular case of planar simply connected domains, it is shown in [3] that being a John domain is also a necessary condition in the case $1 < p < 2$. In particular, there exist bounded domains and values of p for which solutions of (1.2) satisfying (1.4) do not exist.

Actually, this fact was previously well known, indeed, several arguments have been given to show it. For example, in the old paper [20], Friedrichs proved that, for smooth planar domains, the L^2-norm of the conjugate of a harmonic function f (normalized in an appropriate way) is bounded by the L^2-norm of f times a constant depending only on the domain. Moreover, he showed that this inequality is not valid if the domain has an external cusp of quadratic type. It is easy to see that the Friedrichs inequality can be deduced from the existence of $\mathbf{u}$ satisfying (1.2) and (1.3). Therefore, such a $\mathbf{u}$ cannot exist for that kind of domains. More recently other examples have been given in [23] and in an unpublished work of Gabriel Acosta. Acosta's examples are very elementary and applies to external cusps of power type with any power $\gamma > 1$ and any $1 < p < \infty$. We refer the reader to [14] where a particular case has been reproduced.

On the other hand, existence and regularity of solutions of elliptic equations in domains with external cusps have been considered by several authors. Let us mention for example [2, 12, 34, 35, 36, 37, 38]. However, as far as we know, there are no results for the Stokes equations on this kind of domains.

Therefore, it seems natural to ask whether the existence of solutions of (1.2) satisfying an estimate weaker than (1.3) can be proved for the above mentioned domains. Moreover, if this is the case, can these results be applied to show the well posedness of the Stokes problem in appropriate Hilbert spaces?

In this paper we give some partial answers to these questions in the particular case of planar domains. We consider simply connected planar Hölder α domains Ω, with $0 < \alpha \le 1$, i. e., $\partial\Omega$ is locally the graph of a Hölder α function. For these domains we prove the existence of solutions of (1.2) satisfying weaker estimates than (1.4) involving weighted norms where the weights are powers of the distance to the boundary. For general Hölder α domains the zero boundary condition will be imposed in a weak way. Afterwards, in some particular examples, we will show that this weak boundary condition agrees with the usual one.

As an application we will prove the well posedness, in appropriate weighted norms, of the Stokes equations for some cuspidal domains. As a consequence we will obtain in particular that, for this kind of domains, given $\mathbf{f} \in H^{-1}(\Omega)^2$ there exists a unique solution $(\mathbf{u}, p) \in$

$H_0^1(\Omega)^2 \times L_0^r(\Omega)$ of the Stokes equations (1.1) satisfying

$$\|\mathbf{v}\|_{H_0^1(\Omega)} + \|p\|_{L^r(\Omega)} \leq C\|\mathbf{f}\|_{H^{-1}(\Omega)}$$

for some $1 < r \leq 2$ which depends on the power of the cusp.

Our approach use some of the ideas of the papers [1, 24]. The existence of solutions of the divergence is derived from appropriate Korn type inequalities. The weighted Korn inequalities that we need are slight variants of those obtained in [1] but we include the proofs for the sake of completeness.

Although our arguments to prove the existence of solutions of the divergence are two dimensional, we write the proofs of the Korn type inequalities in the general n-dimensional case because they have interest in themselves.

The rest of the paper is organized as follows. In Section 2 we introduce some notations and prove the weighted Korn inequalities. Section 3 deals with our main results concerning the existence of solutions of the divergence in appropriate weighted Sobolev spaces for Hölder α domains. In Section 4 we apply the results of the previous section for the particular case of domains having power type external cusps. We show that in this case our weak zero boundary condition agrees with the usual one. Also in this section we prove optimality of our results. In Section 5 we show how our results can be applied to prove the well posedness of the Stokes equations in appropriate Hilbert spaces.

2. Preliminaries and Korn type inequalities

Let Ω be a bounded open subset of $\mathbb{R}^n$ and $d(x)$ the distance of $x \in \Omega$ to the boundary $\partial\Omega$. We will denote by $L^p(\Omega, \gamma)$ the Banach space given by the norm

$$\|u\|_{L^p(\Omega,\gamma)} := \|u\,d^\gamma\|_{L^p(\Omega)}$$

and, analogously, $W^{1,p}(\Omega, \gamma)$ will be the Banach space with norm

$$\|u\|_{W^{1,p}(\Omega,\gamma)} := \|u\,d^\gamma\|_{L^p(\Omega)} + \|\nabla u\,d^\gamma\|_{L^p(\Omega)}. \tag{2.1}$$

Whenever $L^p(\Omega, \gamma) \subset L^1(\Omega)$ we will call $L_0^p(\Omega, \gamma)$ the subspace of $L^p(\Omega, \gamma)$ formed by functions of vanishing mean value. Since no confusion is possible we will use the same notations for the norms of vector or tensor fields.

For a vector field $\mathbf{u} = (u_1, \cdots, u_n)$ defined in Ω we denote by $D\mathbf{u}$ the jacobian matrix, namely, $(D\mathbf{u})_{ij} = \frac{\partial u_i}{\partial x_j}$ and by $\varepsilon(\mathbf{u})$ its symmetric part (i.e., the linear strain tensor associated with $\mathbf{u}$), that is $\varepsilon(\mathbf{u})_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$.

We start by giving a weighted Korn inequality for Hölder α domains. The statement given in the following theorem is slightly stronger than the result in Theorem 3.1 of [1]. Therefore, we include the proof for the sake of completeness although the arguments are essentially those given in that reference. In particular we will make use of the following improved Poincaré inequality proved in [1, Theorem 2.1]. If Ω is a Hölder α domain, $0 < \alpha \leq 1$, $B \subset \Omega$ a ball and $\phi \in C_0^\infty(B)$ is such that $\int_B \phi = 1$ then, for $\alpha \leq \beta \leq 1$ and f such that $\int_B f\phi = 0$ there exists a constant C depending only on Ω, B and ϕ such that,

$$\|f\|_{L^p(\Omega,1-\beta)} \leq C\,\|\nabla f\|_{L^p(\Omega,1+\alpha-\beta)}. \tag{2.2}$$

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a Hölder α domain, $B \subset \Omega$ a ball and $1 < p < \infty$. Then, for $\alpha \leq \beta \leq 1$ the following inequality holds,*

$$\|D\mathbf{u}\|_{L^p(\Omega,1-\beta)} \leq C\Big\{\|\varepsilon(\mathbf{u})\|_{L^p(\Omega,\alpha-\beta)} + \|\mathbf{u}\|_{L^p(B)}\Big\}$$

where the constant C depends only on Ω, B and p.

Proof. Following [31], we can show that there exists $\mathbf{v} \in W^{1,p}(\Omega)^n$ such that

$$\Delta \mathbf{v} = \Delta \mathbf{u} \qquad \text{in } \Omega \tag{2.3}$$

and

$$\|\mathbf{v}\|_{W^{1,p}(\Omega)} \leq C \|\varepsilon(\mathbf{u})\|_{L^p(\Omega)}. \tag{2.4}$$

Now, let $\phi \in C_0^\infty(B)$ be such that $\int_B \phi \, dx = 1$. For $i = 1, ..., n$ define the linear functions

$$L_i(x) := \left(\int_B \nabla(u_i - v_i)\phi \right) \cdot x$$

and $\mathbf{L}(x)$ as the vector with components $L_i(x)$.

Then,

$$D\mathbf{L} = \int_B D(\mathbf{u} - \mathbf{v})\phi$$

and, integrating by parts and applying the Hölder inequality we obtain

$$|D\mathbf{L}| \leq \|\mathbf{u} - \mathbf{v}\|_{L^p(B)} \|\nabla\phi\|_{L^{p'}(B)}$$

where p' is the dual exponent of p.

Therefore, it follows from (2.4) that there exists a constant C depending only on Ω, p and ϕ such that

$$\|D\mathbf{L}\|_{L^p(\Omega)} \leq C \left\{ \|\mathbf{u}\|_{L^p(B)} + \|\varepsilon(\mathbf{u})\|_{L^p(\Omega)} \right\}. \tag{2.5}$$

Let us now introduce

$$\mathbf{w} := \mathbf{u} - \mathbf{v} - \mathbf{L}.$$

Then, in view of the bounds (2.4) and (2.5), it only remains to estimate $\mathbf{w}$. But, from (2.3) and the fact that $\mathbf{L}$ is linear we know that

$$\Delta \mathbf{w} = 0$$

and consequently,

$$\Delta \varepsilon_{ij}(\mathbf{w}) = 0.$$

But, if f is a harmonic function in Ω, the following estimate holds

$$\|\nabla f\|_{L^p(\Omega, 1-\mu)} \leq C \|f\|_{L^p(\Omega, -\mu)}$$

for all $\mu \in \mathbb{R}$. Indeed, this estimate was proved in [13] (see also Lema 3.1 in [1], and [31] for a different proof in the case $p = 2$ and $\mu = 0$).

Therefore, taking $\mu = \beta - \alpha$ we obtain

$$\|\nabla \varepsilon_{ij}(\mathbf{w})\|_{L^p(\Omega, 1+\alpha-\beta)} \leq C \|\varepsilon_{ij}(\mathbf{w})\|_{L^p(\Omega, \alpha-\beta)}$$

and using the well known identity

$$\frac{\partial^2 w_i}{\partial x_j \partial x_k} = \frac{\partial \varepsilon_{ik}(\mathbf{w})}{\partial x_j} + \frac{\partial \varepsilon_{ij}(\mathbf{w})}{\partial x_k} - \frac{\partial \varepsilon_{jk}(\mathbf{w})}{\partial x_i}$$

we conclude that

$$\left\| \frac{\partial^2 w_i}{\partial x_j \partial x_k} \right\|_{L^p(\Omega, 1+\alpha-\beta)} \leq C \|\varepsilon(\mathbf{w})\|_{L^p(\Omega, \alpha-\beta)} \tag{2.6}$$

for any i, j and k.

Since $\int \frac{\partial w_i}{\partial x_j} \phi = 0$ (indeed, we have defined $\mathbf{L}$ in order to have this property), it follows from the improved Poincaré inequality (2.2) that

$$\left\| \frac{\partial w_i}{\partial x_j} \right\|_{L^p(\Omega, 1-\beta)} \leq C \left\| \nabla \frac{\partial w_i}{\partial x_j} \right\|_{L^p(\Omega, 1+\alpha-\beta)}.$$

Therefore, using (2.6), we obtain

$$\|D\mathbf{w}\|_{L^p(\Omega,1-\beta)} \leq C\|\varepsilon(\mathbf{w})\|_{L^p(\Omega,\alpha-\beta)} \leq C\|\varepsilon(\mathbf{u})\|_{L^p(\Omega,\alpha-\beta)}$$

concluding the proof. $\qquad\square$

In the following corollary we give a weighted Korn inequality for Hölder α domains which can be seen as a generalization of the so-called second case of Korn inequality. To state this inequality we need to introduce the space of infinitesimal rigid motions, namely,

$$\mathcal{N} = \{\mathbf{v} \in W^{1,p}(\Omega)^n : \varepsilon(\mathbf{v}) = 0\}.$$

Corollary 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a Hölder α domain and $1 < p < \infty$. Then, for $\alpha \leq \beta \leq 1$ the following inequality holds,*

$$\inf_{\mathbf{v}\in\mathcal{N}} \|\mathbf{u} - \mathbf{v}\|_{W^{1,p}(\Omega,1-\beta)} \leq C\|\varepsilon(\mathbf{u})\|_{L^p(\Omega,\alpha-\beta)}. \tag{2.7}$$

Proof. Take B and ϕ as in the previous theorem with $\overline{B} \subset \Omega$. Define $\overline{x}_i = \int_B x_i\phi(x)\,dx$ and $\mathbf{v} \in W^{1,p}(\Omega)^n$ defined by

$$v_i(x) = a_i + \sum_{j=1}^{n} b_{ij}(x_j - \overline{x}_j)$$

with

$$a_i = \int_B u_i\phi \qquad \text{and} \qquad b_{ij} = \frac{1}{2|B|}\int_B \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right).$$

It is easy to check that $\mathbf{v} \in \mathcal{N}$. Now, since $\int_B(\mathbf{u} - \mathbf{v})\phi = 0$, it follows from (2.2) (actually we are using only a weaker standard Poincaré inequality with weights) and Theorem 2.1 that

$$\|\mathbf{u} - \mathbf{v}\|_{W^{1,p}(\Omega,1-\beta)} \leq C\Big\{\|\varepsilon(\mathbf{u} - \mathbf{v})\|_{L^p(\Omega,\alpha-\beta)} + \|\mathbf{u} - \mathbf{v}\|_{L^p(B)}\Big\}$$

and using now the Poincaré inequality in B we have

$$\|\mathbf{u} - \mathbf{v}\|_{W^{1,p}(\Omega,1-\beta)} \leq C\Big\{\|\varepsilon(\mathbf{u} - \mathbf{v})\|_{L^p(\Omega,\alpha-\beta)} + \|D(\mathbf{u} - \mathbf{v})\|_{L^p(B)}\Big\}. \tag{2.8}$$

But,

$$\int_B \left(\frac{\partial(u-v)_i}{\partial x_j} - \frac{\partial(u-v)_j}{\partial x_i}\right) = 0$$

and therefore, the so-called second case of Korn inequality applied in B gives

$$\|D(\mathbf{u} - \mathbf{v})\|_{L^p(B)} \leq C\|\varepsilon(\mathbf{u} - \mathbf{v})\|_{L^p(B)}.$$

Using this inequality in (2.8) and that $\varepsilon(\mathbf{v}) = 0$ we obtain

$$\|\mathbf{u} - \mathbf{v}\|_{W^{1,p}(\Omega,1-\beta)} \leq C\Big\{\|\varepsilon(\mathbf{u})\|_{L^p(\Omega,\alpha-\beta)} + \|\varepsilon(\mathbf{u})\|_{L^p(B)}\Big\}$$

which implies (2.7) because $\overline{B} \subset \Omega$. $\qquad\square$

Remark 2.1. *It is possible to prove the above corollary directly, i.e., without using the Korn inequality in the ball B, by using a standard compactness argument. Indeed, assuming that (2.7) does not hold and using that $W^{1,p}(\Omega, 1 - \beta)$ is compactly embedded in $L^p(\Omega, \gamma)$ for any $\gamma > (1 - \beta - \alpha)/\alpha$ (see [33, Theorem 19.11]) and Theorem 2.1 one obtains a contradiction.*

3. Solutions of the divergence in Hölder α domains

This section deals with solutions of divergence in planar simply connected Hölder α domains. In what follows we restrict ourselves to the case $n = 2$.

For regular enough bounded domains Ω (for example Lipschitz) it is known that, if $f \in L_0^p(\Omega)$, $1 < p < \infty$, there exists $\mathbf{u} \in W_0^{1,p}(\Omega)^2$ such that

$$div\, \mathbf{u} = f \tag{3.1}$$

and

$$\|\mathbf{u}\|_{W_0^{1,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)} \tag{3.2}$$

where the constant C depends only on Ω and p.

On the other hand, as we have mentioned in the introduction, it is known that for general Hölder α domains this result is not valid. Our main goal is to prove a similar result for this kind of domains but using weighted norms.

We will use the following notation. For a scalar function ψ we write $\mathbf{curl}\,\psi = (\frac{\partial \psi}{\partial x_2}, -\frac{\partial \psi}{\partial x_1})$ and for a vector field $\Psi = (\psi_1, \psi_2)$, $Curl\,\Psi$ denotes the matrix which has $curl\,\psi_i$ as it rows. Furthermore, if $\sigma \in L^p(\Omega)^{2\times 2}$, $Div\,\sigma$ denotes the vector field with components obtained by taking the divergence of the rows of σ.

We will impose the boundary condition in a weak form. To explain this weak condition observe first that to solve the problem it is enough to find a solution $\mathbf{u}$ of (3.1) such that the restriction to $\partial\Omega$ of both components of $\mathbf{u}$ are constant (whenever the domain is such that this restriction makes sense). Of course, we should replace the estimate (3.2) by

$$\|D\mathbf{u}\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}. \tag{3.3}$$

Afterwards, (3.2) would follow by applying the Poincaré inequality to the solution obtained by adding an appropriate constant vector field to $\mathbf{u}$ in order to obtain the vanishing boundary condition.

Now, assume that Ω is a Lipschitz domain. Then, if $\psi \in W^{1,p}(\Omega)$ satisfies

$$\int_\Omega \mathbf{curl}\,\psi \cdot \nabla\phi = 0 \qquad \forall\phi \in W^{1,p'}(\Omega) \tag{3.4}$$

it follows by integration by parts that

$$\int_{\partial\Omega} \frac{\partial\psi}{\partial t}\phi = 0 \qquad \forall\phi \in W^{1,p'}(\Omega) \tag{3.5}$$

where $\frac{\partial\psi}{\partial t}$ indicates the tangential derivative of ψ. Therefore $\frac{\partial\psi}{\partial t} = 0$ and then the restriction of ψ to $\partial\Omega$ is constant.

For a general domain Ω the tangential derivative on the boundary might not even be defined and therefore (3.5) would not make sense. However, condition (3.4) is well defined in any domain and this is the condition that we will use. Therefore we introduce the space

$$W_{const}^{1,p}(\Omega) \subset W^{1,p}(\Omega)$$

defined by

$$W_{const}^{1,p}(\Omega) = \left\{ \psi \in W^{1,p}(\Omega) : \int_\Omega \mathbf{curl}\,\psi \cdot \nabla\phi = 0 \qquad \forall\phi \in W^{1,p'}(\Omega) \right\}$$

and more generally, for any $\gamma \in \mathbb{R}$,

$$W_{const}^{1,p}(\Omega, \gamma) = \left\{ \psi \in W^{1,p}(\Omega, \gamma) : \int_\Omega \mathbf{curl}\,\psi \cdot \nabla\phi = 0 \qquad \forall\phi \in W^{1,p'}(\Omega, -\gamma) \right\}.$$

The proof of the following lemma uses ideas introduced in [24] with different goals.

For $1 < p < \infty$ and $\gamma \in \mathbb{R}$, $L_{sym}^p(\Omega, \gamma)^{2\times 2}$ denotes the subspace of symmetric tensors in $L^p(\Omega, \gamma)^{2\times 2}$.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a Hölder α domain and $\mathbf{u} \in W^{1,p}(\Omega, \beta - 1)^2$, with $\alpha \leq \beta \leq 1$, such that $\int_\Omega div\, \mathbf{u} = 0$. Then, there exists $\sigma \in L^p_{sym}(\Omega, \beta - \alpha)^{2\times 2}$ satisfying*

$$\int_\Omega \sigma : D\, \mathbf{w} = \int_\Omega Curl\, \mathbf{u} : D\mathbf{w}, \qquad \forall \mathbf{w} \in W^{1,p'}(\Omega, \alpha - \beta)^2$$

and

$$\|\sigma\|_{L^p(\Omega, \beta - \alpha)^{2\times 2}} \leq C \|Curl\, \mathbf{u}\|_{L^p(\Omega, \beta - 1)^{2\times 2}}.$$

Proof. Let $H \subset L^{p'}_{sym}(\Omega, \alpha - \beta)^{2\times 2}$ the subspace defined as

$$H = \{\tau \in L^{p'}_{sym}(\Omega, \alpha - \beta)^{2\times 2} : \tau = \varepsilon(\mathbf{w}) \text{ with } \mathbf{w} \in W^{1,p'}(\Omega, \alpha - \beta)^2\}.$$

Let us see that the application

$$T : \varepsilon(\mathbf{w}) \mapsto \int_\Omega Curl\, \mathbf{u} : D\mathbf{w} \tag{3.6}$$

defines a continuous linear functional on H.

First of all observe that T is well defined. Indeed, it is enough to check that the expression on the right of (3.6) vanishes whenever $\varepsilon(\mathbf{w}) = 0$. But, it is known that in that case $\mathbf{w}(x, y) = (a - cy, b + cx)$ and therefore

$$\int_\Omega Curl\, \mathbf{u} : D\mathbf{w} = c \int_\Omega div\, \mathbf{u} = 0.$$

Now, we want to show that T is continuous on H. Using again that $\int_\Omega Curl\, \mathbf{u} : D\mathbf{v} = 0$ if $\varepsilon(\mathbf{v}) = 0$ and applying Corollary 2.1 we have, for $\tau = \varepsilon(\mathbf{w}) \in H$,

$$
\begin{aligned}
|T(\tau)| &= \left| \int_\Omega Curl\, \mathbf{u} : D\mathbf{w} \right| \\
&\leq \|Curl\, \mathbf{u}\|_{L^p(\Omega, \beta - 1)^{2\times 2}} \inf_{\mathbf{v} \in \mathcal{N}} \|D(\mathbf{w} - \mathbf{v})\|_{L^{p'}(\Omega, 1 - \beta)^{2\times 2}} \\
&\leq C \|Curl\, \mathbf{u}\|_{L^p(\Omega, \beta - 1)^{2\times 2}} \|\varepsilon(\mathbf{w})\|_{L^{p'}(\Omega, \alpha - \beta)^{2\times 2}} \\
&= C \|Curl\, \mathbf{u}\|_{L^p(\Omega, \beta - 1)^{2\times 2}} \|\tau\|_{L^{p'}(\Omega, \alpha - \beta)^{2\times 2}}.
\end{aligned}
$$

By the Hahn-Banach theorem the functional T can be extended to $L^{p'}_{sym}(\Omega, \alpha - \beta)^{2\times 2}$ and therefore, by the Riesz representation theorem, there exists $\sigma \in L^p_{sym}(\Omega, \beta - \alpha)^{2\times 2}$ such that

$$T(\tau) = \int_\Omega \sigma : \tau \qquad \forall \tau \in L^{p'}_{sym}(\Omega, \alpha - \beta)^{2\times 2}$$

and

$$\|\sigma\|_{L^p(\Omega, \beta - \alpha)^{2\times 2}} \leq C \|Curl\, \mathbf{u}\|_{L^p(\Omega, \beta - 1)^{2\times 2}},$$

where C depends on the constant in Corollary 2.1. In particular,

$$\int_\Omega \sigma : \varepsilon(\mathbf{w}) = \int_\Omega Curl\, \mathbf{u} : D\mathbf{w} \tag{3.7}$$

for every $\mathbf{w} \in W^{1,p'}(\Omega, \alpha - \beta)^2$. Then, we conclude the proof observing that, since σ is symmetric, we can replace $\varepsilon(\mathbf{w})$ in (3.7) by $D\mathbf{w}$. $\qquad \square$

It is a very well known result that a divergence free vector field is a rotational of a scalar function ϕ. Indeed, for smooth vector fields the proof is usually given at elementary courses on calculus in several variables. On the other hand, if the vector field is only in $L^p(\Omega)^2$ but $\partial\Omega$ is Lipstchiz, it is not difficult to see that the vector field can be extended to a divergence free vector field defined in $\mathbb{R}^2$ and then, the existence of ϕ can be proved by using the Fourier transform. However, we need to use the existence of ϕ in the case where the domain and the

vector field are both non-smooth. We have not been able to find a proof of this result in the literature and so we include the following lemma.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected Hölder α domain and $\alpha \leq \beta \leq 1$. Given a vector field $\mathbf{v} \in L^p(\Omega, 1 - \beta)^2$ such that $div\,\mathbf{v} = 0$, there exists $\phi \in W^{1,p}(\Omega, 1 - \beta)$ such that*

$$\mathbf{curl}\,\phi = \mathbf{v} \qquad \text{and} \qquad \|\phi\|_{W^{1,p}(\Omega, 1-\beta)} \leq C\|\mathbf{v}\|_{L^p(\Omega, 1-\beta)}$$

where C is a constant depending only on Ω.

Proof. Take $\psi \in C_0^\infty(B_1)$ satisfying $\int \psi = 1$, where B_1 is the unit ball centered at the origin. For $k \geq 1$, define $\psi_k(x) = k^2\psi(kx)$ and, extending $\mathbf{v}$ by zero to $\mathbb{R}^2$, $\mathbf{v}_k = \psi_k * \mathbf{v}$.

Let Ω_n be a sequence of Lipschitz simply connected open subsets of Ω such that

$$\overline{\Omega}_n \subset \left\{ x \in \Omega : d(x) > \frac{1}{n} \right\} \qquad \text{and} \qquad \Omega_n \nearrow \Omega.$$

Using that the distance between Ω_n and $\partial\Omega$ is greater than $1/n$ and $\operatorname{supp}\psi_k \subset B(0, \frac{1}{k})$, it is not difficult to see that $\operatorname{div}\mathbf{v}_k = 0$ in Ω_n for every $k \geq n$.

Then, since $\mathbf{v}_n \in C_0^\infty(\mathbb{R}^2)^2$, there exists $\phi_n \in C_0^\infty(\Omega_n)$ such that $\mathbf{curl}\,\phi_n = \mathbf{v}_n$. Moreover, adding a constant we can take ϕ_n such that $\int_{\Omega_1}\phi_n = 0$.

Now, by the Poincaré inequality we have, for any n, there exists a constant C depending only on n such that

$$\|\phi_k - \phi_{k'}\|_{L^p(\Omega_n)} \leq C\|\mathbf{curl}\,(\phi_k - \phi_{k'})\|_{L^p(\Omega_n)} = C\|\mathbf{v}_k - \mathbf{v}_{k'}\|_{L^p(\Omega_n)} \to 0$$

for $k, k' \to \infty$.

Then, there exists $\phi \in L_{loc}^1(\Omega)$ such that $\phi_k|_{\Omega_n} \to \phi$ in $W^{1,p}(\Omega_n)$ and so $\mathbf{curl}\,\phi = \mathbf{v}$ in $\Omega_n, \forall n$ and consequently in Ω.

Finally, using Theorem 2.1 of [1] we have

$$\|\phi\|_{L^p(\Omega, 1-\beta)} \leq C\|\mathbf{curl}\,\phi\|_{L^p(\Omega, 1-\beta+\alpha)} \leq C\|\mathbf{v}\|_{L^p(\Omega, 1-\beta)}$$

and the Lemma is proved. $\qquad\qquad\square$

We can now state and prove our results on solutions of the divergence on Hölder-α domains. As we mentioned above, it is known that for this kind of domains a solution of (3.1) satisfying (3.2) does not exist in general. Therefore, it is natural to look for solutions of (3.1) satisfying a weaker estimate. There are two possibilities: to use a stronger norm on the right of (3.2) or a weaker norm on the left. We will prove both kind of results but, to avoid technical complications while presenting the arguments, we give first a particular case of our results and postpone the generalization.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected Hölder-α domain, $0 < \alpha \leq 1$. Given $f \in L_0^p(\Omega)$, $1 < p < \infty$, there exists $\mathbf{u} \in W_{const}^{1,p}(\Omega, 1 - \alpha)^2$ such that*

$$div\,\mathbf{u} = f$$

and

$$\|D\mathbf{u}\|_{L^p(\Omega, 1-\alpha)} \leq C\|f\|_{L^p(\Omega)} \tag{3.8}$$

Proof. Take $\mathbf{v} \in W^{1,p}(\Omega)^2$ such that

$$\operatorname{div}\mathbf{v} = f \tag{3.9}$$

and

$$\|\mathbf{v}\|_{W^{1,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}. \tag{3.10}$$

The existence of such a $\mathbf{v}$ is well known, for example, since no boundary condition on $\mathbf{v}$ is required, we can extend f by zero and take the solution of problem (3.1) and (3.2) in a ball containing Ω.

To prove the theorem it is enough to show that there exists $\mathbf{w} \in W^{1,p}(\Omega, 1-\alpha)^2$ satisfying $\mathrm{div}\,\mathbf{w} = 0$ and such that

$$\mathbf{v} - \mathbf{w} \in W_{const}^{1,p}(\Omega, 1-\alpha)^2$$

and

$$\|D\mathbf{w}\|_{L^p(\Omega, 1-\alpha)} \leq C\|f\|_{L^p(\Omega)}. \tag{3.11}$$

Indeed, in view of (3.9), $\mathbf{u} := \mathbf{v} - \mathbf{w}$ will be the desired solution.

But, since $\mathrm{div}\,\mathbf{v}$ has vanishing mean value, we know from Lemma 3.1 that there exists $\sigma \in L^p_{sym}(\Omega, 1-\alpha)^{2 \times 2}$ satisfying

$$\|\sigma\|_{L^p(\Omega, 1-\alpha)} \leq C\,\|Curl\,\mathbf{v}\|_{L^p(\Omega)} \tag{3.12}$$

and

$$\int_\Omega \sigma : D\mathbf{r} = \int_\Omega Curl\,\mathbf{v} : D\mathbf{r} \quad , \quad \forall \mathbf{r} \in W^{1,p'}(\Omega, \alpha - 1)^2.$$

Then,

$$\int_\Omega Div\,\sigma \cdot \mathbf{r} = -\int_\Omega \sigma : D\mathbf{r} = -\int_\Omega Curl\,\mathbf{v} : D\mathbf{r} = \int_\Omega Div\,Curl\,\mathbf{v} \cdot \mathbf{r} = 0$$

for every $\mathbf{r} \in C_0^\infty(\Omega)^2$ and therefore $Div\,\sigma = 0$.

Now, from Lemma 3.2 we know that there exists $\mathbf{w} \in W^{1,p}(\Omega, 1-\alpha)^2$ such that

$$Curl\,\mathbf{w} = \sigma \qquad \text{and} \qquad \|\mathbf{w}\|_{W^{1,p}(\Omega, 1-\alpha)} \leq C\|\sigma\|_{L^p(\Omega, 1-\alpha)}. \tag{3.13}$$

We have to check that $\mathrm{div}\,\mathbf{w} = 0$, but since σ is a symmetric tensor we have

$$\mathrm{div}\,\mathbf{w} = \frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2} = -\sigma_{12} + \sigma_{21} = 0.$$

To conclude the proof observe that in view of (3.10), (3.12) and (3.13) we have (3.11) which together with (3.10) yields (3.8). $\qquad\square$

Now, it is natural to ask whether part or all the weight in the estimate (3.8) can be moved to the right hand side. We will give a positive answer to this question. As we will show, the proof of this more general result is similar to that of Theorem 3.2 but it requires some non-trivial preliminary results. In particular, we will need an extra hypothesis on the domain.

We are going to use that some singular integral operators are continuous in weighted L^p-norms, $1 < p < \infty$, for weights in the Muckenhoupt class A_p. This well known result as well as the definition of the A_p classes can be seen for example in the book [40].

In what follows we consider the distance to $\partial\Omega$, $d(x)$, defined for every $x \in \mathbb{R}^n$ and not only for $x \in \Omega$. We will give sufficient conditions on $\partial\Omega$ and on the exponent μ such that d^μ belongs to A_p. We state and prove the lemma in the more general situation of the distance to a compact set F contained in $\mathbb{R}^n$ since this result can be of interest in other situations and its proof does not require any extra effort.

The following lemma generalizes the result proved in [18] for smooth domains. Since the proof is too technical we postpone it for an appendix and continue now with our main results.

Given a compact set $F \subset \mathbb{R}^n$, we will denote with $d_F(x)$ the distance from x to F.

Definition 3.1. *For $0 \leq m \leq n$, a compact set $F \subset \mathbb{R}^n$ is an **m-regular set**, if there exists a positive constant C such that*

$$C^{-1}r^m < \mathcal{H}^m(B(x,r) \cap F) < Cr^m,$$

for every $x \in F$ and $0 < r \leq diam\,F$, where $\mathcal{H}^m$ is the m-dimensional Hausdorff measure and $B(x,r)$ is the ball with radius r and center x. The restriction $0 < r \leq diam\,F$ is eliminated if F is a set with only one point.

The reader who is not familiar with Hausdorff measure can think in the particular case that F is a rectifiable curve in $\mathbb{R}^2$ and $m = 1$ (this will be the case in the example of application to Stokes that we will give). In that case $\mathcal{H}^1$ is the length.

Lemma 3.3. *Let $F \subset \mathbb{R}^n$ be a compact set included in a **m-regular set** K. If*

$$-(n-m) < \mu < (n-m)(p-1),$$

then d_F^μ belongs to the class A_p.

Proof. See Apendix. $\qquad\square$

As a consequence we have the following result on weighted estimates for solutions of the divergence.

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain such that its boundary is contained in a **1-regular set**. Given $f \in L^p(\Omega, \gamma)$, $1 < p < \infty$, with $-1/p < \gamma \le 1 - 1/p$ there exists $\mathbf{v} \in W^{1,p}(\Omega, \gamma)^2$ such that*

$$div\,\mathbf{v} = f$$

and

$$\|\mathbf{v}\|_{W^{1,p}(\Omega,\gamma)^2} \le C\|f\|_{L^p(\Omega,\gamma)}$$

Proof. Extend f by zero to $\mathbb{R}^2$. Then, it is well known that

$$\phi(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^n} \log |x - y|\, f(y)\, dy$$

is a solution of $\Delta\phi = f$. Moreover, it follows from the theory of singular integral operators (see for example [40]) that, if $w \in A_p$,

$$\int_{\mathbb{R}^2} \left| \frac{\partial^2 \phi(x)}{\partial x_i \partial x_j} \right|^p w(x)\, dx \le \int_{\mathbb{R}^2} |f(x)|^p\, w(x)\, dx.$$

But, since $\mu = \gamma p$ satisfies the hypothesis of Lemma 3.3 with $n = 2$ and $m = 1$, $d^\mu \in A_p$ and therefore $\mathbf{v} := \nabla\phi$ is the desired solution. $\qquad\square$

We can now give our more general result on solutions of the divergence.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain such that its boundary is contained in a **1-regular set**. Given $f \in L_0^p(\Omega, \beta - 1)$, $1 < p < \infty$, if $\alpha \le \beta \le 1$ and $-1/p < \beta - 1$, there exists $\mathbf{u} \in W_{const}^{1,p}(\Omega, \beta - \alpha)^2$ such that*

$$div\,\mathbf{u} = f$$

and

$$\|D\mathbf{u}\|_{L^p(\Omega,\beta-\alpha)} \le C\|f\|_{L^p(\Omega,\beta-1)} \tag{3.14}$$

Proof. Since $-1/p < \beta - 1$, it follows from Lemma 3.4 that there exists $\mathbf{v} \in W^{1,p}(\Omega, \beta - 1)^2$ such that

$$\mathrm{div}\,\mathbf{v} = f \tag{3.15}$$

and

$$\|\mathbf{v}\|_{W^{1,p}(\Omega,\beta-1)} \le C\|\mathbf{v}\|_{W^{1,p}(\Omega,\beta-\alpha)} \le C\|f\|_{L^p(\Omega,\beta-1)}. \tag{3.16}$$

The rest of the proof follows as that of Theorem 3.1. Now we have to show that there exists $\mathbf{w} \in W^{1,p}(\Omega, \beta - \alpha)^2$ satisfying $\mathrm{div}\,\mathbf{w} = 0$ and such that

$$\mathbf{v} - \mathbf{w} \in W_{const}^{1,p}(\Omega, \beta - \alpha)^2$$

and

$$\|D\mathbf{w}\|_{L^p(\Omega,\beta-\alpha)} \le C\|f\|_{L^p(\Omega,\beta-1)}.$$

The reader can easily check that the existence of $\mathbf{w}$ follows by using Lemma 3.1 as in Theorem 3.2. $\qquad\square$

4. Domains with external cusps

In this section we consider the particular case of the Hölder-α domain defined as

$$\Omega = \left\{ (x,y) \in \mathbb{R}^2 \, : \, 0 < x < 1 \, , \, 0 < |y| < x^{1/\alpha} \right\} \tag{4.1}$$

with $0 < \alpha \leq 1$.

We are going to show that in this case the weaker boundary condition imposed in Theorem 3.2 is equivalent to the standard one, i.e., that the solution of the divergence obtained in that theorem can be modified, by adding a constant vector field, to obtain a solution which vanishes on the boundary in the classic sense.

We will consider the particular case $\beta = \alpha$ of our general Theorem 3.2. Extension of the arguments to other cases might be possible but it is not straightforward.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^2$ be the domain defined in (4.1) and $1 < p < \infty$. If $1 - 1/p < \alpha \leq 1$ then, given $f \in L_0^p(\Omega, \alpha - 1)$ there exists $\mathbf{u} \in W_0^{1,p}(\Omega)^2$ such that*

$$div\, \mathbf{u} = f \tag{4.2}$$

and

$$\|\mathbf{u}\|_{W_0^{1,p}(\Omega)} \leq C\|f\|_{L^p(\Omega, \alpha-1)} \tag{4.3}$$

with a constant depending only on Ω, p and α.

Proof. It is easy to see that Ω satisfies the hypotheses of Theorem 3.2. Therefore, it follows from that theorem that there exists $\mathbf{u} \in W_{const}^{1,p}(\Omega)^2$ which verifies (4.2).

We are going to prove that, for any $\psi \in W_{const}^{1,p}(\Omega)$, there exists a constant $\psi_0 \in \mathbb{R}$ such that

$$\psi - \psi_0 \in W_0^{1,p}(\Omega) := \overline{C_0^\infty(\Omega)}.$$

Consequently, $\mathbf{u}$ can be modified by adding a constant to each of its components to obtain the desired solution. Indeed, the estimate (4.3) will follow form (3.14) by the Poincaré inequality.

Given $\psi \in W_{const}^{1,p}(\Omega)$, let us show first that ψ is constant on $\partial\Omega$. From the definition of $W_{const}^{1,p}(\Omega)$ we have that

$$\int_\Omega \mathbf{curl}\,\psi \cdot \nabla\phi = 0 \qquad \forall \phi \in W^{1,p'}(\Omega).$$

Now, let (x_0, y_0) be a point in $\partial\Omega$ different from the origin and B an open ball centered in (x_0, y_0) such that $0 \notin B$. Taking $\phi \in C^\infty(B)$ we have

$$0 = \int_\Omega \mathbf{curl}\,\psi \cdot \nabla\phi = -\int_{B \cap \partial\Omega} \psi \frac{\partial\phi}{\partial t} \qquad \forall \phi \in C^\infty(B)$$

where $\frac{\partial\phi}{\partial t}$ indicates the tangential derivative of ϕ. Consequently $\frac{\partial\psi}{\partial t} = 0$ in the distributional sense on $B \cap \partial\Omega$ and then, since $\partial\Omega - (0,0)$ is a connected set, we conclude that there exists a constant ψ_0 such that $\psi = \psi_0$ on $\partial\Omega$. To simplify notation we assume in what follows that $\psi_0 = 0$ and so, we have to see that $\psi \in W_0^{1,p}(\Omega)$.

Now, let $\zeta \in C^\infty(\mathbb{R}_+)$ be such that

$$\zeta \equiv 1 \text{ in } [0,1] \qquad \zeta \equiv 0 \text{ in } \mathbb{R}_+ - (0,2) \qquad 0 \leq \zeta \leq 1.$$

We decompose ψ as

$$\psi(x,y) = \zeta(3x)\psi(x,y) + (1 - \zeta(3x))\,\psi(x,y) =: \psi_1 + \psi_2.$$

It is easy to see that $\psi_2 \in W_0^{1,p}(\Omega_2)$ where Ω_2 is the Lipschitz domain

$$\Omega_2 := \Omega \cap \left\{ x > \frac{1}{3} \right\}.$$

Thus, we can suppose that $\psi = \psi_1$. Let now $\phi_n \in C^\infty(\Omega)$ be a sequence satisfying $\phi_n \to \psi$ in $W^{1,p}(\Omega)$ and let $\gamma := 1/\alpha$.

It is easy to check that, for $y \in (0,1)$,

$$
|\phi_n(x, x^\gamma - y)| \le |\phi_n(x, x^\gamma)| + \int_0^y \left| \frac{\partial \phi_n}{\partial y}(x, x^\gamma - t) \right| dt.
$$

Therefore, integrating and using the Hölder inequality we have

$$
\int_{y^\alpha}^1 |\phi_n(x, x^\gamma - y)|^p \, dx \le C \left(\int_{y^\alpha}^1 |\phi_n(x, x^\gamma)|^p \, dx + y^{p-1} \int_{y^\alpha}^1 \int_0^y \left| \frac{\partial \phi_n}{\partial y}(x, x^\gamma - t) \right|^p dt \, dx \right).
$$

Thus, using the continuity of the trace in the Lipschitz domain $\Omega \cap \{x > y^\alpha\}$ we have

$$
\begin{aligned}
\int_{y^\alpha}^1 |\psi(x, x^\gamma - y)|^p \, dx &= \lim_{n \to \infty} \int_{y^\alpha}^1 |\phi_n(x, x^\gamma - y)|^p \, dx \\
&\le C \lim_{n \to \infty} \left(\int_{y^\alpha}^1 |\phi_n(x, x^\gamma)|^p \, dx + y^{p-1} \int_{y^\alpha}^1 \int_0^y \left| \frac{\partial \phi_n}{\partial y}(x, x^\gamma - t) \right|^p dt \, dx \right) \\
&= C \, y^{p-1} \int_{y^\alpha}^1 \int_0^y \left| \frac{\partial \psi}{\partial y}(x, x^\gamma - t) \right|^p dt \, dx. \tag{4.4}
\end{aligned}
$$

Now we will show that the sequence ψ_m defined by

$$
\psi_m(x, y) := \psi(x, y) \left(1 - \zeta_m(x^\gamma - |y|) \right),
$$

where $\zeta_m(t) := \zeta(mt)$, converges to ψ in $W^{1,p}(\Omega)$. Moreover, it is easy to see that $\operatorname{supp} \psi_m \subset \Omega$.

By symmetry we can assume that $\Omega = \Omega \cap \{y > 0\}$. Using the dominated convergence theorem we obtain

$$
\lim_{m \to \infty} \|\psi - \psi_m\|_{L^p(\Omega)}^p = \lim_{m \to \infty} \int_\Omega |\psi(x, y) \zeta_m(x^\gamma - y)|^p = 0.
$$

On the other hand,

$$
\frac{\partial \psi_m}{\partial x}(x, y) = \frac{\partial \psi}{\partial x}(x, y) \left(1 - \zeta_m(x^\gamma - y) \right) - m \, \psi(x, y) \, \zeta' \gamma x^{\gamma - 1}
$$

and then,

$$
\begin{aligned}
\int_\Omega \left| \frac{\partial \psi}{\partial x} - \frac{\partial \psi_m}{\partial x} \right|^p &\le \int_\Omega \left| \frac{\partial \psi}{\partial x}(x, y) \, \zeta_m(x^\gamma - y) \right|^p + C m^p \int_\Omega \left| \psi(x, y) \chi_{\{y > \psi(x) - 2/m\}} \right|^p \\
&=: \ I + II.
\end{aligned}
$$

Thus, using again dominated convergence, it is easy to check that $I \to 0$. So, it only remains to analyze II.

Now, by the change of variables defined by $(x,y) \longmapsto (x, x^\gamma - y)$ and using (4.4) it follows that

$$
\begin{aligned}
II &= C\,m^p \int_0^{2/m} \int_{y^\alpha}^1 |\psi(x, x^\gamma - y)|^p \, dx \, dy \\
&\leq C\,m^p \int_0^{2/m} y^{p-1} \int_{y^\alpha}^1 \int_0^y \left| \frac{\partial\psi}{\partial y}(x, x^\gamma - t) \right|^p dt \, dx \, dy \\
&\leq C\,m^p \int_0^{2/m} y^{p-1} \int_0^{2/m} \int_{t^\alpha}^1 \left| \frac{\partial\psi}{\partial y}(x, x^\gamma - t) \right|^p dx \, dt \, dy \\
&\leq C\,m^p \left(\frac{2}{m}\right)^p \int_0^{2/m} \int_{t^\alpha}^1 \left| \frac{\partial\psi}{\partial y}(x, x^\gamma - t) \right|^p dx \, dt \\
&\leq C \int_\Omega \left| \frac{\partial\psi}{\partial y}(x,y)\chi_{\{y>\psi(x)-2/m\}} \right|^p \longrightarrow 0
\end{aligned}
$$

An analogous argument can be applied to prove that $\frac{\partial\psi_m}{\partial y} \to \frac{\partial\psi}{\partial y}$ in $L^p(\Omega)$.

Consequently, we conclude the proof by observing that ψ_m belongs to $W_0^{1,p}(\Omega)$. $\qquad\square$

In the following theorem we show that the estimate (4.3) is optimal in the sense that it is not possible to improve the power of the distance in the right hand side. Recall that $p' = \frac{p}{p-1}$ is the dual exponent of p.

Theorem 4.2. *Let Ω be the domain defined in (4.1). If for some $\beta \leq 0$ and for any $f \in L_0^p(\Omega, \beta)$ there exists $\mathbf{u} \in W_0^{1,p}(\Omega)^2$ such that*

$$
div\,\mathbf{u} = f
$$

and

$$
\|\mathbf{u}\|_{W_0^{1,p}(\Omega)} \leq C\|f\|_{L^p(\Omega,\beta)}
$$

with a constant C independent of f then, $\beta \leq \alpha - 1$.

Proof. For $s < \frac{1-\beta p' + \alpha}{\alpha p'}$ define $f_s(x,y) = x^{-\frac{s}{p-1}} d(x,y)^{-p'\beta}$. Then, calling $\Omega_+ = \Omega \cap \{y > 0\}$, we have

$$
\|f_s\|_{L^p(\Omega,\beta)}^p = 2 \int_{\Omega_+} x^{-sp'} d(x,y)^{-\beta pp' + \beta p} \, dxdy = 2 \int_{\Omega_+} x^{-sp'} d(x,y)^{-\beta p'} \, dxdy
$$

and therefore, using that for $y > 0$, $d(x,y) \simeq x^{1/\alpha} - y$, we obtain

$$
\|f_s\|_{L^p(\Omega,\beta)}^p \simeq 2 \int_{\Omega_+} x^{-sp'} (x^{1/\alpha} - y)^{-\beta p'} \, dxdy
$$

but,

$$
\begin{aligned}
\int_{\Omega_+} x^{-sp'}(x^{1/\alpha} - y)^{-\beta p'} \, dxdy &= \int_0^1 \int_0^{x^{\frac{1}{\alpha}}} x^{-sp'}(x^{1/\alpha} - y)^{-\beta p'} \, dydx \\
&= \frac{1}{1 - \beta p'} \int_0^1 x^{-sp'} x^{(1-\beta p')/\alpha} \, dx = \frac{1}{1 - \beta p'} \frac{1}{p'(\frac{1-\beta p'+\alpha}{\alpha p'} - s)}
\end{aligned}
$$

where we have used $s < \frac{1-\beta p'+\alpha}{\alpha p'}$. Therefore,

$$
\|f_s\|_{L^p(\Omega,\beta)}^p \simeq \frac{1}{A - s} \tag{4.5}
$$

where $A := \frac{1-\beta p'+\alpha}{\alpha p'}$ and with constants in the equivalence independent of s.

Now, let B be a ball such that $\overline{B} \subset \Omega$ and $\omega \in C_0^\infty(B)$ such that $\int_B \omega = 1$. From our hypothesis we know that, if $c_s = \int_\Omega f_s$, there exists $\mathbf{v}_s \in W_0^{1,p}(\Omega)^2$ such that

$$\operatorname{div} \mathbf{v}_s = f_s - c_s\omega \qquad \text{and} \qquad \|\mathbf{v}_s\|_{W^{1,p}(\Omega)} \leq C\|f_s - c_s\omega\|_{L^p(\Omega,\beta)}.$$

But, since $\beta \leq 0$,

$$|c_s| = \|f_s\|_{L^1(\Omega)} \leq C\|f_s\|_{L^p(\Omega,\beta)} \tag{4.6}$$

and so,

$$\|\mathbf{v}_s\|_{W^{1,p}(\Omega)} \leq C\|f_s\|_{L^p(\Omega,\beta)} \tag{4.7}$$

where we have used that $\|\omega\|_{L^p(\Omega,\beta)} \leq C$ because the support of ω is contained in B. Then,

$$
\begin{aligned}
\|f_s\|_{L^p(\Omega,\beta)}^p &= \int_\Omega f_s^{p-1}\,(f_s - c_s\omega)\,d^{p\beta} + \int_\Omega f_s^{p-1}\,c_s\omega\,d^{p\beta} \\
&= \int_\Omega f_s^{p-1}\operatorname{div}\mathbf{v}_s\,d^{p\beta} + \int_\Omega f_s^{p-1}\,c_s\omega\,d^{p\beta} \\
&= \int_\Omega x^{-s}\operatorname{div}\mathbf{v}_s + \int_\Omega f_s^{p-1}\,c_s\omega\,d^{p\beta}.
\end{aligned}
$$

Using again that the support of ω is at a positive distance from the boundary, together with (4.6), it follows that

$$\int_\Omega f_s^{p-1}\,c_s\omega\,d^{p\beta} \leq C\|f_s\|_{L^p(\Omega,\beta)}.$$

On the other hand,

$$
\begin{aligned}
\int_\Omega x^{-s}\operatorname{div}\mathbf{v}_s &= s\int_\Omega x^{-s-1}\,\mathbf{v}_{s,1} = s\int_\Omega \frac{\partial(y\,x^{-s-1})}{\partial y}\mathbf{v}_{s,1} \\
&= -s\int_\Omega y\,x^{-s-1}\frac{\partial\mathbf{v}_{s,1}}{\partial y} \leq s\|y\,x^{-s-1}\|_{L^{p'}(\Omega)}\,\|\mathbf{v}_s\|_{W^{1,p}(\Omega)} \\
&\leq Cs\|y\,x^{-s-1}\|_{L^{p'}(\Omega)}\,\|f_s\|_{L^p(\Omega,\beta)}
\end{aligned}
$$

where for the last inequality we have used (4.7).

Therefore,

$$\|f_s\|_{L^p(\Omega,\beta)}^{p-1} \leq C\{s\|y\,x^{-s-1}\|_{L^{p'}(\Omega)} + 1\} \tag{4.8}$$

But, an elementary computation shows that

$$\|y\,x^{-s-1}\|_{L^{p'}(\Omega)}^{p'} \simeq \frac{1}{B-s} \tag{4.9}$$

where $B := \frac{1-(\alpha-1)p'+\alpha}{\alpha p'}$ and with constants in the equivalence independent of s.

Thus, from (4.5), (4.8) and (4.9) we conclude that there exists a constant independent of s such that

$$\frac{1}{A-s} \leq C\frac{1}{B-s}$$

therefore, $B \leq A$ and it follows immediately that $\beta \leq \alpha - 1$. $\qquad\square$

5. An application to the Stokes equations

In this section we show how our results can be applied to the analysis of the Stokes equations when Ω is the domain defined in (4.1).

We are going to use the well known theory developed by Brezzi (see for example [9, 15, 26]) but modifying the usual Hilbert spaces and the bilinear form corresponding to the divergence free restriction in the weak formulation of the Stokes equations.

Theorem 5.1. *Let Ω be the domain defined in (4.1) with $1/2 < \alpha \leq 1$. Then, if $\mathbf{f} \in H^{-1}(\Omega)^2$, there exists a unique weak solution $(u, p) \in H_0^1(\Omega)^2 \times L_0^2(\Omega, 1 - \alpha)$ of the Stokes equations (1.1). Moreover, there exists a constant C depending only on α such that*

$$\|\mathbf{v}\|_{H_0^1(\Omega)} + \|p\|_{L^2(\Omega, 1-\alpha)} \leq C\|\mathbf{f}\|_{H^{-1}(\Omega)}. \tag{5.1}$$

Proof. Let us introduce the spaces

$$V = \left\{ \mathbf{v} \in H_0^1(\Omega)^2 \, : \, \operatorname{div} \mathbf{v} \in L^2(\Omega, \alpha - 1) \right\}$$

which is a Hilbert space with the norm

$$\|\mathbf{v}\|_V^2 := \|\mathbf{v}\|_{H_0^1(\Omega)}^2 + \|\operatorname{div} \mathbf{v}\|_{L^2(\Omega, \alpha-1)}^2,$$

and

$$Q = L_0^2(\Omega, \alpha - 1).$$

Define the bilinear forms $a : V \times V \to \mathbb{R}$ and $b : V \times Q \to \mathbb{R}$ by

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega D\mathbf{u} : D\mathbf{v}$$

and

$$b(\mathbf{v}, q) = \int_\Omega \operatorname{div} \mathbf{v} \, q \, d^{2\alpha-2}.$$

We are going to show that the problem

$$a(\mathbf{u}, \mathbf{v}) - b(\mathbf{v}, q) = \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \forall \mathbf{v} \in V \tag{5.2}$$

$$b(\mathbf{u}, r) = 0 \qquad \forall r \in Q \tag{5.3}$$

has a unique solution $(\mathbf{u}, q) \in V \times Q$.

Using the Schwarz inequality it is easy to check that the bilinear forms a and b are continuous and, since $\mathbf{f} \in H^{-1}(\Omega)^2$, that the linear functional defined by the right hand side of (5.2) is continuous.

Let

$$W = \left\{ \mathbf{v} \in V \, : \, b(\mathbf{v}, r) = 0 \ \forall r \in Q \right\}.$$

According to Brezzi's theory it is enough to see that a is coercive in W and b satisfies the inf-sup condition

$$\inf_{r \in Q} \sup_{\mathbf{v} \in V} \frac{b(\mathbf{v}, r)}{\|r\|_Q \|\mathbf{v}\|_V} > 0 \tag{5.4}$$

Since $\operatorname{div} V \subset Q$ we can take $r = \operatorname{div} \mathbf{v}$ in the equation $b(\mathbf{v}, r) = 0$ and conclude that $W = \{\mathbf{v} \in H_0^1(\Omega)^2 \, : \, \operatorname{div} \mathbf{v} = 0\}$. Therefore, coerciveness of a in W follows from the Poincaré inequality.

On the other hand, from Theorem 4.1 we know that given $r \in L_0^2(\Omega, \alpha - 1)$ there exists $\mathbf{w} \in H_0^1(\Omega)$ such that

$$\operatorname{div} \mathbf{w} = r \qquad \text{and} \qquad \|\mathbf{w}\|_{H_0^1(\Omega)} \leq C\|r\|_{L_0^2(\Omega, \alpha-1)}$$

where C is a positive constant which depends only on α. Moreover, from the definition of the norm in V it follows immediately that

$$\|\mathbf{w}\|_V \leq C_1 \|r\|_Q$$

for another constant depending only on α. Then,

$$\sup_{\mathbf{v} \in V} \frac{b(\mathbf{v}, r)}{\|r\|_Q \|\mathbf{v}\|_V} \geq \frac{\int_\Omega \operatorname{div} \mathbf{w} \, r \, d^{2\alpha - 2}}{\|r\|_Q \|\mathbf{w}\|_V} = \frac{\|r\|_Q}{\|\mathbf{w}\|_V} \geq C_1^{-1}$$

and therefore the inf-sup condition (5.4) is proved.

Summing up we have proved that the problem given in (5.2) and (5.3) has a unique solution $(\mathbf{u}, q) \in V \times Q$. Moreover, it follows also from the general theory that there exists a constant C depending only on C_1 such that

$$\|\mathbf{v}\|_V + \|q\|_Q \leq C \|\mathbf{f}\|_{H^{-1}(\Omega)}. \tag{5.5}$$

Now, define $p = q \, d^{2\alpha - 2}$. It is easy to see that $p \in L^2(\Omega, 1 - \alpha)$ and moreover, it follows from (5.3) that $\operatorname{div} \mathbf{u} = 0$ and from (5.2) that $(\mathbf{u}, p)$ verifies

$$\int_\Omega D\mathbf{u} : D\mathbf{v} - \int_\Omega \operatorname{div} \mathbf{v} \, p = \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \forall \mathbf{v} \in V.$$

Therefore, since $C_0^\infty(\Omega) \subset V$, $(\mathbf{u}, p)$ is a solution of the Stokes equations (1.1) in the sense of distributions as we wanted to prove. Finally, since $\|p\|_{L^2(\Omega, 1-\alpha)} = \|q\|_Q$, (5.1) follows immediately from (5.5). $\qquad \square$

We end this section with a corollary which gives an estimate for the pressure in a standard L^r-norm.

Corollary 5.1. *Let Ω be the domain defined in (4.1) with $1/2 < \alpha \leq 1$ and $(\mathbf{u}, p) \in H_0^1(\Omega)^2 \times L_0^2(\Omega, 1 - \alpha)$ be the solution of the Stokes equations (1.1). If $\mathbf{f} \in H^{-1}(\Omega)^2$ and $1 \leq r < 2/(3 - 2\alpha)$ then $(\mathbf{u}, p) \in H_0^1(\Omega)^2 \times L^r(\Omega)$. Moreover, there exists a constant C depending only on α such that*

$$\|\mathbf{u}\|_{H_0^1(\Omega)} + \|p\|_{L^r(\Omega)} \leq C \|\mathbf{f}\|_{H^{-1}(\Omega)}$$

Proof. We only have to prove that $p \in L^r(\Omega)$ and that

$$\|p\|_{L^r(\Omega)} \leq C \|\mathbf{f}\|_{H^{-1}(\Omega)}. \tag{5.6}$$

Observe that $\int_o d^\beta < +\infty$ for any $\beta > -1$. Indeed, this follows easily by using that $d(x, y) \simeq x^{1/\alpha} - |y|$. Then, applying the Hölder inequality with exponent $2/r$, we have

$$\|p\|_{L^r(\Omega)}^r = \int_\Omega |p|^r d^{(1-\alpha)r} d^{(\alpha - 1)r} \leq \|p\|_{L^2(\Omega, 1-\alpha)}^r \left(\int_\Omega d^{\frac{2(\alpha-1)r}{2-r}} \right)^{\frac{2-r}{2}}$$

but the integral in the right hand side is finite because $(2(\alpha - 1)r)/(2 - r) > -1$. So $\|p\|_{L^r(\Omega)} \leq C \|p\|_{L^2(\Omega, 1-\alpha)}$ and therefore, (5.6) follows immediately from (5.1). $\qquad \square$

6. Appendix

To prove Lemma 3.3 we will work with Whitney decompositions. If F is a compact nonempty subset of $\mathbb{R}^n$, then $\mathbb{R}^n \setminus F$ can be represented as a union of closed dyadic cubes with pairwise disjoint interior Q_j^k satisfying

$$\mathbb{R}^n \setminus F = \bigcup_{k \in \mathbb{Z}} \bigcup_{j=1}^{N_k} Q_j^k \tag{6.1}$$

where the edge length of Q_j^k is 2^{-k}. The decomposition (6.1) is called a *Whitney decomposition of* $\mathbb{R}^n \setminus F$ and the collection $\{Q_j^k \ : \ j = 1, ..., N_k\}$ is called the kth generation of Whitney cubes. Furthermore, the Whitney cubes satisfy

$$\ell_k \leq d(Q_j^k, F) \leq 4\ell_k$$

where $d(Q_j^k, F)$ denotes the distance of the cube to F and ℓ_k the diameter of Q_j^k (see for example [39]).

For $x_0 \in F$ and $R > 0$, $N_k(B(x_0, R))$ denotes the number of Whitney cubes of F^c in the kth generation contained in $B(x_0, R)$.

Lemma 6.1. *Let $F \subset \mathbb{R}^n$ be a compact set included in an **m-regular set** K. Given $x_0 \in F$ and $0 < R < \mathrm{diam}\, F/3$, there exists a constant C depending only on K such that*

$$N_k(B(x_0, R)) \leq C\, R^m\, 2^{km}.$$

Proof. We can assume that $2^{-k} \leq R$, if not $N_k(B(x_0, R) = 0$. The number of Whitney cubes of F^c in the kth generation contained in the ball $B(x_0, R)$ can be estimated in terms of the number of balls of radius 2^{-k} and center contained in F necessary to cover $F \cap B(x_0, 2R)$. Indeed, suppose there exist balls $B(x_i, 2^{-k})$ with $x_i \in F$, $1 \leq i \leq N$, such that

$$F \cap B(x_0, 2R) \subseteq \bigcup_{i=1}^{N} B(x_i, 2^{-k}) \tag{6.2}$$

and let Q^k be a Whitney cube in the kth generation contained in $B(x_0, R)$. Then, it is easy to check that

$$d(Q^k, F) = d(Q^k, F \cap B(x_0, 2R)).$$

Thus, if $y_Q \in F$ is a point satisfying $d(Q^k, F) = d(Q^k, y_Q)$, there exists some i, $1 \leq i \leq N$, such that $y_Q \in B(x_i, 2^{-k})$. So, using that Q^k is a Whitney cube in the kth generation it follows that

$$Q^k \subset B(x_i, 6\ell_k).$$

But, $B(x_i, 6\ell_k)$ cannot contain more than a finite number $c(n)$ of Whitney cubes Q^k. Then, by (6.2) it follows that

$$N_k(B(x_0, R)) \leq c(n)N$$

Therefore, to complete the proof we have to show that there exists N balls satisfying (6.2) with $N \leq C\, R^m\, 2^{km}$.

Let $r = 2^{-(k+1)}$. For $K_0 := K \cap B(x_0, 2R)$ we define the numbers

$$H_m(K_0, r) := \min\left\{Nr^m \ : \ K_0 \subseteq \bigcup_{i=1}^{N} B(x_i, r), \text{with } x_i \in K_0\right\}$$

and

$$P(K_0, r) := \max\left\{N \ : \ \text{there exists disjoint balls } B(x_i, r),\, i = 1, \ldots, N, \text{with } x_i \in K_0\right\}.$$

Then, using that K is an m-regular set we have

$$
\begin{aligned}
H_m(K_0, r) \ &\leq \ P\left(K_0, \frac{r}{2}\right) r^m \ = \ 2^m P\left(K_0, \frac{r}{2}\right)\left(\frac{r}{2}\right)^m \\
&< \ 2^m C \sum_{i=1}^{P(K_0, r/2)} \mathcal{H}^m\left(B\left(x_i, \frac{r}{2}\right) \cap K\right) \\
&= \ 2^m C \sum_{i=1}^{P(K_0, r/2)} \mathcal{H}^m\left(B\left(x_i, \frac{r}{2}\right) \cap K \cap B(x_0, 3R)\right) \\
&\leq \ 2^m C\, \mathcal{H}^m\left(K \cap B(x_0, 3R)\right) \ < \ C^2 6^m R^m.
\end{aligned}
$$

Thus, using the definition of $H_m(K_0, r)$ we obtain

$$K \cap B(x_0, 2R) \subseteq \bigcup_{i=1}^{N} B(x_i, 2^{-(k+1)}) \qquad \text{and} \qquad N \leq C R^m 2^{(k+1)m}. \qquad (6.3)$$

Now, F is contained in K and therefore it is possible to cover $F \cap B(x_0, 2R)$ with $\bigcup_{i=1}^{N} B(x_i, 2^{-(k+1)})$. Then, if $B(x_i, 2^{-(k+1)})$ intersects F, for $x_i' \in F \cap B(x_i, 2^{-(k+1)})$ we have that

$$B(x_i, 2^{-(k+1)}) \subseteq B(x_i', 2^{-k}).$$

Thus, it is easy to see that the balls $B(x_i', 2^{-k})$ satisfy (6.2), concluding the proof. $\qquad \square$

Before proving Lemma 3.3 let us recall the definition of the Muckenhoupt class A_p. For $1 < p < \infty$ a non-negative function w is in A_p if

$$\sup_{B \subset \mathbb{R}^n} \left(\frac{1}{|B|} \int_B w(x)\, dx \right) \left(\frac{1}{|B|} \int_B w(x)^{-\frac{1}{p-1}}\, dx \right)^{p-1} < \infty. \qquad (6.4)$$

where the supremum is taken over all the balls B.

Proof of Lemma 3.3: Let B be a ball in $\mathbb{R}^n$, r_B its radius and $d(B)$ the distance of B to F.

If $r_B \leq d(B)$, given x in B we have $d(B) \leq d(x) \leq 3d(B)$. Then,

$$\left(\frac{1}{|B|} \int_B d^{\mu} \right) \left(\frac{1}{|B|} \int_B d^{-\frac{\mu}{p-1}} \right)^{p-1} \leq C \left(\frac{1}{|B|} \int_B d(B)^{\mu} \right) \left(\frac{1}{|B|} \int_B d(B)^{-\frac{\mu}{p-1}} \right)^{p-1} \leq C$$

On the other hand, if $r_B \geq d(B)$, there exists $x_0 \in \partial\Omega$ such that $B \subseteq B(x_0, 3r_B)$. Then, without loss of generality, we can assume that B is centered at a point in F.

We consider two cases:

a) If $r_B < diam\, F/6$, from the Whitney decomposition of F we have

$$\left(\frac{1}{|B|} \int_B d^{\mu} \right) \left(\frac{1}{|B|} \int_B d^{-\frac{\mu}{p-1}} \right)^{p-1} \leq C r_B^{-np} \left(\sum_{Q^k} \int_{Q^k} d^{\mu} \right) \left(\sum_{Q^k} \int_{Q^k} d^{-\frac{\mu}{p-1}} \right)^{p-1} =: \mathrm{I}$$

where the sum is taken over all Whitney cubes Q^k intersecting B. But it is enough to consider the Whitney cubes contained in the ball $2B$.

Observe that if Q^k is contained in $2B$ then the diagonal of Q^k is smaller than the diagonal of $2B$. So, $2^{-k} \leq \frac{4}{\sqrt{n}} r_B$. Thus, if we call k_0 the minimum k such that there exists Q^k contained in B, it satisfies that $2^{-k_0} \leq C r_B$.

Now, using that $d(x) \simeq d(Q^k) \simeq 2^{-k}$ for every $x \in Q^k$ and Lemma 6.1 we obtain

$$\begin{aligned}
\mathrm{I} \;\leq\;& C r_B^{-np} \left(\sum_{Q_k} 2^{-k\mu} 2^{-kn} \right) \left(\sum_{Q_k} 2^{\frac{\mu k}{p-1}} 2^{-kn} \right)^{p-1} \\
\leq\;& C r_B^{-np} \left(\sum_{k=k_0}^{\infty} N_k(B(x_0, 2r_B)) 2^{-k\mu} 2^{-kn} \right) \left(\sum_{k=k_0}^{\infty} N_k(B(x_0, 2r_B)) 2^{\frac{\mu k}{p-1}} 2^{-kn} \right)^{p-1} \\
\leq\;& C r_B^{-np} \left(\sum_{k=k_0}^{\infty} r_B^m 2^{-k(\mu+n-m)} \right) \left(\sum_{k=k_0}^{\infty} r_B^m 2^{-k\left(n-m-\frac{\mu}{p-1} \right)} \right)^{p-1} = \mathrm{II}.
\end{aligned}$$

Then, since $-(n-m) < \mu < (p-1)(n-m)$, we obtain

$$\text{II} \quad \leq \quad C\, r_B^{-p(n-m)} \left(2^{-k_0(\mu+n-m)}\right) \left(2^{-k_0\left(n-m-\frac{\mu}{p-1}\right)}\right)^{p-1} \leq C\, r_B^{-p(n-m)} \left(2^{-k_0}\right)^{p(n-m)} \leq C$$

b) If $r_B \geq \operatorname{diam} F/6$, let x_F be a point in F independent of B. Then, since $x_0 \in F$ and $r_B > \operatorname{diam} F/6$ we can assume that B is the ball with radius $r_B > 3\operatorname{diam} F$ and center x_F. On the other hand, if B_1 denotes the ball of radius $2\operatorname{diam} K$ centered at x_F we can see that d^μ and $d^{-\frac{\mu}{p-1}}$ are locally integrable. Then,

$$\int_{B_1} d^\mu \leq C \qquad \text{y} \qquad \int_{B_1} d^{-\frac{\mu}{p-1}} \leq C.$$

But, it is easy to see that $d(x) \simeq d(x, x_F)$ for all $x \in B \setminus B_1$. Therefore,

$$\int_B d^\mu = \int_{B_1} d^\mu + \int_{B\setminus B_1} d^\mu \leq C\left(1 + \int_{B\setminus B_1} |x - x_F|^\mu\right)$$

$$\leq C\left(1 + \int_{2\operatorname{diam} F}^{r_B} \rho^\mu \rho^{n-1}\right) \leq C r_B^{\mu+n}.$$

Analogously we can show that

$$\int_B d^{-\frac{\mu}{p-1}} \leq C r_B^{-\frac{\mu}{p-1}+n}$$

and therefore,

$$\left(\frac{1}{|B|}\int_B d^\mu\right)\left(\frac{1}{|B|}\int_B d^{-\frac{\mu}{p-1}}\right)^{p-1} \leq C \frac{1}{|B|^p} r_B^{\mu+n} r_B^{-\mu+n(p-1)} \leq C.$$

and the lemma is proved. $\square$

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