FINITE ELEMENT APPROXIMATION OF CONVECTION DIFFUSION PROBLEMS USING GRADED MESHES

RICARDO G. DURÁN AND ARIEL L. LOMBARDI

Abstract. We consider the numerical approximation of a model convection-diffusion equation by standard bilinear finite elements.

Using appropriately graded meshes we prove optimal order error estimates in the ε-weighted H^1-norm valid uniformly, up to a logarithmic factor, in the singular perturbation parameter.

Finally, we present some numerical examples showing the good behavior of our method.

1. Introduction

As is well known, the numerical approximation of convection-diffusion equations requires some special treatment in order to obtain good results when the problem is convection dominated due to the presence of boundary or interior layers. A lot of work has been done in this direction (see for example the books [4, 5] and their references). There are in principle two ways to proceed: to use some kind of upwind or to use adapted meshes appropriately refined. This last possibility seems very reasonable when the layers are due to the boundary conditions and so, their location is known a priori.

In this paper we analyze the approximation of the solution of a model convection-diffusion equation. We prove that, using appropriate graded meshes, the solution is well approximated by the standard piecewise bilinear finite element method in the ε-weighted H^1-norm $\| \,.\, \|_\varepsilon$ defined as

$$\|v\|_\varepsilon^2 = \|v\|_{L^2(\Omega)}^2 + \varepsilon \|\nabla v\|_{L^2(\Omega)}^2.$$

Precisely, we consider the problem

$$\begin{aligned} -\varepsilon \Delta u + b \cdot \nabla u + cu &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

where $\Omega = (0,1)^2$ and $\varepsilon > 0$ is a small parameter. We prove that, on appropriate graded meshes,

$$\|u - u_h\|_\varepsilon \leq C \frac{(\log(1/\varepsilon))^2}{\sqrt{N}},$$

where u_h is the standard piecewise bilinear approximation of u on a graded mesh $\mathcal{T}_h$ (where $h > 0$ is a parameter arising in the definition of the mesh), N denotes the number of nodes, and C is a constant independent of ε and N.

Observe that this error estimate is almost optimal, i. e., the order with respect to the number of nodes is the same as that obtained for a smooth function on uniform meshes and, up to a logarithmic factor, the estimate is valid uniformly in the perturbation parameter.

Consequently, the graded meshes seems an interesting alternative to the well known Shishkin meshes which provide also optimal order [6]. Indeed, from some numerical experiments the graded meshes procedure seems to be more robust in the sense that the numerical results are not strongly affected by variations of parameters defining the meshes.

The rest of the paper is organized as follows. In Section 2 we introduce the graded meshes and prove the error estimates and in Section 3 we present some numerical results. We end the paper with some conclusions.

1991 *Mathematics Subject Classification.* 65N30.

Key words and phrases. Convection-diffusion, graded meshes, finite elements.

Supported by ANPCyT under grant PICT 03-05009, by Universidad de Buenos Aires under grant X052 and by Fundación Antorchas. The first author is a member of CONICET, Argentina.

2. Error estimates

In this section we consider the numerical approximation of problem (1.1). We assume that the functions $b = (b_1, b_2), c$ and f are smooth on $\overline{\Omega}$, and that

$$b_i < -\gamma \qquad \text{with} \qquad \gamma > 0 \qquad \text{for} \quad i = 1, 2. \tag{2.1}$$

Then, the solution will have a boundary layer of width $O(\varepsilon \log \frac{1}{\varepsilon})$ at the outflow boundary $\{(x_1, x_2) \in \partial\Omega : x_1 = 0 \text{ or } x_2 = 0\}$ [5]. We also assume the following compatibility conditions

$$f(0,0) = f(1,0) = f(1,1) = f(0,1) = 0$$
$$\frac{\partial^{i+j} f}{\partial x_1^i \partial x_2^j}(1,1) = 0 \qquad \text{for } 0 \le i + j \le 2.$$

Under these conditions it can be proved that $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ and precise estimates of the derivatives that will be useful for our purposes are known.

Using the graded meshes defined below we will obtain an almost optimal estimate in the ε-weighted H^1-norm for the error between the solution of Problem (1.1) and its standard piecewise bilinear finite element approximation.

Given a parameter $h > 0$ and a constant $\sigma > 0$, we introduce the partition $\{\xi_i\}_{i=0}^M$ of the interval $[0, 1]$ given by

$$\begin{cases} \xi_0 = 0 \\ \xi_1 = \sigma h \varepsilon \\ \xi_{i+1} = \xi_i + \sigma h \xi_i \qquad \text{for} \quad 1 \le i \le M-2 \\ \xi_M = 1 \end{cases} \tag{2.2}$$

where M is such that $\xi_{M-1} < 1$ and $\xi_{M-1} + \sigma h \xi_{M-1} \ge 1$. We assume that the last interval $(\xi_{M-1}, 1)$ is not too small in comparison with the previous one (ξ_{M-2}, ξ_{M-1}) (if this is not the case we just eliminate the node ξ_{M-1}).

In practice it is natural to take $h_i := \xi_i - \xi_{i-1}$ to be monotonically increasing. Therefore, it is convenient to modify the partition by taking $h_i = h_1$ for i such that $\xi_{i-1} < \varepsilon$ and starting with the graded mesh after that. In this way we obtain the following alternative partition,

$$\begin{cases} \xi_0 = 0 \\ \xi_i = i\sigma h \varepsilon \qquad \text{for} \quad 1 \le i < \frac{1}{\sigma h} + 1 \\ \xi_{i+1} = \xi_i + \sigma h \xi_i \qquad \text{for} \quad \frac{1}{\sigma h} + 1 \le i \le M-2 \\ \xi_M = 1 \end{cases} \tag{2.3}$$

with M as in the other case.

For any of these choices of ξ_i we introduce the partitions $\mathcal{T}_h$ of Ω defined as

$$\mathcal{T}_h = \{R_{ij}\}_{i,j=1}^M,$$

where $R_{ij} = (\xi_{i-1}, \xi_i) \times (\xi_{j-1}, \xi_j)$.

Associated with $\mathcal{T}_h$ we introduce the standard piecewise bilinear finite element space V_h and its corresponding Lagrange interpolation operator Π.

First, we have to prove some weighted *a priori* estimates for the solution u. The following pointwise estimates are immediate consequences of Theorem 2.1 of [6]:

$$\left| \frac{\partial^k u}{\partial x_1^k}(x_1, x_2) \right| \le C \left(1 + \frac{1}{\varepsilon^k} e^{-\frac{\gamma x_1}{\varepsilon}} \right) \qquad 0 \le k \le 2, \tag{2.4}$$

$$\left| \frac{\partial^k u}{\partial x_2^k}(x_1, x_2) \right| \le C \left(1 + \frac{1}{\varepsilon^k} e^{-\frac{\gamma x_2}{\varepsilon}} \right) \qquad 0 \le k \le 2, \tag{2.5}$$

$$\left| \frac{\partial^2 u}{\partial x_1 \partial x_2}(x_1, x_2) \right| \le C \left(1 + \frac{1}{\varepsilon} e^{-\frac{\gamma x_1}{\varepsilon}} + \frac{1}{\varepsilon} e^{-\frac{\gamma x_2}{\varepsilon}} + \frac{1}{\varepsilon^2} e^{-\frac{\gamma x_1}{\varepsilon}} e^{-\frac{\gamma x_2}{\varepsilon}} \right). \tag{2.6}$$

for all $(x_1, x_2) \in \Omega$ and $0 \le k \le 2$.

For the proof of our error estimates we will need to decompose $\overline{\Omega}$ as $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \overline{\Omega}_3$, where

$$
\begin{aligned}
\overline{\Omega}_1 &= \bigcup\left\{\overline{R}_{ij} : \xi_{i-1} < c_1\varepsilon\log\frac{1}{\varepsilon}\right\}, \\
\overline{\Omega}_2 &= \bigcup\left\{\overline{R}_{ij} : \xi_{i-1} \geq c_1\varepsilon\log\frac{1}{\varepsilon},\ \xi_{j-1} < c_1\varepsilon\log\frac{1}{\varepsilon}\right\}, \\
\overline{\Omega}_3 &= \bigcup\left\{\overline{R}_{ij} : \xi_{i-1} \geq c_1\varepsilon\log\frac{1}{\varepsilon},\ \xi_{j-1} \geq c_1\varepsilon\log\frac{1}{\varepsilon}\right\},
\end{aligned}
\tag{2.7}
$$

with a constant $c_1 > 1/2\gamma$.

As a consequence of the estimates (2.4), (2.5), and (2.6) we obtain the following lemma.

Lemma 2.1. *There exists a constant C such that we have the following a priori estimates*

$$\varepsilon^{\frac{3}{2}}\left\|\frac{\partial^2 u}{\partial x_1^2}\right\|_{L^2(\Omega)} \leq C \qquad \text{and} \qquad \varepsilon^{\frac{3}{2}}\left\|\frac{\partial^2 u}{\partial x_2^2}\right\|_{L^2(\Omega)} \leq C, \tag{2.8}$$

$$\varepsilon^{\frac{1}{2}}\left\|x_1\frac{\partial^2 u}{\partial x_1^2}\right\|_{L^2(\Omega)} \leq C \qquad \text{and} \qquad \varepsilon^{\frac{1}{2}}\left\|x_2\frac{\partial^2 u}{\partial x_2^2}\right\|_{L^2(\Omega)} \leq C, \tag{2.9}$$

$$\left\|x_1^{\frac{3}{2}}\frac{\partial^2 u}{\partial x_1^2}\right\|_{L^2(\Omega)} \leq C \qquad \text{and} \qquad \left\|x_2^{\frac{3}{2}}\frac{\partial^2 u}{\partial x_2^2}\right\|_{L^2(\Omega)} \leq C, \tag{2.10}$$

$$\varepsilon^{\frac{1}{2}}\left\|x_1^{\frac{1}{2}}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(\Omega)} \leq C \qquad \text{and} \qquad \varepsilon^{\frac{1}{2}}\left\|x_2^{\frac{1}{2}}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(\Omega)} \leq C, \tag{2.11}$$

$$\varepsilon\left\|\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(\Omega)} \leq C, \tag{2.12}$$

$$\left\|x_i\frac{\partial^2 u}{\partial x_i\partial x_j}\right\|_{L^2(\Omega_3)} \leq C \qquad \text{if} \quad c_1 > 1/2\gamma \quad , \quad i,j = 1,2. \tag{2.13}$$

Proof. Let us prove for example the first inequality in (2.11). From (2.6) we have,

$$
\begin{aligned}
\int_\Omega x_1\left|\frac{\partial^2 u}{\partial x_1\partial x_2}\right|^2 dx_1\,dx_2 &\leq C\int_0^1\int_0^1 x_1\left(1+\frac{1}{\varepsilon^2}e^{-\frac{2\gamma x_1}{\varepsilon}}+\frac{1}{\varepsilon^2}e^{-\frac{2\gamma x_2}{\varepsilon}}+\frac{1}{\varepsilon^4}e^{-\frac{2\gamma x_1}{\varepsilon}}e^{-\frac{2\gamma x_2}{\varepsilon}}\right)dx_1\,dx_2 \\
&\leq C\left(\frac{1}{2}+\int_0^\infty se^{-2\gamma s}\,ds+\frac{1}{\varepsilon}\int_0^\infty e^{-2\gamma s}\,ds+\right. \\
&\qquad\left.+\frac{1}{\varepsilon}\int_0^\infty se^{-2\gamma s}\,ds\int_0^\infty e^{-2\gamma s}\,ds\right) \\
&\leq C\left(1+\frac{1}{\varepsilon}\right),
\end{aligned}
$$

obtaining the desired inequality.

For inequality (2.13) with $i=1, j=2$ we have also from (2.6)

$$
\begin{aligned}
\int_{\Omega_3} x_1^2\left|\frac{\partial^2 u}{\partial x_1\partial x_2}\right|^2 dx_1\,dx_2 &\leq \int_{c_1\varepsilon\log\frac{1}{\varepsilon}}^1\int_{c_1\varepsilon\log\frac{1}{\varepsilon}}^1 x_1^2\left(1+\frac{1}{\varepsilon^2}e^{-\frac{2\gamma x_1}{\varepsilon}}+\frac{1}{\varepsilon^2}e^{-\frac{2\gamma x_2}{\varepsilon}}+\right. \\
&\qquad\left.+\frac{1}{\varepsilon^4}e^{-\frac{2\gamma x_1}{\varepsilon}}e^{-\frac{2\gamma x_2}{\varepsilon}}\right)dx_1\,dx_2 \\
&\leq C\left[\frac{1}{3}+\frac{2}{\varepsilon}\int_{c_1\log\frac{1}{\varepsilon}}^\infty e^{-2\gamma s}\,ds+\left(\frac{1}{\varepsilon}\int_{c_1\log\frac{1}{\varepsilon}}^\infty e^{-2\gamma s}\,ds\right)^2\right] \\
&\leq C,
\end{aligned}
$$

if $c_1 \geq 1/2\gamma$.

For inequality (2.13) with $i = j = 1$ we have from the estimate (2.4) that

$$\begin{aligned}\int_{\Omega_3} x_1^2 \left|\frac{\partial^2 u}{\partial x_1^2}\right|^2 dx_1\, dx_2 &\le C \int_{c_1\varepsilon \log\frac{1}{\varepsilon}}^1 x_1^2 \left(1 + \frac{1}{\varepsilon^4} e^{-\frac{2\gamma x_1}{\varepsilon}}\right) dx_1 \\ &\le C \left(\frac{1}{3} + \frac{1}{\varepsilon}\int_{c_1 \log\frac{1}{\varepsilon}}^\infty s^2 e^{-2\gamma s}\, ds\right) \\ &= C \left[\frac{1}{3} + \frac{1}{8\gamma^3}\left(4\gamma^2 c_1^2 \log^2\varepsilon + 4\gamma \log\frac{1}{\varepsilon} + 2\right)\varepsilon^{2\gamma c_1 - 1}\right] \\ &\le C\end{aligned}$$

whenever $c_1 > 1/2\gamma$.

The other inequalities follow by similar arguments and so we omit the details. □

We can now prove the error estimates for the Lagrange interpolation on our graded meshes.

Theorem 2.1. *Let u be the solution of Problem (1.1). If $\mathcal{T}_h$ are the meshes given by the partitions (2.2) or (2.3) then,*

$$\|u - \Pi u\|_{L^2(\Omega)} \le Ch^2 \qquad \text{and,} \qquad \varepsilon^{\frac{1}{2}} \|\nabla(u - \Pi u)\|_{L^2(\Omega)} \le Ch$$

with a constant C independent of h and ε. In particular,

$$\|u - \Pi u\|_\varepsilon \le Ch. \tag{2.14}$$

Proof. For simplicity we give the proof for the case of the partitions given in (2.2). However, it is not difficult to see that the same arguments apply in the other case.

Recalling that $R_{ij} = (\xi_{i-1}, \xi_i) \times (\xi_{j-1}, \xi_j)$ for $1 \le i, j \le N$ and $h_i = \xi_i - \xi_{i-1}$ we have, from standard error estimates (see for example [2]),

$$\|u - \Pi u\|_{L^2(R_{ij})} \le C \left\{ h_i^2 \left\|\frac{\partial^2 u}{\partial x_1^2}\right\|_{L^2(R_{ij})} + h_j^2 \left\|\frac{\partial^2 u}{\partial x_2^2}\right\|_{L^2(R_{ij})} \right\}. \tag{2.15}$$

Now, we decompose the error as

$$\|u - \Pi u\|_{L^2(\Omega)}^2 = \sum_{j=1}^N \|u - \Pi u\|_{L^2(R_{1j})}^2 + \sum_{i=2}^N \|u - \Pi u\|_{L^2(R_{i1})}^2 + \sum_{i,j=2}^N \|u - \Pi u\|_{L^2(R_{ij})}^2.$$

Then, using (2.15) and the definition of the mesh we have

$$\|u - \Pi u\|_{L^2(R_{11})}^2 \le Ch^4 \left\{ \varepsilon^4 \left\|\frac{\partial^2 u}{\partial x_1^2}\right\|_{L^2(R_{11})}^2 + \varepsilon^4 \left\|\frac{\partial^2 u}{\partial x_2^2}\right\|_{L^2(R_{11})}^2 \right\},$$

$$\|u - \Pi u\|_{L^2(R_{1j})}^2 \le Ch^4 \left\{ \varepsilon^4 \left\|\frac{\partial^2 u}{\partial x_1^2}\right\|_{L^2(R_{1j})}^2 + \left\|x_2^2 \frac{\partial^2 u}{\partial x_2^2}\right\|_{L^2(R_{1j})}^2 \right\} \qquad \text{for } j \ge 2,$$

$$\|u - \Pi u\|_{L^2(R_{ij})}^2 \le Ch^4 \left\{ \left\|x_1^2 \frac{\partial^2 u}{\partial x_1^2}\right\|_{L^2(R_{ij})}^2 + \left\|x_2^2 \frac{\partial^2 u}{\partial x_2^2}\right\|_{L^2(R_{ij})}^2 \right\} \qquad \text{for } i, j \ge 2,$$

and

$$\|u - \Pi u\|_{L^2(R_{i1})}^2 \le Ch^4 \left\{ \left\|x_1^2 \frac{\partial^2 u}{\partial x_1^2}\right\|_{L^2(R_{i1})}^2 + \varepsilon^4 \left\|\frac{\partial^2 u}{\partial x_2^2}\right\|_{L^2(R_{i1})}^2 \right\} \qquad \text{for } i \ge 2,$$

and therefore, putting all together and using the a priori estimates (2.8) and (2.10) we obtain

$$\|u - \Pi u\|_{L^2(\Omega)} \le Ch^2.$$

Now, to bound the other part of the norm we use the known estimate (see for example [1]),

$$\left\|\frac{\partial(u - \Pi u)}{\partial x_1}\right\|_{L^2(R_{ij})} \le C \left\{ h_i \left\|\frac{\partial^2 u}{\partial x_1^2}\right\|_{L^2(R_{ij})} + h_j \left\|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right\|_{L^2(R_{ij})} \right\}. \tag{2.16}$$

Then, proceeding as in the case of the L^2 norm we can easily obtain

$$\left\| \frac{\partial(u-\Pi u)}{\partial x_1} \right\|^2_{L^2(R_{11})} \le Ch^2 \left\{ \varepsilon^2 \left\| \frac{\partial^2 u}{\partial x_1^2} \right\|^2_{L^2(R_{11})} + \varepsilon^2 \left\| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|^2_{L^2(R_{11})} \right\},$$

$$\left\| \frac{\partial(u-\Pi u)}{\partial x_1} \right\|^2_{L^2(R_{1j})} \le Ch^2 \left\{ \varepsilon^2 \left\| \frac{\partial^2 u}{\partial x_1^2} \right\|^2_{L^2(R_{1j})} + \left\| x_2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|^2_{L^2(R_{1j})} \right\} \qquad \text{for } j \ge 2,$$

$$\left\| \frac{\partial(u-\Pi u)}{\partial x_1} \right\|^2_{L^2(R_{ij})} \le Ch^2 \left\{ \left\| x_1 \frac{\partial^2 u}{\partial x_1^2} \right\|^2_{L^2(R_{ij})} + \left\| x_2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|^2_{L^2(R_{ij})} \right\} \qquad \text{for } i,j \ge 2,$$

and

$$\left\| \frac{\partial(u-\Pi u)}{\partial x_1} \right\|^2_{L^2(R_{i1})} \le Ch^2 \left\{ \left\| x_1 \frac{\partial^2 u}{\partial x_1^2} \right\|^2_{L^2(R_{i1})} + \varepsilon^2 \left\| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|^2_{L^2(R_{i1})} \right\} \qquad \text{for } i \ge 2, .$$

Therefore, multiplying by ε, summing up, and using the a priori estimates (2.8), (2.9), (2.11), and (2.12), we obtain

$$\varepsilon^{\frac{1}{2}} \left\| \frac{\partial(u-\Pi u)}{\partial x_1} \right\|_{L^2(\Omega)} \le Ch.$$

Clearly, a similar estimate holds for $\frac{\partial(u-\Pi u)}{\partial x_2}$ and so, the theorem is proved. □

Now we consider the numerical approximation of problem (1.1). The weak form of this problem consists in finding $u \in H_0^1(\Omega)$ such that

$$a(u,v) = \int_\Omega fv \, dx \qquad \forall v \in H_0^1(\Omega),$$

where

$$a(v,w) = \int_\Omega (\varepsilon \nabla v \nabla w + b \cdot \nabla v \, w + c \, vw) \, dx.$$

Assume that there exists a constant μ independent of ε such that

$$c - \frac{\operatorname{div} b}{2} \ge \mu > 0. \tag{2.17}$$

Observe that, as pointed out in [5, page 67], this is not an important restriction because, under the assumption (2.1), a change of variable $u \to e^{\eta \cdot x} v$, for suitable chosen η, leads to a problem satisfying (2.17).

It is known that the bilinear form a is coercive in the ε-weighted H^1-norm uniformly in ε [5], i. e., there exists $\beta > 0$, independent of ε, such that

$$\beta \|v\|_\varepsilon \le a(v,v) \qquad \forall v \in H_0^1(\Omega). \tag{2.18}$$

However, the continuity of a is not uniform in ε and this is why the standard theory based on Cea's lemma can not be applied to obtain error estimates valid uniformly in ε.

The finite element approximation $u_h \in V_h$ is given by

$$a(u_h,v) = \int_\Omega fv \, dx \qquad \forall v \in V_h.$$

In the following theorem we prove an almost optimal error estimate in the ε-weighted H^1-norm. The constants will depend on the coercivity constant β, on the constants in the estimates given in Lemma 2.1, on the L^∞- norms of b and c, on the constant σ introduced in the definition of the meshes, and on the constant γ appearing in the estimates (2.4), (2.5) and (2.6) (also on the c_1 used in the partition of Ω introduced in (2.7), but this constant depends only on γ). We will not state all these dependencies explicitly.

Theorem 2.2. *Let u be the solution of Problem (1.1) and $u_h \in V_h$ its finite element approximation. If $\mathcal{T}_h$ are the meshes given by the partitions (2.2) or (2.3) then,*

$$\|u - u_h\|_\varepsilon \le Ch \log \frac{1}{\varepsilon} \tag{2.19}$$

with a constant C independent of h and ε.

Proof. Let $e = u_h - \Pi u \in V_h$. In view of Theorem 2.1 it is enough to bound the norm of e. Again, we consider only the case of $\mathcal{T}_h$ given by (2.2). In the other case the result can be obtained by simple modifications.

From (2.18) and using the error equation $a(u - u_h, e) = 0$ we have

$$\beta \|e\|_\varepsilon^2 \le a(e,e) = a(u - \Pi u, e) \le C \left\{ \|u - \Pi u\|_\varepsilon \|e\|_\varepsilon + \int_\Omega b \cdot \nabla(u - \Pi u) e \right\}$$
$$\le C\|u - \Pi u\|_\varepsilon^2 + \frac{\beta}{2}\|e\|_\varepsilon^2 + C \int_\Omega b \cdot \nabla(u - \Pi u)e,$$

where we have used the generalized arithmetic-geometric mean inequality. Then, it is enough to prove that

$$\left| \int_\Omega b \cdot \nabla(u - \Pi u)e \right| \le Ch^2 \log^2 \frac{1}{\varepsilon} + \delta \|e\|_\varepsilon^2 \tag{2.20}$$

for a small δ to be chosen.

To prove this estimate we use the decomposition of Ω introduced in (2.7). Since e vanishes at the boundary, we know from the Poincaré inequality that

$$\|e\|_{L^2(\Omega_1)} \le C\varepsilon \log \frac{1}{\varepsilon} \left\| \frac{\partial e}{\partial x_1} \right\|_{L^2(\Omega_1)}.$$

Therefore, since b is bounded, we have

$$\int_{\Omega_1} |b \cdot \nabla(u - \Pi u)e| \le C\|\nabla(u - \Pi u)\|_{L^2(\Omega_1)} \varepsilon \log \frac{1}{\varepsilon} \|\nabla e\|_{L^2(\Omega_1)}$$
$$\le C\varepsilon \log^2 \frac{1}{\varepsilon} \|\nabla(u - \Pi u)\|_{L^2(\Omega_1)}^2 + \delta\varepsilon \|\nabla e\|_{L^2(\Omega_1)}^2$$

and so, using Theorem 2.1, we obtain

$$\int_{\Omega_1} |b \cdot \nabla(u - \Pi u)e| \le Ch^2 \log^2 \frac{1}{\varepsilon} + \delta\varepsilon \|\nabla e\|_{L^2(\Omega_1)}^2. \tag{2.21}$$

Clearly, the same argument can be applied to obtain an analogous estimate in Ω_2.

Finally, for $R_{ij} \subset \Omega_3$, we have shown in the proof of Theorem 2.1 that

$$\left\| \frac{\partial(u - \Pi u)}{\partial x_1} \right\|_{L^2(R_{ij})}^2 \le Ch^2 \left\{ \left\| x_1 \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2(R_{ij})}^2 + \left\| x_2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(R_{ij})}^2 \right\},$$

and

$$\left\| \frac{\partial(u - \Pi u)}{\partial x_2} \right\|_{L^2(R_{ij})}^2 \le Ch^2 \left\{ \left\| x_1 \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(R_{ij})}^2 + \left\| x_2 \frac{\partial^2 u}{\partial x_2^2} \right\|_{L^2(R_{ij})}^2 \right\},$$

and so, using (2.13), we obtain

$$\int_{\Omega_3} |b \cdot \nabla(u - \Pi u)e| \le Ch^2 + \delta \|e\|_{L^2(\Omega_3)}^2.$$

Therefore, putting together the estimates in Ω_1, Ω_2, and Ω_3 we obtain (2.20), and the proof concludes by choosing δ small enough depending on the coercivity constant β. □

To show that the error estimate is almost optimal we have to restate the inequality (2.19) in terms of the number of nodes in the mesh. This is the objective of the following corollary. As we mentioned before, the use of the partition given in (2.2) may produce too small intervals in the boundary layer region increasing the number of nodes in an unnecessary way. Therefore, in practice it is more natural to use the meshes based on the partitions given in (2.3) and so, we will consider only this case.

Corollary 2.3. *Let u be the solution of Problem (1.1) and $u_h \in V_h$ its finite element approximation. If $\mathcal{T}_h$ are the meshes given by the partitions (2.3) and N is the number of nodes in $\mathcal{T}_h$ then,*

$$\|u - u_h\|_\varepsilon \le C \frac{\log^2(1/\varepsilon)}{\sqrt{N}}$$

with a constant C independent of h and ε.

Proof. We have to show that

$$h \leq C\frac{\log(1/\varepsilon)}{\sqrt{N}}.$$

Let M_1 be the number of points ξ_i in the partition given in (2.3) such that $\xi_i \leq \varepsilon$ and M_2 be the number of points such that $\xi_i > \varepsilon$.

Clearly M_1 is bounded by C/h. To bound M_2, let us call ξ_k the smallest of the points such that $\xi_i > \varepsilon$. Assuming $\sigma h \leq 1$ we have

$$M_2 - 2 = \sum_{i=k}^{M-2} (\xi_{i+1} - \xi_i)^{-1} \int_{\xi_i}^{\xi_{i+1}} d\xi = \sum_{i=k}^{M-2} (\sigma h \xi_i)^{-1} \int_{\xi_i}^{\xi_{i+1}} d\xi \leq \sum_{i=k}^{M-2} 2(\sigma h \xi_{i+1})^{-1} \int_{\xi_i}^{\xi_{i+1}} d\xi$$
$$= \frac{2}{\sigma h} \sum_{i=k}^{M-2} \int_{\xi_i}^{\xi_{i+1}} \xi^{-1} d\xi \leq \frac{2}{\sigma h} \int_{\varepsilon}^{1} \xi^{-1} d\xi = \frac{2}{\sigma h} \log\frac{1}{\varepsilon}.$$

Therefore, the proof concludes by recalling that $M = M_1 + M_2$ and $N = M^2$. □

Observe that the estimate given in the corollary is almost optimal. Indeed, the order is the same as that obtained in the approximation of a smooth function by piecewise bilinear elements on a uniform mesh and the factor $\log(1/\varepsilon)^2$ is not significant in practice.

Remark 2.4. Some slight variations of the meshes $\mathcal{T}_h$ could be more convenient. For example, the following grading giving a lower number of nodes can be used,

$$\xi_{i+1} = \xi_i + \sigma h \xi_i^{\alpha} \qquad \text{with} \qquad \alpha = 1 - \frac{1}{\log\frac{1}{\varepsilon}}.$$

Of course, also in this case we can take a uniform partition at the beginning and start with the grading after ε.

With this choice of α the same error estimates can be proved by a simple modification of our arguments assuming that $\alpha > 1/2$ (which is valid for small ε), using now the estimates

$$\varepsilon^{\frac{1}{2}} \left\| x_1^{\alpha} \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2(\Omega)} \leq C\varepsilon^{\alpha-1} \qquad , \qquad \varepsilon^{\frac{1}{2}} \left\| x_2^{\alpha} \frac{\partial^2 u}{\partial x_2^2} \right\|_{L^2(\Omega)} \leq C\varepsilon^{\alpha-1},$$

and

$$\left\| x_i \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L^2(\Omega_3)} \leq C \qquad \text{if} \quad c_1 > 3/2\gamma.$$

which can be proved using the same arguments of Lemma 2.1, and observing that $\varepsilon^{\alpha-1} = e^{-1}$.

3. Numerical experiments

In this section we present some numerical examples which show the good behavior of the standard piecewise bilinear finite element method on graded meshes for convection-diffusion equations. Although, for simplicity, we have restricted the analysis in the previous section to Dirichlet boundary conditions, in our examples we have considered more general boundary conditions. Also, we have included two examples which do not satisfy condition (2.17) (Examples (1) and (2)) to test the method in cases not covered by the theory.

We have solved the problem

$$\begin{aligned} -\varepsilon\Delta u + b\cdot\nabla u + cu &= f \qquad \text{in } \Omega \\ u &= u_D \quad \text{in } \Gamma_D \\ \frac{\partial u}{\partial n} &= g \qquad \text{in } \Gamma_N, \end{aligned}$$

with different choices of coefficients b and c, and data u_D and g, namely,

(1) $b = (0, -1)$, $c = 0$, $\Gamma_D = [0, 1] \times \{0, 1\}$, $\Gamma_N = \{0, 1\} \times [0, 1]$, $u_D = 0$, $g = 0$ and $f = 1$,

(2) $b = (0, -1)$, $c = 0$, $\Gamma_D = [0, 1] \times \{0, 1\}$, $\Gamma_N = \{0, 1\} \times [0, 1]$, $u_D = 0$ on $\{0\} \times [0, 1]$ and $u_D = 1$ on $\{1\} \times [0, 1]$, $g = 0$ and $f = 0$,

(3) $b = (-\frac{1}{2}, -1)$, $c = 2$, $\Gamma_D = \partial\Omega$, $u_D = 0$, and $f = 1$,

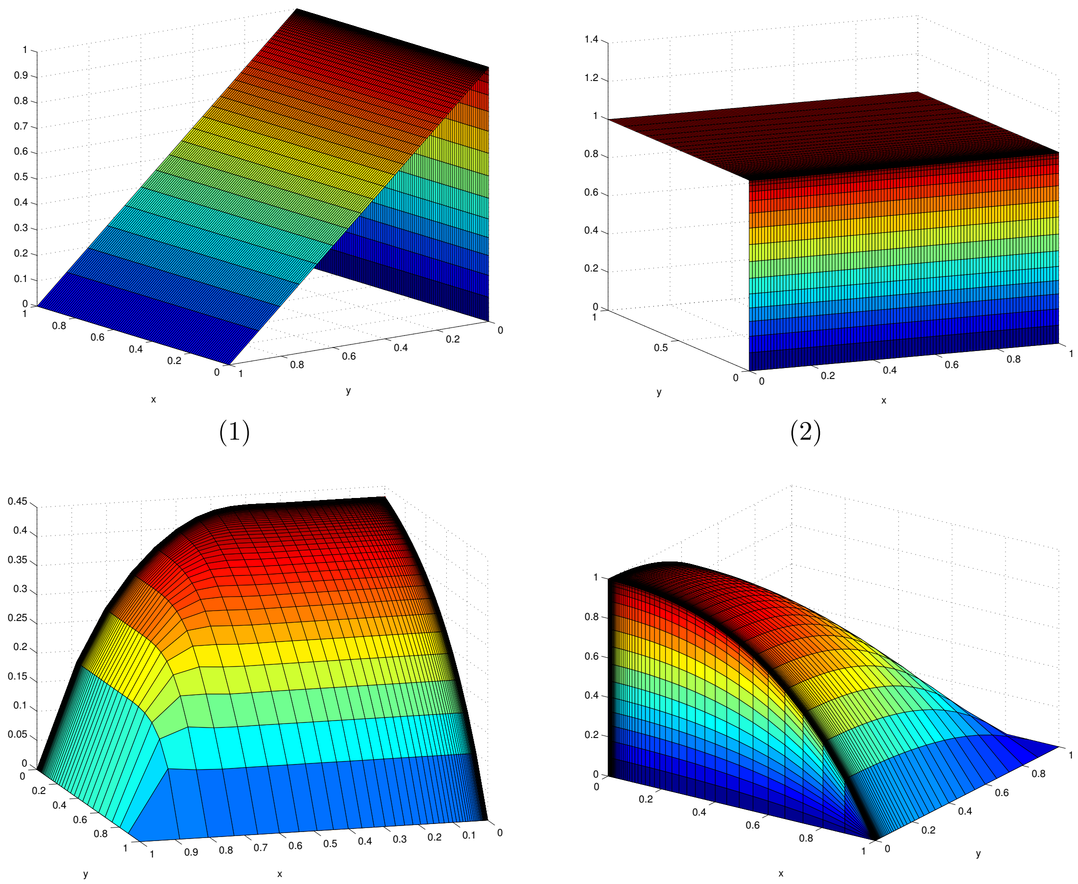
(4) $b = (1-2\varepsilon)(-1,-1)$, $c = 2(1-\varepsilon)$, $\Gamma_D = \partial\Omega$, $u_D = 0$ and

$$f(x,y) = -\left[x - \left(\frac{1-e^{-\frac{x}{\varepsilon}}}{1-e^{-\frac{1}{\varepsilon}}}\right) + y - \left(\frac{1-e^{-\frac{y}{\varepsilon}}}{1-e^{-\frac{1}{\varepsilon}}}\right)\right] e^{x+y}.$$

For the examples (1) and (2), which present a boundary layer only on $0 \times [0,1]$ we have used meshes graded only along the x_1 axis.

We have made several experiments using the gradings giving in (2.2) and (2.3) and the variant explained in Remark 2.4. Also, we have considered different values of the constant σ appearing in the definition of the meshes and we have observed that the results do not change significantly for values of σ of the order of 1, therefore we have taken $\sigma = 1$ for our examples.

No significant differences were observed with the different choices of meshes and, in all cases, no oscillations in the approximate solutions have been observed (see Figure 1 where the numerical solutions obtained in the four cases for $\varepsilon = 10^{-6}$ are shown). Therefore, we will show only results obtained with the meshes graded with $\alpha = 1 - \frac{1}{\log\frac{1}{\varepsilon}}$ defined in Remark 2.4 and starting the grading after ε.



(1) (2) (3) (4)

FIGURE 1

For the example (4) we know the exact solution which is given by

$$u(x,y) = \left[\left(x - \frac{1-e^{-\frac{x}{\varepsilon}}}{1-e^{-\frac{1}{\varepsilon}}}\right)\left(y - \frac{1-e^{-\frac{y}{\varepsilon}}}{1-e^{-\frac{1}{\varepsilon}}}\right)\right] e^{x+y},$$

therefore, we know the exact error and so the order of convergence in terms of the number of nodes can be computed.

In Table 1 we show the ε-weighted H^1-norm of the error for different values of N for Problem (4) with $\varepsilon = 10^{-4}$ and $\varepsilon = 10^{-6}$. The orders computed from this tables is 0.513738 for the first case and 0.507040 for the second one as predicted by the theoretical results.

N	Error
324	0.16855
961	0.097606
3249	0.052696
12100	0.025912
45796	0.013419

$\varepsilon = 10^{-4}$

N	Error
676	0.16494
2025	0.094645
6889	0.050256
25281	0.026023
96100	0.013427

$\varepsilon = 10^{-6}$

Table 1

With the following results we want to point out an advantage of the graded meshes over the Shishkin meshes: the graded meshes designed for a given ε work well also for larger values of ε. Indeed, this follows from the error analysis. This is not the case for the Shishkin meshes as shown by the following example. This might be of interest if one want to solve a problem with a variable ε.

Table 2 shows the values of the ε-weighted H^1-norm of the error for different values of ε solving the problem with the mesh corresponding to $\varepsilon = 10^{-6}$, using graded meshes and Shishkin meshes.

ε	Error
10^{-6}	0.040687
10^{-5}	0.033103
10^{-4}	0.028635
10^{-3}	0.024859
10^{-2}	0.02247
10^{-1}	0.027278

Graded meshes, $N = 10404$

ε	Error
10^{-6}	0.0404236
10^{-5}	0.249139
10^{-4}	0.623650
10^{-3}	0.718135
10^{-2}	0.384051
10^{-1}	0.0331733

Shishkin meshes, $N = 10609$

Table 2

We consider one more example, similar to tests (1) and (2). Precisely, we take $\varepsilon = 10^{-6}$, $b = (-1, 0)$, $c = 1$, and $f(x, y) = w(x) + v(y)$, where

$$\begin{aligned} w(t) &= 1 - t - \frac{e^{-\frac{1}{\varepsilon}} - e^{\frac{-t}{\varepsilon}}}{e^{-\frac{1}{\varepsilon}} - 1}; \\ v(t) &= e^{\frac{2}{\sqrt{\varepsilon}}} \left(e^{-\frac{1-t}{\sqrt{\varepsilon}}} - e^{-\frac{2-t}{\sqrt{\varepsilon}}} + e^{-\frac{1+t}{\sqrt{\varepsilon}}} - e^{-\frac{t}{\sqrt{t}}} \right) + 1. \end{aligned}$$

The solution of this problem is $u(x, y) = w(x)v(y)$ and it presents exponential boundary layers along $x = 0$ of width $O(\varepsilon \log \frac{1}{\varepsilon})$ and along $y = 0, 1$ of width $O(\sqrt{\varepsilon} \log \frac{1}{\varepsilon})$. Although the boundary layers presented near $y = 0, 1$ are weaker than the one near $x = 0$ we have used the grading indicated in Remark 2.4 for the three boundary layers. Of course, a different refinement with a lowest number of nodes could be used near the weaker layers but we wanted to show that our procedure works well also for cases in which the equation becomes reaction dominant. In Table 3 we show the results obtained in this case. We observe that the numerical order is 0.489546.

Finally, just to see the different structures, we show in Figure 2 a Shishkin mesh and one of our graded meshes having the same number of nodes. For the sake of clarity we have pictured only the part of the meshes corresponding to $(0, 1/2) \times (0, 1/2)$ and $\varepsilon = 10^{-\frac{3}{2}}$.

N	Error
3735	0.0820324
7488	0.0578917
12699	0.0449325
19278	0.0367152
26983	0.0310352
36260	0.0268979

TABLE 3

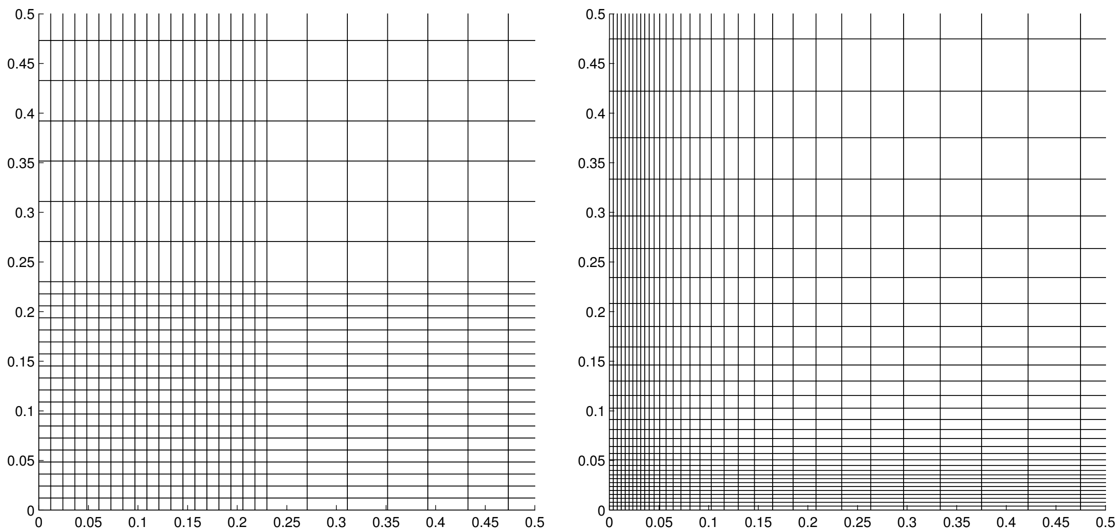


FIGURE 2

4. CONCLUSIONS

We have proved that the optimal order in the ε-weighted H^1-norm, up to logarithmic factors, is obtained using appropriate graded meshes and standard finite element methods of lowest order for convection diffusion problems. In both, theoretical and numerical experiments, we have worked with rectangular elements but it is not difficult to see that analogous results can be obtained for linear triangular elements.

The numerical experiments showed that no oscillations appear in the numerical solution and the predicted order of convergence is observed.

We believe that graded meshes are an interesting alternative to the Shishkin meshes that have been widely analyzed for this kind of problems. In particular, numerical experiments show that the graded mesh method is more robust in the sense that the numerical results does not depend strongly on parameters defining the mesh, instead the results obtained with Shishkin meshes depend in a significant way on the parameter defining the point where the mesh change its size, indeed, if this parameter is slightly moved from its optimal choice the numerical solution may present oscillations. Also, we have observed that the graded meshes designed for some value of the singular perturbation parameter work well also for larger values of this parameter while this is not the case for the Shishkin meshes. This might be of interest in problems with a variable ε.

We have performed the analysis and the numerical experiments for a model problem in a square domain. However, we believe that similar results could be obtained for more general domains. Also, similar problems in three dimensional domains can be analyzed in a similar way. However, in that case, a mean average interpolation should be used to prove the error estimates because, as it is known, the estimates for the Lagrange interpolation in H^1 are not independent of the relations between different edges of an element in $3D$. On the other hand, the graded meshes defined in Remark 2.4 and used in our numerical experiments satisfy the local regularity conditions required in [3] for the error estimates proved in that paper for the mean average interpolant introduced there, and therefore, that operator could be used for the analysis. These generalizations will be the object of further research.

Acknowledgment: We want to thank Thomas Apel for helpful comments on a previous version of our results which lead to a considerable simplification and improvement of our paper. In particular he pointed out to us the argument used in the proof of Corollary 2.3.

References

1. T. Apel, Anisotropic finite elements: Local estimates and applications, Series "Advances in Numerical Mathematics", Teubner, Stuttgart, 1999.
2. S. C. Brenner and L. R. Scott, The Mathematical Theory of Finite Element Methods, Texts in Applied Mathematics 15, Springer, Berlin, 1994.
3. R. G. Durán, A. L. Lombardi, *Error estimates on anisotropic $\mathcal{Q}_1$ elements for functions in weighted Sobolev spaces*, Math. Comp., to appear.
4. K. W. Morton, Numerical Solutions fo Convection-Diffusion Problems, Chapman & Hall, London, 1996.
5. H. G. Roos, M. Stynes, L. Tobiska, Numerical Methods for Singularly Perturbed Differential Equations, Springer-Verlag, Berlin, 1996.
6. M. Stynes, E. O'Riordan, *A uniformly convergent Galerkin Method on a Shishkin mesh for a convection-diffusion problem*, J. Math. Anal. Appl. 214 (1997) 36-54.

Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, 1428 Buenos Aires, Argentina.
E-mail address: rduran@dm.uba.ar

Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, 1428 Buenos Aires, Argentina.
E-mail address: aldoc7@dm.uba.ar