Error estimates for the Raviart-Thomas interpolation under the maximum angle condition

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Abstract. The classical error analysis for the Raviart-Thomas interpolation on triangular elements requires the so-called regularity of the elements, or equivalently, the minimum angle condition.

However, in the lowest order case, optimal order error estimates have been obtained in [1] replacing the regularity hypothesis by the maximum angle condition, which was known to be sufficient to prove estimates for the standard Lagrange interpolation.

In this paper we prove error estimates on triangular elements for the Raviart-Thomas interpolation of any order under the maximum angle condition. Also, we show how our arguments can be extended to the three dimensional case to obtain error estimates for tetrahedral elements under the regular vertex property introduced in [1].

Key words. mixed finite elements, Raviart-Thomas, anisotropic finite elements.

AMS subject classifications. 65N30.

1 Introduction

The classical error analysis for finite element approximations is based on the so-called regularity assumption on the elements. In other words, the constants in the error estimates obtained depend on the ratio between outer and inner diameter of the elements and blow up when this ratio goes to infinity (see for example [4, 5]).

However, it is well known that the regularity assumption can be relaxed for standard finite element approximations. For example, in the 2d case, optimal order error estimates have been proved for triangular elements under the weaker maximum angle condition (i.e. angles bounded away from π). This condition allows the

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use of the so-called anisotropic elements which is of interest in several applications, for example in problems with boundary or interior layers.

Error estimates under the maximum angle condition were first obtained in [3, 9]). After these pioneering works many papers have considered different generalizations of their results (see for example [2] and its references).

The usual error analysis for mixed finite element methods also makes use of the regularity assumption [12, 14]. In view of the results for the standard method mentioned above, it is a natural question whether the regularity hypothesis on the elements can be relaxed in this case also. A positive answer for the Raviart-Thomas space of lowest order $\mathcal{RT}_0$ is given in [1], where an optimal error estimate is proved under the maximum angle condition. However, it is not straightforward to extend the arguments given in that paper to higher order approximations. In [7] it is proved that the maximum angle condition is also sufficient to obtain optimal error estimates for the Raviart-Thomas space $\mathcal{RT}_1$.

The goal of this paper is to prove that the maximum angle condition is also sufficient to obtain optimal error estimates for the approximation using the Raviart-Thomas space $\mathcal{RT}_k$ for arbitrary k.

The 3d case presents some important differences with the 2d case. Results for the standard finite element approximations in 3d were obtained in several papers (see for example [2, 6, 8, 10, 13]). In [10] Krízek proved optimal order error estimates for the Lagrange interpolation on a tetrahedron K for smooth functions, namely $u \in W^{2,\infty}$, provided the angles between faces and the angles in the faces are bounded away from π.

The Krízek condition seems to be a natural extension of the 2d maximum angle condition. However, there is another possible extension: the regular vertex property introduced in [1]. Roughly speaking, a family of tetrahedral elements satisfies this condition if for each element there is at least one vertex such that the unit vectors in the direction of the edges sharing that vertex are "uniformly" linearly independent, in the sense that the volume determined by them is uniformly bounded away from zero.

It is easy to see that in the 2d case the maximum angle condition and the regular vertex property are equivalent. However, in 3d this is not the case. Indeed, the Krízek condition allows for more general elements. To understand better these two possible extensions of the maximum angle condition consider the two families of elements given in Figure 1, where h_1, h_2 and h_3 are arbitrary positive numbers.

It is easy to see that both families satisfy the maximum angle condition, but the second family does not satisfy the regular vertex property. Moreover, it was proved in [1] that the family of all elements satisfying the Krízek condition with a constant $\psi < \pi$ (i.e., angles between faces and angles in the faces less than or equal to ψ) can be obtained transforming both families in the figure by affine transformations with bounded condition number. On the other hand, the family of all elements satisfying the regular vertex property with a given constant (see Section 3 for the formal definition of this condition) is obtained by transforming in the same way only the first family in the figure.

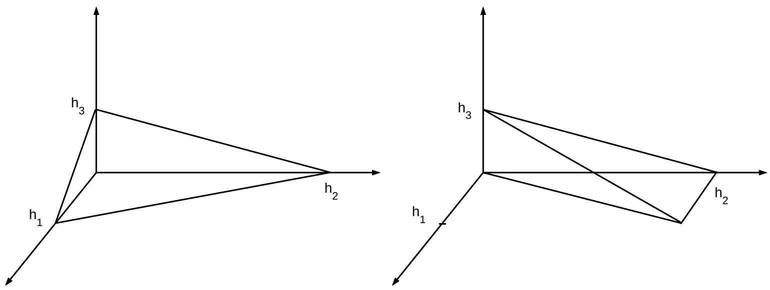

Figure 1

In [1], optimal error estimates under the Krízek condition were obtained for the $\mathcal{RT}_0$ spaces in 3d. In this paper we prove optimal error estimates for the general $\mathcal{RT}_k$ spaces under the regular vertex property. The more general case of elements satisfying the Krízek condition requires a different argument and will be the subject of future research.

To simplify the exposition we present first the proofs in the 2d case in Section 2. Then, in Section 3 we show how the arguments can be easily generalized to the 3d case for elements satisfying the regular vertex property.

2 Error estimates in the two-dimensional case

We use the standard notation $\mathcal{P}_k$ for the space of polynomials of degree less than or equal to k. Then the local Raviart-Thomas space of order $k \geq 0$ on a triangle T [12] is defined by

$$\mathcal{RT}_k(T) = \mathcal{P}_k^2(T) + (x, y)\mathcal{P}_k(T). \tag{2.1}$$

It is known that there exists an operator

$$\Pi_k : H^1(T)^2 \longrightarrow \mathcal{RT}_k(T)$$

such that

$$\int_\ell \Pi_k \mathbf{v} \cdot \mathbf{n} p_k = \int_\ell \mathbf{v} \cdot \mathbf{n} p_k \quad \forall p_k \in \mathcal{P}_k(\ell) \quad \forall \ell \,\text{side of } T \tag{2.2}$$

and, if $k \geq 1$,

$$\int_T \Pi_k \mathbf{v} \cdot \mathbf{p}_{k-1} = \int_T \mathbf{v} \cdot \mathbf{p}_{k-1} \quad \forall \mathbf{p}_{k-1} \in \mathcal{P}_{k-1}^2(T) \tag{2.3}$$

Introducing the L^2 orthogonal projection

$$P_k : L^2(T) \longrightarrow \mathcal{P}_k(T)$$

it is not difficult to check that Π_k and P_k satisfy the following commutative diagram:

$$\begin{array}{ccccc} H^1(T)^2 & \stackrel{div}{\longrightarrow} & L^2(T) & & \\ \Pi_k \Big\downarrow & & \Big\downarrow P_k & & \\ \mathcal{RT}_k(T) & \stackrel{div}{\longrightarrow} & \mathcal{P}_k(T) & \longrightarrow 0 & \end{array} \tag{2.4}$$

where the right bottom arrow in this diagram indicates that the divergence operator is surjective.

In the rest of the paper the letter C will denote a generic constant independent of the element T and of the functions involved in the estimates.

The error estimates for Π_k will be a consequence of the generalized Poincaré inequality given in the next lemma.

Lemma 2.1 *Let T be a triangle and ξ_1 and ξ_2 unit vectors in the directions of two sides of T that have lengths h_1 and h_2. Assume that for one side ℓ of T and $f \in H^{k+1}(T)$ we have*

$$\int_\ell f\, q = 0 \qquad \forall q \in \mathcal{P}_k(\ell). \tag{2.5}$$

and, if $k \geq 1$,

$$\int_T f\, p = 0 \qquad \forall p \in \mathcal{P}_{k-1}(T). \tag{2.6}$$

Then, there exists a constant C independent of T and f such that

$$\|f\|_{L^2(T)} \leq C \sum_{i+j=k+1} h_1^i h_2^j \left\| \frac{\partial^{k+1} f}{\partial \xi_1^i \partial \xi_2^j} \right\|_{L^2(T)}. \tag{2.7}$$

Proof. First we prove the result for the reference element $\widehat{T}$ which has vertices at $(0,0)$, $(1,0)$ and $(0,1)$. Namely, we will show that if $\hat{f} \in H^{k+1}(\widehat{T})$ satisfies

$$\int_{\hat{\ell}} \hat{f}\, q = 0 \qquad \forall q \in \mathcal{P}_k(\hat{\ell}). \tag{2.8}$$

and, if $k \geq 1$,

$$\int_T \hat{f}\, p = 0 \qquad \forall p \in \mathcal{P}_{k-1}(\widehat{T}), \tag{2.9}$$

where $\hat{\ell}$ is one of the sides of $\widehat{T}$, then

$$\|\hat{f}\|_{L^2(\widehat{T})} \leq C \sum_{i+j=k+1} \left\| \frac{\partial^{k+1} \hat{f}}{\partial \hat{x}^i \partial \hat{y}^j} \right\|_{L^2(\widehat{T})}. \tag{2.10}$$

Let $\mathcal{S}_0 = \mathcal{P}_0(\widehat{T})$ and, for $k \geq 1$, $\mathcal{S}_k$ be the orthogonal complement of $\mathcal{P}_{k-1}(\widehat{T})$ in $\mathcal{P}_k(\widehat{T})$. Then, $\|\,.\,\|_{L^2(\hat{\ell})}$ is a norm on $\mathcal{S}_k$. For $k = 0$ this is trivial and to see this for $k \geq 1$ it is enough to check that if $p \in \mathcal{S}_k$ satisfies $\|p\|_{L^2(\hat{\ell})} = 0$, then $p = 0$. But, $\|p\|_{L^2(\hat{\ell})} = 0$ implies that p vanishes on $\hat{\ell}$ and therefore $p = \lambda_1 q$ with $q \in \mathcal{P}_{k-1}(\widehat{T})$, where $\lambda_1 \in \mathcal{P}_1$ is the function such that $\lambda_1 = 0$ is the equation of the line containing the side $\hat{\ell}$ (i.e., the so called barycentric coordinate). Now, since $p \in \mathcal{S}_k$ we have

$$\int_{\widehat{T}} \lambda_1 \, q^2 = 0$$

and, since λ_1 does not change sign in $\hat{T}$, it follows that $q = 0$ and so $p = 0$ as we wanted to see.

In view of (2.9), the L^2 orthogonal projection of $\hat{f}$ on $\mathcal{P}_k(\widehat{T})$, namely $\widehat{P}_k \hat{f}$, belongs to $\mathcal{S}_k$. Then, since $\mathcal{S}_k$ is a finite dimensional space, there exists a constant C depending only on k and $\widehat{T}$ such that

$$\|\widehat{P}_k \hat{f}\|_{L^2(\widehat{T})} \leq C \|\widehat{P}_k \hat{f}\|_{L^2(\hat{\ell})}. \tag{2.11}$$

On the other hand, using (2.8) we have

$$\|\widehat{P}_k \hat{f}\|^2_{L^2(\hat{\ell})} = \int_{\hat{\ell}} \widehat{P}_k \hat{f} \, (\widehat{P}_k \hat{f} - \hat{f}) \leq \|\widehat{P}_k \hat{f}\|_{L^2(\hat{\ell})} \|\widehat{P}_k \hat{f} - \hat{f}\|_{L^2(\hat{\ell})}$$

which together with (2.11) gives

$$\|\widehat{P}_k \hat{f}\|_{L^2(\widehat{T})} \leq C \|\widehat{P}_k \hat{f} - \hat{f}\|_{L^2(\hat{\ell})}$$

and then, using a standard trace theorem to bound the right hand side and known error estimates for the L^2 projection, we obtain

$$\|\widehat{P}_k \hat{f}\|_{L^2(\widehat{T})} \leq C \|\widehat{P}_k \hat{f} - \hat{f}\|_{H^1(\widehat{T})} \leq C \sum_{i+j=k+1} \left\| \frac{\partial^{k+1} \hat{f}}{\partial \hat{x}^i \partial \hat{y}^j} \right\|_{L^2(\widehat{T})}$$

with another constant C depending only on k and $\widehat{T}$. Therefore, (2.10) follows by using the triangular inequality.

Assume without loss of generality that the vertex shared by the sides of T in the directions ξ_1 and ξ_2 is at the origin. Then, T is the image of $\widehat{T}$ by the linear map

$$\begin{pmatrix} x \\ y \end{pmatrix} = B \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$$

where the columns of B are given by $h_1 \xi_1$ and $h_2 \xi_2$. Therefore, if $\hat{f}(\hat{x}, \hat{y}) = f(x, y)$ then

$$\frac{\partial^{k+1} \hat{f}}{\partial \hat{x}^i \partial \hat{y}^j} = h_1^i h_2^j \frac{\partial^{k+1} f}{\partial \xi_1^i \partial \xi_2^j}$$

Then, (2.7) follows from (2.10) by changing variables.□

Our next goal is to prove a result for the L^2 orthogonal projection which will be needed in our main theorem. For clarity we divide the proof in the following two lemmas.

Lemma 2.2 *Let $\widehat{T}$ be the triangle with vertices at* $(0,0)$, $(1,0)$ *and* $(0,1)$. *Given* (i,j) *such that* $i+j=k$ *we define*

$$q_{ij} := \frac{\partial^k}{\partial x^i \partial y^j}\big(x^i y^j (1-x-y)^k\big).$$

Then, for any $f \in H^k(\widehat{T})$,

$$\int_{\widehat{T}} q_{ij}\, f = (-1)^k \int_{\widehat{T}} \big(x^i y^j (1-x-y)^k\big) \frac{\partial^k f}{\partial x^i \partial y^j}. \tag{2.12}$$

Proof. We want to prove that

$$\int_{\widehat{T}} \frac{\partial^k}{\partial x^i \partial y^j}\big(x^i y^j\, (1-x-y)^k\big)\, f = (-1)^k \int_{\widehat{T}} \big(x^i y^j (1-x-y)^k\big) \frac{\partial^k f}{\partial x^i \partial y^j}$$

so, it is enough to see that the boundary terms arising in the integration by parts vanish. For example, when we integrate by parts in the x variable the boundary terms will be of the form

$$\int_{\partial\widehat{T}} \frac{\partial^{l+j}}{\partial x^l \partial y^j}\big(x^i y^j (1-x-y)^k\big)\, \frac{\partial^{i-1-l} f}{\partial x^{i-1-l}}\, n_1$$

where n_1 is the first component of the unit normal at the boundary and $l = 0,\dots,i-1$. Since $n_1 = 0$ on the side contained in the line $\{y=0\}$, it is enough to see that

$$\frac{\partial^{l+j}}{\partial x^l \partial y^j}\big(x^i y^j (1-x-y)^k\big) = 0 \tag{2.13}$$

on the other two sides of $\widehat{T}$. This derivative can be written as a sum of terms which, up to a multiplicative constant, are of the form

$$x^{i-m} y^{j-n} (1-x-y)^{k+m+n-l-j}$$

with $m = 0,\dots,l$ and $n = 0,\dots,j$. But, since $l < i$, we have $m < i$ and $l + j < i + j = k$ and so in all of these terms the exponents of x and $(1-x-y)$ are positive. Therefore, (2.13) holds on the sides of $\widehat{T}$ contained in $\{x=0\}$ and $\{1-x-y=0\}$ as we wanted to show.$\square$

Lemma 2.3 *If T is the right triangle with vertices at* $(0,0)$, $(h_1,0)$ *and* $(0,h_2)$ *then, for any* $f \in H^k(T)$ *and any* i,j *such that* $i+j=k$,

$$\Big\|\frac{\partial^k P_k f}{\partial x^i \partial y^j}\Big\|_{L^2(T)} \le C \Big\|\frac{\partial^k f}{\partial x^i \partial y^j}\Big\|_{L^2(T)}$$

with a constant C independent of the right triangle T.

Proof. We prove the estimate for the reference triangle $\widehat{T}$ with vertices at $(0,0)$, $(1,0)$ and $(0,1)$. Then, the result follows by making the change of variables $(x,y) = (h_1\hat{x}, h_2\hat{y})$.

Since the result is trivial for $k=0$ we can assume that $k \geq 1$. Fix (i,j) such that $i+j=k$ and let $\{p_1,\dots,p_N\}$ be an orthonormal basis of the subspace $\mathcal{S} \subset \mathcal{P}_k(\widehat{T})$ expanded by

$$\{x^m y^n \,:\, m+n \leq k, (m,n) \neq (i,j)\}.$$

From the definition of q_{ij} given in the previous lemma it is easy to check that $q_{ij} \in \mathcal{P}_k$. Moreover, since $\frac{\partial^k p}{\partial x^i \partial y^j} = 0$ for all $p \in \mathcal{S}$, it follows from (2.12) that q_{ij} is orthogonal to $\mathcal{S}$.

Then, defining $p_{N+1} := q_{ij}/\|q_{ij}\|_{L^2(\widehat{T})}$, we have that $\{p_1,\dots,p_{N+1}\}$ is an orthonormal basis of $\mathcal{P}_k(\widehat{T})$. Then

$$P_k f = \sum_{s=1}^{N+1} \left(\int_{\widehat{T}} f p_s \right) p_s$$

and, since $\frac{\partial^k p_s}{\partial x^i \partial y^j} = 0$, for $s = 1,\dots,N$, we have

$$\frac{\partial^k P_k f}{\partial x^i \partial y^j} = \left(\int_{\widehat{T}} f p_{N+1} \right) \frac{\partial^k p_{N+1}}{\partial x^i \partial y^j} \tag{2.14}$$

but, it follows from (2.12) that

$$\int_{\widehat{T}} f p_{N+1} = \frac{(-1)^k}{\|q_{ij}\|_{L^2(\widehat{T})}} \int_{\widehat{T}} \left(x^i y^j (1-x-y)^k \right) \frac{\partial^k f}{\partial x^i \partial y^j}$$

which together with (2.14) and an application of Schwarz inequality gives

$$\Big\| \frac{\partial^k P_k f}{\partial x^i \partial y^j} \Big\|_{L^2(\widehat{T})} \leq \widehat{C} \Big\| \frac{\partial^k f}{\partial x^i \partial y^j} \Big\|_{L^2(\widehat{T})}$$

as we wanted to prove. □

To end with the preliminary results we give in the next lemma a relation between derivatives of a function in $\mathcal{RT}_k(T)$ and derivatives of its divergence.

Lemma 2.4 *If T is a triangle and $\mathbf{u} \in \mathcal{RT}_k(T)$ then*

$$\frac{\partial^{k+1}\mathbf{u}}{\partial x^{k+1}} = \Big(\frac{k+1}{k+2} \frac{\partial^k (\operatorname{div} \mathbf{u})}{\partial x^k}, 0 \Big) \quad , \qquad \frac{\partial^{k+1}\mathbf{u}}{\partial y^{k+1}} = \Big(0\,, \frac{k+1}{k+2} \frac{\partial^k (\operatorname{div} \mathbf{u})}{\partial y^k} \Big)$$

and for $i+j = k+1$, with $i > 0$ and $j > 0$,

$$\frac{\partial^{k+1}\mathbf{u}}{\partial x^i \partial y^j} = \Big(\frac{i}{2+k} \frac{\partial^k \operatorname{div} \mathbf{u}}{\partial x^{i-1} \partial y^j} \,, \frac{j}{2+k} \frac{\partial^k \operatorname{div} \mathbf{u}}{\partial x^i \partial y^{j-1}} \Big).$$

Proof. Any function $\mathbf{u} \in \mathcal{RT}_k(T)$ can be written as $\mathbf{u} = \mathbf{p} + (xq, yq)$ with $\mathbf{p} \in \mathcal{P}_k^2$ and $q \in \mathcal{P}_k$. An elementary computation shows that

$$\frac{\partial^{k+1}\mathbf{u}}{\partial x^{k+1}} = \left((k+1)\frac{\partial^k q}{\partial x^k}\,,\,0\right) \quad , \qquad \frac{\partial^{k+1}\mathbf{u}}{\partial y^{k+1}} = \left(0\,,\,(k+1)\frac{\partial^k q}{\partial y^k}\right)$$

and, for any positive i, j such that $i + j = k + 1$,

$$\frac{\partial^{k+1}\mathbf{u}}{\partial x^i \partial y^j} = \left(i\frac{\partial^k q}{\partial x^{i-1}\partial y^j}\,,\,j\frac{\partial^k q}{\partial x^i \partial y^{j-1}}\right).$$

On the other hand, for $l + m = k$

$$\frac{\partial^k(\operatorname{div}\mathbf{u})}{\partial x^l \partial y^m} = (k+2)\frac{\partial^k q}{\partial x^l \partial y^m}\,.$$

Then the Lemma is easily obtained. □

Now we are ready to prove our main result. Given a triangle T we call h_T the longest side of T. We will also use the notation

$$D^m f = \sum_{i+j=m} \left|\frac{\partial^m f}{\partial x^i \partial y^j}\right|$$

and a tilde over operators will indicate that the derivatives are taken with respect to $(\tilde{x}, \tilde{y})$.

Theorem 2.5 *Let T be a triangle with maximum angle α. Let h_1 and h_2 be the lengths of the sides adjacent to α and ξ_1 and ξ_2 be unit vectors in the directions of those sides. Then, for any function $\mathbf{v} \in H^{k+1}(T)^2$,*

$$\|\mathbf{v} - \Pi_k\mathbf{v}\|_{L^2(T)} \leq \frac{C}{\sin\alpha}\left\{\sum_{i+j=k+1} h_1^i h_2^j \left\|\frac{\partial^{k+1}\mathbf{v}}{\partial\xi_1^i \partial\xi_2^j}\right\|_{L^2(T)} + h_T^{k+1}\|D^k \operatorname{div}\mathbf{v}\|_{L^2(T)}\right\} \tag{2.15}$$

with a constant C independent of T.

Proof. In view of (2.2) and (2.3), for any side ℓ_s of T the function $(\mathbf{v} - \Pi_k\mathbf{v}) \cdot \mathbf{n}_s$, where $\mathbf{n}_s$ denotes the unit exterior normal on ℓ_s, satisfies conditions (2.6) and (2.5) of Lemma 2.1, and therefore

$$\|(\mathbf{v} - \Pi_k\mathbf{v}) \cdot \mathbf{n}_s\|_{L^2(T)} \leq C \sum_{i+j=k+1} h_1^i h_2^j \left\|\frac{\partial^{k+1}\left[(\mathbf{v} - \Pi_k\mathbf{v}) \cdot \mathbf{n}_s\right]}{\partial\xi_1^i \partial\xi_2^j}\right\|_{L^2(T)}.$$

But, choosing ℓ_1 and ℓ_2 as the sides with directions ξ_1 and ξ_2, it is not difficult to see that

$$\|\mathbf{v} - \Pi_k\mathbf{v}\|_{L^2(T)} \leq \frac{C}{\sin\alpha}\{\|(\mathbf{v} - \Pi_k\mathbf{v}) \cdot \mathbf{n}_1\|_{L^2(T)} + \|(\mathbf{v} - \Pi_k\mathbf{v}) \cdot \mathbf{n}_2\|_{L^2(T)}\}$$

and then,

$$\|\mathbf{v} - \Pi_k \mathbf{v}\|_{L^2(T)} \le \frac{C}{\sin\alpha} \sum_{i+j=k+1} h_1^i h_2^j \left\| \frac{\partial^{k+1}(\mathbf{v} - \Pi_k \mathbf{v})}{\partial \xi_1^i \partial \xi_2^j} \right\|_{L^2(T)} . \tag{2.16}$$

Then (2.15) is obtained from the inequality

$$\sum_{i+j=k+1} \left\| \frac{\partial^{k+1}\Pi_k \mathbf{v}}{\partial \xi_1^i \partial \xi_2^j} \right\|_{L^2(T)} \le C \left\| D^k \mathrm{div}\, \mathbf{v} \right\|_{L^2(T)} , \tag{2.17}$$

that we will prove in what follows.

Without loss of generality we can assume that T has vertices at $(0,0)$, $(h_1, 0)$ and (a, b), and that the vertex corresponding to the maximum angle is at (0,0). Then $\xi_1 = (1, 0)$ and $\xi_2 = (\frac{a}{h_2}, \frac{b}{h_2})$.

Consider the right triangle $\widetilde{T}$ with vertices at $(0,0)$, $(h_1, 0)$ and $(0, h_2)$. Then, the linear transformation $(x, y) = F(\tilde{x}, \tilde{y})$ defined as

$$\begin{pmatrix} x \\ y \end{pmatrix} = B \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

with

$$B = \begin{pmatrix} 1 & \frac{a}{h_2} \\ 0 & \frac{b}{h_2} \end{pmatrix}$$

maps $\widetilde{T}$ onto T (see Figure 2). We have

$$\|B\|_2 \le \sqrt{2} \tag{2.18}$$

where $\| \,.\, \|_2$ denotes the matrix norm associated with the euclidean norm.

Now, given $\mathbf{v} \in H^{k+1}(T)^2$, we define the corresponding function $\tilde{\mathbf{v}} \in H^{k+1}(\widetilde{T})^2$ via the Piola transform, that is,

$$\mathbf{v}(x, y) = \frac{1}{|\det B|} B\, \tilde{\mathbf{v}}(\tilde{x}, \tilde{y}) \quad , \quad (x, y) \in T.$$

Then, we have

$$\mathrm{div}\, \mathbf{v} = \frac{1}{|\det B|} \widetilde{\mathrm{div}}\, \tilde{\mathbf{v}} \tag{2.19}$$

Denoting by $\widetilde{\Pi}_k$ the Raviart-Thomas interpolation on $\widetilde{T}$, it is proved in [12, page 303] that

$$\Pi_k \mathbf{v}(x, y) = \frac{1}{|\det B|} B\, \widetilde{\Pi}_k \tilde{\mathbf{v}}(\tilde{x}, \tilde{y}) \quad , \quad (x, y) \in T.$$

Then, we have

$$\frac{\partial^{k+1}\Pi_k \mathbf{v}}{\partial \xi_1^i \partial \xi_2^j}(x, y) = \frac{1}{|\det B|}\, B\, \frac{\partial^{k+1}\widetilde{\Pi}_k \tilde{\mathbf{v}}}{\partial \tilde{x}^i \partial \tilde{y}^j}(\tilde{x}, \tilde{y}),$$

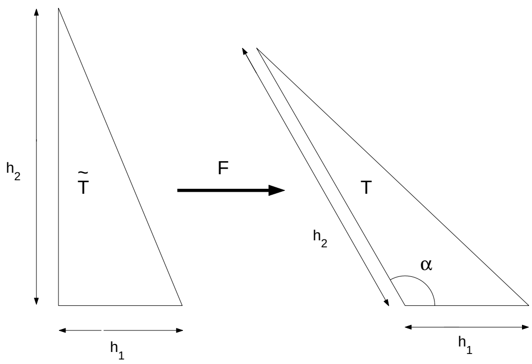

Figure 2

and, therefore,

$$\left\| \frac{\partial^{k+1}\Pi_k \mathbf{v}}{\partial\xi_1^i \partial\xi_2^j} \right\|_{L^2(T)} \leq \frac{\|B\|_2}{|\det B|^{\frac{1}{2}}} \left\| \frac{\partial^{k+1}\widetilde{\Pi}_k \tilde{\mathbf{v}}}{\partial\tilde{x}^i \partial\tilde{y}^j} \right\|_{L^2(\widetilde{T})} .$$

Now, since $\widetilde{\Pi}_k \tilde{\mathbf{v}} \in \mathcal{RT}_k(\widetilde{T})$, using Lemma 2.4, the commutative diagram property (2.4) for $\widetilde{\Pi}_k$, Lemma 2.3 in $\widetilde{T}$ and (2.19), we have

$$\begin{aligned}
\sum_{i+j=k+1} \left\| \frac{\partial^{k+1}\Pi_k \mathbf{v}}{\partial\xi_1^i \partial\xi_2^j} \right\|_{L^2(T)} &\leq \frac{\|B\|_2}{|\det B|^{\frac{1}{2}}} \sum_{i+j=k+1} \left\| \frac{\partial^{k+1}\widetilde{\Pi}_k \tilde{\mathbf{v}}}{\partial\tilde{x}^i \partial\tilde{y}^j} \right\|_{L^2(\widetilde{T})} \\
&\leq C \frac{\|B\|_2}{|\det B|^{\frac{1}{2}}} \left\| \widetilde{D}^k \widetilde{\operatorname{div}}\, \widetilde{\Pi}_k \tilde{\mathbf{v}} \right\|_{L^2(\widetilde{T})} \\
&\leq C \frac{\|B\|_2}{|\det B|^{\frac{1}{2}}} \left\| \widetilde{D}^k \widetilde{P}_k \widetilde{\operatorname{div}}\, \tilde{\mathbf{v}} \right\|_{L^2(\widetilde{T})} \\
&\leq C \frac{\|B\|_2}{|\det B|^{\frac{1}{2}}} \left\| \widetilde{D}^k \widetilde{\operatorname{div}}\, \tilde{\mathbf{v}} \right\|_{L^2(\widetilde{T})} \\
&\leq C \|B\|_2^{k+1} \left\| D^k \operatorname{div} \mathbf{v} \right\|_{L^2(T)} ,
\end{aligned}$$

then (2.17) follows by using (2.18). □

Remark 2.1 *It is possible to prove error estimates for the Raviart-Thomas interpolation under the maximum angle condition without using the Piola transform. However, the dependence of the constant on the maximum angle of T obtained in this way is worse than that in Theorem 2.5.*

Indeed, using the same change of variables from a right triangle $\widetilde{T}$ onto T as in Theorem 2.5, and the estimate given in Lemma 2.3 we can prove, for any $f \in H^k(T)$,

$$\|D^k P_k f\|_{L^2(T)} \le \frac{C}{(\sin\alpha)^k} \|D^k f\|_{L^2(T)} \tag{2.20}$$

where α is the maximum angle of T.

Then, using Lemma 2.4, the commutative diagram property for Π_k (2.4) and (2.20) we have

$$\begin{aligned}
\sum_{i+j=k+1} \left\| \frac{\partial^{k+1}\Pi_k \mathbf{v}}{\partial\xi_1^i \partial\xi_2^j} \right\|_{L^2(T)} &\le \|B\|_2^{k+1} \sum_{i+j=k+1} \left\| \frac{\partial^{k+1}\Pi_k \mathbf{v}}{\partial x^i \partial y^j} \right\|_{L^2(T)} \\
&\le C\|B\|_2^{k+1} \sum_{i+j=k+1} \left\| D^k \mathrm{div}\, \Pi_k \mathbf{v} \right\|_{L^2(T)} \\
&= C\|B\|_2^{k+1} \sum_{i+j=k+1} \left\| D^k P_k \mathrm{div}\, \mathbf{v} \right\|_{L^2(T)} \\
&\le \frac{C}{(\sin\alpha)^k} \|B\|_2^{k+1} \sum_{i+j=k+1} \left\| D^k \mathrm{div}\, \mathbf{v} \right\|_{L^2(T)}.
\end{aligned}$$

This inequality, together with (2.16) and (2.18) gives

$$\|\mathbf{v} - \Pi_k \mathbf{v}\|_{L^2(T)} \le \frac{C}{\sin\alpha} \left\{ \sum_{i+j=k+1} h_1^i h_2^j \left\| \frac{\partial^{k+1}\mathbf{v}}{\partial\xi_1^i \partial\xi_2^j} \right\|_{L^2(T)} + \frac{h_T^{k+1}}{(\sin\alpha)^k} \|D^k \mathrm{div}\, \mathbf{v}\|_{L^2(T)} \right\}.$$

3 The three-dimensional case

The Raviart-Thomas spaces have been generalized to the three dimensional case by Nedelec [11]. In this section we briefly explain how to extend the results previously obtained to a class of tetrahedral elements satisfying a condition which is weaker than regularity.

Following [1], we say that a tetrahedron K satisfies the *regular vertex property* with a constant $\bar{c} > 0$ if there exist a vertex $\mathbf{Z}$ of K, called the *regular vertex*, unit vectors ξ_i and scalars h_i, $i = 1, 2, 3$, such that K is the convex hull of $\{\mathbf{Z}\} \cup \{\mathbf{Z} + h_i\xi_i, i = 1, 2, 3\}$, and the matrix B made up with ξ_i as its columns verifies $|\det B| \ge \bar{c}$.

Given a tetrahedron K the spaces introduced in [11] are given, for $k \ge 0$, by

$$\mathcal{RT}_k(K) = \mathcal{P}_k(K)^3 + (x, y, z)\mathcal{P}_k(K).$$

Also in [11] it is proved that there exists an operator

$$\Pi_k : H^1(K)^3 \longrightarrow \mathcal{RT}_k(K)$$

such that

$$\int_F \Pi_k \mathbf{v}\cdot\mathbf{n}p_k = \int_F \mathbf{v}\cdot\mathbf{n}p_k \quad \forall p_k \in \mathcal{P}_k(F),\ \forall F \ \text{ face of } \ K$$

and, if $k \geq 1$,

$$\int_K \Pi_k \mathbf{v}\cdot\mathbf{p}_{k-1} = \int_K \mathbf{v}\cdot\mathbf{p}_{k-1} \quad \forall \mathbf{p}_{k-1} \in \mathcal{P}^3_{k-1}(K).$$

In what follows we state, for tetrahedral elements satisfying the regular vertex property, an error estimate for Π_k analogous to that given in Theorem 2.5. We will mention without proof the 3d versions of Lemmas 2.1 and 2.3 that can be obtained following the same arguments as in the 2d case.

We define the tetrahedron $\widetilde{K}$ with vertices at $(0,0,0)$, $(h_1,0,0)$, $(0,h_2,0)$ and $(0,0,h_3)$. We denote by $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ the normals to the faces of $\widetilde{K}$ contained in the planes $x=0$, $y=0$ and $z=0$, respectively. For a tetrahedron K we call h_K its diameter.

Theorem 3.1 *Let K be a tetrahedron satisfying the regular vertex property with a constant $\bar{c}$ and let h_i and ξ_i, $i=1,2,3$, be the lengths and unit directions corresponding to the edges sharing the regular vertex $\mathbf{Z}$. Then, for any function $\mathbf{v}\in H^{k+1}(K)^3$,*

$$\|\mathbf{v}-\Pi_l\mathbf{v}\|_{L^2(K)} \leq C\left\{\sum_{i+j+l=k+1} h_1^i h_2^j h_3^l \left\|\frac{\partial^{k+1}\mathbf{v}}{\partial\xi_1^i\partial\xi_2^j\partial\xi_3^l}\right\|_{L^2(K)} + h_K^{k+1}\|D^k \operatorname{div}\mathbf{v}\|_{L^2(T)}\right\}$$

with a constant C depending only on $\bar{c}$.

Proof. From [1, Lemma 5.2] there exists a constant C depending only on $\bar{c}$ such that

$$\|\mathbf{v}-\Pi_k\mathbf{v}\|_{L^2(K)} \leq C\sum_{s=1}^{3}\|(\mathbf{v}-\Pi_k\mathbf{v})\cdot\mathbf{n}_s\|_{L^2(K)}.$$

We can apply to $(\mathbf{v}-\Pi_l\mathbf{v})\cdot\mathbf{n}_s$ a straightforward generalization of Lemma 2.1 to obtain

$$\|\mathbf{v}-\Pi_k\mathbf{v}\|_{L^2(K)} \leq C\sum_{i+j+l=k+1} h_1^i h_2^j h_3^l \left\|\frac{\partial^{k+1}(\mathbf{v}-\Pi_k\mathbf{v})}{\partial\xi_1^i\partial\xi_2^j\partial\xi_3^l}\right\|_{L^2(K)}. \tag{3.21}$$

Hence, we have to prove

$$\sum_{i+j+l=k+1}\left\|\frac{\partial^{k+1}\Pi_k\mathbf{v}}{\partial\xi_1^i\partial\xi_2^j\partial\xi_3^l}\right\|_{L^2(K)} \leq C\left\|D^k\operatorname{div}\mathbf{v}\right\|_{L^2(K)}.$$

Without loss of generality, we can assume that K has vertices $(0,0,0)$, $(h_1,0,0)$, $(a_2,b_2,0)$ and (a_3,b_3,c_3) and that it satisfies the definition of the regular vertex property with $\mathbf{Z}=(0,0,0)$, $\xi_1=(1,0,0)$, $\xi_2=\frac{1}{h_2}(a_2,b_2,0)$ and $\xi_3=\frac{1}{h_3}(a_3,b_3,c_3)$.

The linear transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = B \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix}$$

with

$$B = \begin{pmatrix} 1 & \frac{a_2}{h_2} & \frac{a_3}{h_3} \\ 0 & \frac{b_2}{h_2} & \frac{b_3}{h_3} \\ 0 & 0 & \frac{c_3}{h_3} \end{pmatrix}$$

maps $\widetilde{K}$ onto K.

Moreover, it is easy to check that

$$\|B\|_2 \leq 3\sqrt{3}.$$

For $\mathbf{v} \in H^{k+1}(K)^3$ we can consider $\tilde{\mathbf{v}} \in H^{k+1}(\widetilde{K})^3$ defined by the Piola Transform

$$\mathbf{v} = \frac{1}{|\det B|} B\,\tilde{\mathbf{v}}.$$

Then, as in the proof of Theorem 2.15, we have

$$\left\| \frac{\partial^{k+1}\Pi_k \mathbf{v}}{\partial\xi_1^i \partial\xi_2^j \partial\xi_3^l} \right\|_{L^2(K)} \leq \frac{\|B\|_2}{|\det B|^{\frac{1}{2}}} \left\| \frac{\partial^{k+1}\widetilde{\Pi}_l \tilde{\mathbf{v}}}{\partial\tilde{x}^i \partial\tilde{y}^j \partial\tilde{z}^l} \right\|_{L^2(\widetilde{K})}.$$

But, as in the 2d case, for any tetrahedral element T we can write the derivatives of order $k+1$ of $\mathbf{u} \in \mathcal{RT}_k(T)$ in terms of the derivatives of its divergence, namely,

$$\begin{aligned} \frac{\partial^{k+1}\mathbf{u}}{\partial x^{k+1}} &= \Big(\frac{k+1}{k+3} \frac{\partial^k(\operatorname{div}\mathbf{u})}{\partial x^k}, 0, 0 \Big) \\ \frac{\partial^{k+1}\mathbf{u}}{\partial y^{k+1}} &= \Big(0, \frac{k+1}{k+3} \frac{\partial^k(\operatorname{div}\mathbf{u})}{\partial y^k}, 0 \Big) \\ \frac{\partial^{k+1}\mathbf{u}}{\partial z^{k+1}} &= \Big(0, 0, \frac{k+1}{k+3} \frac{\partial^k(\operatorname{div}\mathbf{u})}{\partial z^k} \Big) \end{aligned}$$

and for $i+j+l = k+1$, with $i > 0$, $j > 0$ and $l > 0$,

$$\frac{\partial^{k+1}\mathbf{u}}{\partial x^i \partial y^j \partial z^l} = \Big(\frac{i}{k+3} \frac{\partial^l \operatorname{div}\mathbf{u}}{\partial x^{i-1} \partial y^j \partial z^l}, \frac{j}{k+3} \frac{\partial^l \operatorname{div}\mathbf{u}}{\partial x^i \partial y^{j-1} \partial z^l}, \frac{l}{k+3} \frac{\partial^k \operatorname{div}\mathbf{u}}{\partial x^i \partial y^j \partial z^{l-1}} \Big).$$

We conclude, as in the last part of the proof of Theorem 2.5, by using the commutative diagram property of $\widetilde{\Pi}_k$ and by observing that Lemma 2.3 has a straightforward extension to the element $\widetilde{K}$. $\square$

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