ERROR ESTIMATES ON ANISOTROPIC $\mathcal{Q}_1$ ELEMENTS FOR FUNCTIONS IN WEIGHTED SOBOLEV SPACES

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Abstract. In this paper we prove error estimates for a piecewise $\mathcal{Q}_1$ average interpolation on anisotropic rectangular elements, i.e., rectangles with sides of different orders, in two and three dimensions.

Our error estimates are valid under the condition that neighboring elements have comparable size. This is a very mild assumption that includes more general meshes than those allowed in previous papers. In particular, strong anisotropic meshes arising naturally in the approximation of problems with boundary layers fall under our hypotheses.

Moreover, we generalize the error estimates allowing on the right hand side some weighted Sobolev norms. This extension is of interest in singularly perturbed problems.

Finally, we consider the approximation of functions vanishing on the boundary by finite element functions with the same property, a point that was not considered in previous papers on average interpolations for anisotropic elements.

As an application we consider the approximation of a singularly perturbed reaction-diffusion equation and show that, as a consequence of our results, almost optimal order error estimates in the energy norm, valid uniformly in the perturbation parameter, can be obtained.

1. Introduction

In the finite element approximation of functions which have singularities or boundary layers it is necessary to use highly non uniform meshes such that the mesh size is much smaller near the singularities than far from them. In the case of boundary layers these meshes contain very narrow or anisotropic elements.

The goal of this paper is to obtain new error estimates for $\mathcal{Q}_1$ (piecewise bilinear in 2d or trilinear in 3d) approximations on meshes containing anisotropic rectangular elements, i. e., rectangles with sides of different orders. The classic error analysis is based on the so called regularity assumption which excludes this kind of elements (see for example [8, 9]). However, it is now well known that this assumption is not needed. Indeed, many papers have been written to prove error estimates under more general conditions. In particular, for rectangular elements we refer to [1, 12, 18] and their references.

We will prove the error estimates for a mean average interpolation. There are two reasons to work with this kind of approximation instead of the Lagrange interpolation. The first one is to approximate non smooth functions for which the Lagrange interpolation is not even defined, in fact this is the reason that motivated the introduction of average interpolations (see [10]). On the other hand, it has already been observed that, in the three dimensional case, average interpolations have better approximation properties than the Lagrange interpolation even for smooth functions when narrow elements are used (see [1, 12]).

Our estimates extend previous known results in several aspects:

First, our assumptions include more general meshes than those allowed in the previous papers. Indeed, in [12] it was required that the meshes were quasiuniform in each direction. This requirement was relaxed in [1] but not enough to include the meshes that arise naturally in the approximation of boundary layers, which will be included under our assumptions. To prove our error estimates we require only that neighboring elements are of comparable size and so, our results are valid for a rather general family of anisotropic meshes.

Second, we generalize the error estimates allowing weaker norms on the right hand side. These norms are weighted Sobolev norms where the weights are related with the distance to the boundary. The interest of working with these norms arise in the approximation of boundary layers. Indeed,

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1991 *Mathematics Subject Classification.* 65N30.

Key words and phrases. Anisotropic elements, weighted norms.

The research was supported by ANPCyT under grant PICT 03-05009 and by CONICET under grant PIP 0660/98. The first author is a member of CONICET, Argentina.

for many singular perturbed problems it is possible to prove that the solution has first and second derivatives which are bounded, uniformly in the perturbation parameter, in appropriate weighted Sobolev norms.

The use of weighted norms to design appropriate meshes in finite element approximations of singular problems is a well known procedure. In particular, error estimates for functions in weighted Sobolev spaces have been obtained in several works (see for example [4, 5, 6, 14]). In those works, the weights considered are related with the distance to a point or an edge (in the 3d case), instead here we consider weights related with the distance to the boundary.

Finally, we consider the approximation of functions vanishing on the boundary by finite element functions with the same property. This is a non trivial point that was not considered in the above mentioned references.

Our mean average interpolation is similar to that introduced in [12] but the difference is that we define it directly on the given mesh instead of using reference elements. This is important in order to relax the regularity assumptions on the elements.

We will prove our estimates for the domain $\Omega = [0,1]^d$, $d = 2, 3$. It will be clear that the interior estimates derived in Section 2 are valid for any domain which can be decomposed in d-rectangles. However, the extension of our results of Section 3 for interpolations satisfying Dirichlet boundary conditions to other domains is not straightforward and would require a further analysis.

To prove the weighted estimates we will use a result of Boas and Straube [7] which, as we show, can be derived from the classic Hardy inequality in higher dimensions.

In Section 2 we construct the mean average interpolation and prove the error estimates for interior elements. Section 3 deals with the approximation on boundary elements. Since the proofs of this section are rather technical we give them in the two dimensional case. However, it is not difficult (although very tedious!) to see that our arguments apply also in three dimensions.

Finally in Section 4, as an application of our results, we consider the finite element approximation of the reaction diffusion equation

$$\begin{aligned} -\varepsilon^2 \Delta u + u = f \qquad &\text{in } \ \Omega \\ u = 0 \qquad &\text{in } \ \partial\Omega \end{aligned}$$

Using that appropriate weighted norms of the solution are bounded uniformly in the perturbation parameter ε we show that it is possible to design graded meshes independent of ε such that almost optimal (in terms of the degrees of freedom) error estimates in the energy norm, valid uniformly in ε, hold.

2. Error estimates for interior elements

In this section we prove error estimates for a piecewise $\mathcal{Q}_1$ mean average interpolation for functions in weighted Sobolev spaces. The weights considered are powers of the distance to the boundary. This kind of weights arise naturally in problems with boundary layers.

The approximation introduced here is a variant of that considered in [12]. The difference is that we define it directly in the given mesh instead of using a reference one. Working in this way we are able to remove the restrictions used in [1, 12]. In particular, our results apply for the anisotropic meshes arising in the approximation of boundary layers.

Let $\mathcal{T}$ be a partition into rectangular elements of $\Omega = [0,1]^d$, $d = 2, 3$. We call $\mathcal{N}$ the set of nodes of $\mathcal{T}$ and $\mathcal{N}_{in}$ the set of interior nodes.

Given an element $R \in \mathcal{T}$, let $h_{R,i}$ be the length of the side of R in the direction x_i.

We assume that there exists a constant σ such that, for $R, S \in \mathcal{T}$ neighboring elements,

$$\frac{h_{R,i}}{h_{S,i}} \le \sigma \qquad 1 \le i \le d. \tag{2.1}$$

For each $\mathtt{v} \in \mathcal{N}$ we define

$$h_{\mathtt{v},i} = \min\{h_{R,i} : \mathtt{v} \text{ is a vertex of } R\}, \qquad 1 \le i \le d.$$

and $h_{\mathtt{v}} = (h_{\mathtt{v},1}, h_{\mathtt{v},2})$ if $d = 2$ or $h_{\mathtt{v}} = (h_{\mathtt{v},1}, h_{\mathtt{v},2}, h_{\mathtt{v},3})$ if $d = 3$. If $p, q \in \mathbb{R}^d$ we denote by $p : q$ the vector (p_1q_1, p_2q_2) if $d = 2$ or (p_1q_1, p_2q_2, p_3q_3) if $d = 3$. Take $\psi \in C^\infty(\mathbb{R}^d)$ with support in a ball centered at the origin and radius $r \leq 1/\sigma$ and such that $\int \psi = 1$, and for $\mathtt{v} \in \mathcal{N}_{in}$ let

$$\psi_{\mathtt{v}}(x) = \frac{1}{h_{\mathtt{v},1}h_{\mathtt{v},2}}\psi\left(\frac{\mathtt{v}_1 - x_1}{h_{\mathtt{v},1}}, \frac{\mathtt{v}_2 - x_2}{h_{\mathtt{v},2}}\right)$$

if $d = 2$ or

$$\psi_{\mathtt{v}}(x) = \frac{1}{h_{\mathtt{v},1}h_{\mathtt{v},2}h_{\mathtt{v},3}}\psi\left(\frac{\mathtt{v}_1 - x_1}{h_{\mathtt{v},1}}, \frac{\mathtt{v}_2 - x_2}{h_{\mathtt{v},2}}, \frac{\mathtt{v}_3 - x_3}{h_{\mathtt{v},3}}\right)$$

if $d = 3$. Given a function u we call $P(x, y)$ its Taylor polynomial of degree 1 at the point x, namely,

$$P(x, y) = u(x) + \nabla u(x) \cdot (y - x).$$

Then, for $\mathtt{v} \in \mathcal{N}_{in}$ we introduce the regularized average

$$u_{\mathtt{v}}(y) = \int P(x, y)\psi_{\mathtt{v}}(x)dx. \tag{2.2}$$

Now, given $u \in H_0^1(\Omega)$ we define Πu as the unique piecewise (with respect to $\mathcal{T}$) $\mathcal{Q}_1$ function such that, for $\mathtt{v} \in \mathcal{N}_{in}$, $\Pi u(\mathtt{v}) = u_{\mathtt{v}}(\mathtt{v})$ while $\Pi u(\mathtt{v}) = 0$ for boundary nodes $\mathtt{v}$.

Introducing the standard basis functions $\lambda_{\mathtt{v}}$ associated with the nodes $\mathtt{v}$ we can write

$$\Pi u(x) = \sum_{\mathtt{v} \in \mathcal{N}_{in}} u_{\mathtt{v}}(\mathtt{v})\lambda_{\mathtt{v}}(x).$$

For $R \in \mathcal{T}$ and $\mathtt{v} \in \mathcal{N}$ we define (see Figure 1 for the 2d case)

$$\tilde{R} = \bigcup\{S \in \mathcal{T} : S \text{ is a neighboring element of } R\}$$

and

$$R_{\mathtt{v}} = \bigcup\{S \in \mathcal{T} : \mathtt{v} \text{ is a vertex of } S\}.$$

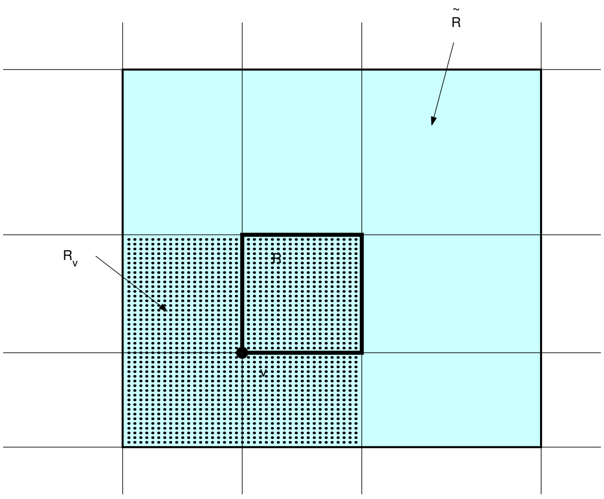

FIGURE 1

In our analysis we will also make use of the regularized average of u, namely,

$$Q_{\mathtt{v}}(u) = \int u(x)\psi_{\mathtt{v}}(x)dx$$

for $\mathtt{v} \in \mathcal{N}_{in}$.

We remark that, since $r \leq 1/\sigma$, it follows from our assumption (2.1) that the support of $\psi_{\mathtt{v}}(x)$ is contained in $R_{\mathtt{v}}$.

Now we prove some weighted estimates which will be useful for our error analysis. For any set D we call $d_D(x)$ the distance of x to the boundary of D. For a d-rectangle $R = \Pi_{i=1}^d (a_i, b_i)$ we have $d_R(x) = \min\{x_i - a_i, b_i - x_i : 1 \le i \le d\}$. For such R we will also consider the following function

$$\delta_R(x) := \min\left\{\frac{x_i - a_i}{h_{R,i}}, \frac{b_i - x_i}{h_{R,i}} : 1 \le i \le d\right\}.$$

We will make use of the following inequality which is known as "Hardy inequality":

$$\left\|\frac{v(x)}{x(x-1)}\right\|_{L^2(0,1)} \le C\|v'\|_{L^2(0,1)} \tag{2.3}$$

for $v \in H_0^1(0,1)$. We will also need the following generalization to higher dimensions: If D is a convex domain and $u \in H_0^1(D)$ then

$$\left\|\frac{u}{d_D}\right\|_{L^2(D)} \le 2\|\nabla u\|_{L^2(D)} \tag{2.4}$$

(see for example [17]).

The following lemma gives an "anisotropic" version of (2.4). It can be proved by standard scaling arguments.

Lemma 2.1. *Let $R = \Pi_{i=1}^d (a_i, b_i)$ be a d-rectangle and $h_i = b_i - a_i, 1 \le i \le d$. For all $u \in H_0^1(R)$*

$$\left\|\frac{u}{\delta_R}\right\|_{L^2(R)} \le 2\sum_{i=1}^d h_i \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2(R)}. \tag{2.5}$$

Another consequence of (2.4) is the inequality that we prove in the following lemma. This inequality was proved for Lipschitz domains by Boas and Straube in [7]. We give a different proof here because we are interested in the dependence of the constant on the domain, which is not stated in [7] because the proof given there is based on compactness arguments.

Lemma 2.2. *Let R be a d-rectangle with sides of lengths h_i, $1 \le i \le d$, such that $\frac{1}{\delta} \le h_i \le \delta$, and let $\psi \in C_0(R)$ be a function such that $\int_R \psi = 1$. Then, there exists a constant C depending only on δ and ψ, such that, for all $u \in H^1(R)$ with $\int_R u\psi = 0$,*

$$\|u\|_{L^2(R)} \le C\|d_R \nabla u\|_{L^2(R)}. \tag{2.6}$$

Proof. Since $v := u - (\int_R u)\psi$ has vanishing mean value, there exists $F \in H_0^1(R)^d$ such that

$$-\mathrm{div}\, F = v \tag{2.7}$$

and such that

$$\|F\|_{H_0^1(R)^2} \le C\|v\|_{L^2(R)}. \tag{2.8}$$

Moreover, from the explicit bound for the constant given in [13] it follows that C can be taken depending only on δ.

Now, since $\int_R u\psi = 0$, we have from (2.7)

$$\|u\|_{L^2(R)}^2 = \int_R uv = -\int_R u \,\mathrm{div}\, F$$

and therefore, integrating by parts and using (2.4) for each component of F, we obtain

$$\|u\|_{L^2(R)}^2 = \int_R \nabla u \cdot F \le \|d_R \nabla u\|_{L^2(R)} \left\|\frac{F}{d_R}\right\|_{L^2(R)} \le 2\|d_R \nabla u\|_{L^2(R)}\|\nabla F\|_{L^2(R)}$$

but,

$$\|v\|_{L^2(R)}^2 \le (1 + |R|\|\psi\|_{L^2(R)}^2)\|u\|_{L^2(R)}^2$$

and so, the proof concludes by using (2.8) and the fact that the constant in that estimate depends only on δ. $\square$

As a consequence of the previous lemma we obtain the following weighted estimates.

Lemma 2.3. *For* $\mathtt{v} \in \mathcal{N}_{in}$ *there exists a constant* C *depending only on* σ *and* ψ *such that, for all* $u \in H^1(R_{\mathtt{v}})$,

$$\|u - Q_{\mathtt{v}}(u)\|_{L^2(R_{\mathtt{v}})} \le C \sum_{i=1}^{d} h_{\mathtt{v},i} \left\| \delta_{R_{\mathtt{v}}} \frac{\partial u}{\partial x_i} \right\|_{L^2(R_{\mathtt{v}})} \tag{2.9}$$

and, for all $u \in H^2(R_{\mathtt{v}})$,

$$\left\| \frac{\partial (u - u_{\mathtt{v}})}{\partial x_j} \right\|_{L^2(R_{\mathtt{v}})} \le C \sum_{i=1}^{d} h_{\mathtt{v},i} \left\| \delta_{R_{\mathtt{v}}} \frac{\partial^2 u}{\partial x_j \partial x_i} \right\|_{L^2(R_{\mathtt{v}})} . \tag{2.10}$$

Proof. Let $K_{\mathtt{v}}$ be the image of $R_{\mathtt{v}}$ by the map $x \to \bar{x}$ with

$$\bar{x}_i = \frac{\mathtt{v}_i - x_i}{h_{\mathtt{v},i}} \quad 1 \le i \le d$$

and, for $\bar{x} \in K_{\mathtt{v}}$, define $\bar{u}$ by $\bar{u}(\bar{x}) = u(x)$. Then, $Q_{\mathtt{v}}(u) = \bar{Q}(\bar{u})$ where

$$\bar{Q}(\bar{u}) = \int \bar{u}(\bar{x}) \psi(\bar{x}) d\bar{x}.$$

Now, in view of our assumption (2.1), the d-rectangle $K_{\mathtt{v}}$ satisfies the hypothesis of Lemma 2.2 with $\delta = 2\sigma$. Moreover, since $r \le \frac{1}{\sigma}$, the support of ψ is contained in $K_{\mathtt{v}}$. Therefore, since $\int (\bar{u} - \bar{Q}(\bar{u}))\psi = 0$, it follows from Lemma 2.2 that there exists a constant C depending only on σ and ψ such that

$$\|\bar{u} - \bar{Q}(\bar{u}))\|_{L^2(K_{\mathtt{v}})} \le C \|d_{K_{\mathtt{v}}} \nabla \bar{u}\|_{L^2(K_{\mathtt{v}})}$$

and (2.9) follows by going back to the variable x.

To prove (2.10), observe that $u_{\mathtt{v}}(y) = \bar{u}_0(\bar{y})$ where

$$\bar{u}_0(\bar{y}) = \int (\bar{u}(\bar{x}) + \nabla(\bar{u})(\bar{x}) \cdot (\bar{y} - \bar{x})) \psi(\bar{x}) d\bar{x}$$

and so, since

$$\int \frac{\partial (\bar{u} - \bar{u}_0)}{\partial \bar{x}_i} \psi = 0,$$

we obtain from Lemma 2.2 that there exists a constant C depending only on σ and ψ such that

$$\left\| \frac{\partial (\bar{u} - \bar{u}_0)}{\partial \bar{x}_i} \right\|_{L^2(K_{\mathtt{v}})} \le C \left\| d_{K_{\mathtt{v}}} \nabla \frac{\partial \bar{u}}{\partial \bar{x}_i} \right\|_{L^2(K_{\mathtt{v}})}$$

and the proof concludes going back to the variable x. $\square$

We can now estimate the approximation error for interior elements in terms of weighted norms. We start with the L^2 norm. From now on C will be a generic constant which depends only on σ and ψ. In view of our hypothesis (2.1), $h_{\mathtt{v},i}$ and $h_{R,i}$ are equivalent up to a constant depending on σ whenever $\mathtt{v}$ is a vertex of R. We will use this fact repeatedly without making it explicitly.

Theorem 2.4. *There exists a constant* C *depending only on* σ *and* ψ *such that*

(i) *For all* $R \in \mathcal{T}$ *and* $u \in H^1(\tilde{R})$ *we have*

$$\|\Pi u\|_{L^2(R)} \le C \|u\|_{L^2(\tilde{R})} . \tag{2.11}$$

(ii) *for all* $R \in \mathcal{T}$ *such that* R *is not a boundary element and* $u \in H^1(\tilde{R})$ *we have*

$$\|u - \Pi u\|_{L^2(R)} \le C \sum_{i=1}^{d} h_{R,i} \left\| \delta_{\tilde{R}} \frac{\partial u}{\partial x_i} \right\|_{L^2(\tilde{R})} . \tag{2.12}$$

Proof. To prove (i) we write

$$(\Pi u)|_R = \sum_{j=1}^{n_R} u_{\mathtt{v}_j}(\mathtt{v}_j)\lambda_{\mathtt{v}_j}$$

where $\{\mathtt{v}_j\}_1^{n_R}$ are the interior nodes of R. Then,

$$\|\Pi u\|_{L^2(R)} \le C \left(\prod_{i=1}^{d} h_{R,i}\right)^{\frac{1}{2}} \sum_{j=1}^{n_R} \|u_{\mathtt{v}_j}\|_{L^\infty(R)} \tag{2.13}$$

and we have to estimate $\|u_{\mathtt{v}_j}\|_{L^\infty(R)}$ for each j. To simplify notation we write $\mathtt{v} = \mathtt{v}_j$ (and so the subindexes denote now the components of $\mathtt{v}$). We have

$$\left|\int u(x)\psi_{\mathtt{v}}(x)dx\right| \le C \left(\prod_{i=1}^{d} h_{R,i}\right)^{-\frac{1}{2}} \|u\|_{L^2(\tilde{R})}. \tag{2.14}$$

On the other hand, since $\psi_{\mathtt{v}} = 0$ on $\partial\tilde{R}$, integration by parts gives

$$\begin{aligned}\left|\int \frac{\partial u}{\partial x_i}(x)(y_i - x_i)\psi_{\mathtt{v}}(x)dx\right| &= \left|\int u(x)\psi_{\mathtt{v}}(x)dx - \int u(x)(y_i - x_i)\frac{\partial \psi_{\mathtt{v}}}{\partial x_i}(x)dx\right| \\ &\le C \left(\prod_{i=1}^{d} h_{R,i}\right)^{-\frac{1}{2}} \|u\|_{L^2(\tilde{R})}\end{aligned} \tag{2.15}$$

where we have used that $|y_i - x_i| \le Ch_{\mathtt{v},i}$. Thus, (2.11) follows from (2.13), (2.14), (2.15) and the definition of $u_{\mathtt{v}}$ given in (2.2).

To prove (ii), choose a node of R, say $\mathtt{v}_1$. Since $Q_{\mathtt{v}_1}(u)$ is a constant function and R is not a boundary element, we have $\Pi Q_{\mathtt{v}_1}(u) = Q_{\mathtt{v}_1}(u)$ on R and so

$$\begin{aligned}\|u - \Pi u\|_{L^2(R)} &\le \|u - Q_{\mathtt{v}_1}(u)\|_{L^2(R)} + \|\Pi(Q_{\mathtt{v}_1}(u) - u)\|_{L^2(R)} \\ &\le C\|u - Q_{\mathtt{v}_1}(u)\|_{L^2(R)}\end{aligned} \tag{2.16}$$

where we have used (2.11). Now, estimate (2.12) follows from (2.16) and (2.9). $\square$

In what follows, we estimate the approximation error for the first derivatives for interior elements. We will use the notation of Figure 2.

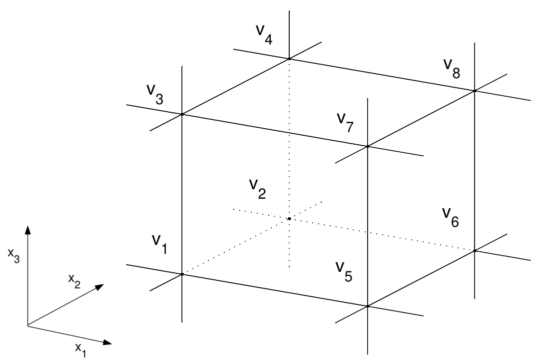

FIGURE 2

Theorem 2.5. *There exists a constant C depending only on σ and ψ such that, if $R \in \mathcal{T}$ is not a boundary element then, for all $u \in H^2(\tilde{R})$ we have*

$$\left\| \frac{\partial}{\partial x_j}(u - \Pi u) \right\|_{L^2(R)} \leq C \sum_{i=1}^{d} h_{R,i} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L^2(\tilde{R})} \qquad 1 \leq j \leq d. \tag{2.17}$$

Proof. We will consider the case $d = 3, j = 1$. Clearly, the other cases are analogous. We have

$$u - \Pi u = (u - u_{\mathtt{v}_1}) + (u_{\mathtt{v}_1} - \Pi u)$$

and from (2.10) we know that $\|\frac{\partial(u - u_{\mathtt{v}_1})}{\partial x_1}\|_{L^2(R)}$ is bounded by the right hand side of (2.17). Therefore, we have to estimate $\|\frac{\partial(u_{\mathtt{v}_1} - \Pi u)}{\partial x_1}\|_{L^2(R)}$. Since $w := u_{\mathtt{v}_1} - \Pi u \in \mathcal{Q}_1$ we have (see for example [18])

$$\frac{\partial w}{\partial x_1} = \sum_{i=1}^{4} (w(\mathtt{v}_i) - w(\mathtt{v}_{i+4})) \frac{\partial \lambda_{\mathtt{v}_i}}{\partial x_1}$$

then,

$$\left\| \frac{\partial w}{\partial x_1} \right\|_{L^2(R)} \leq \sum_{i=1}^{4} |w(\mathtt{v}_i) - w(\mathtt{v}_{i+4})| \left\| \frac{\partial \lambda_{\mathtt{v}_i}}{\partial x_1} \right\|_{L^2(R)}. \tag{2.18}$$

But, it is easy to see that

$$\left\| \frac{\partial \lambda_{\mathtt{v}_i}}{\partial x_1} \right\|_{L^2(R)} \leq C \left(\frac{h_{\mathtt{v}_i,2} h_{\mathtt{v}_i,3}}{h_{\mathtt{v}_i,1}} \right)^{\frac{1}{2}}. \tag{2.19}$$

So, we have to estimate $|w(\mathtt{v}_i) - w(\mathtt{v}_{i+4})|$ for $1 \leq i \leq 4$. We have

$$\begin{aligned} w(\mathtt{v}_1) - w(\mathtt{v}_5) &= u_{\mathtt{v}_5}(\mathtt{v}_5) - u_{\mathtt{v}_1}(\mathtt{v}_5) \\ &= \int P(x, \mathtt{v}_5) \psi_{\mathtt{v}_5}(x) dx - \int P(x, \mathtt{v}_5) \psi_{\mathtt{v}_1}(x) dx. \end{aligned} \tag{2.20}$$

So, changing variables we obtain

$$w(\mathtt{v}_1) - w(\mathtt{v}_5) = \int \left[P(\mathtt{v}_5 - h_{\mathtt{v}_5} : y, \mathtt{v}_5) - P(\mathtt{v}_1 - h_{\mathtt{v}_1} : y, \mathtt{v}_5) \right] \psi(y) dy. \tag{2.21}$$

We introduce the notation $\mathtt{v}_i = (\mathtt{v}_i^1, \mathtt{v}_i^2, \mathtt{v}_i^3)$. Define now

$$\theta = (\theta_1, 0, 0) := \left(\mathtt{v}_5^1 - \mathtt{v}_1^1 + (h_{\mathtt{v}_1,1} - h_{\mathtt{v}_5,1}) y_1, 0, 0 \right)$$

and

$$F_y(t) := P(\mathtt{v}_1 - h_{\mathtt{v}_1} : y + t\theta, \mathtt{v}_5).$$

Then, since $h_{\mathtt{v}_1,2} = h_{\mathtt{v}_5,2}$, $h_{\mathtt{v}_1,3} = h_{\mathtt{v}_5,3}$ and $\mathtt{v}_1^2 = \mathtt{v}_5^2$, $\mathtt{v}_1^3 = \mathtt{v}_5^3$, we have

$$P(\mathtt{v}_5 - h_{\mathtt{v}_5} : y, \mathtt{v}_5) - P(\mathtt{v}_1 - h_{\mathtt{v}_1} : y, \mathtt{v}_5) = F_y(1) - F_y(0)$$

and replacing in (2.21) we obtain

$$w(\mathtt{v}_1) - w(\mathtt{v}_5) = \int \int_0^1 F_y'(t) \psi(y) dt dy = \int_0^1 \left\{ \int F_y'(t) \psi(y) dy \right\} dt$$

and therefore it is enough to estimate

$$I(t) := \int F_y'(t) \psi(y) dy$$

for $0 \leq t \leq 1$. But, from the definition of F_y and P, we have

$$
\begin{aligned}
|I(t)| \le \int \Big\{ \Big| \frac{\partial^2 u}{\partial x_1^2}(\mathtt{v}_1 - h_{\mathtt{v}_1} : y + t\theta) \Big| \times |\mathtt{v}_5^1 - \mathtt{v}_1^1 + h_{\mathtt{v}_1,1} y_1 - t\theta_1| \\
+ \Big| \frac{\partial^2 u}{\partial x_1 \partial x_2}(\mathtt{v}_1 - h_{\mathtt{v}_1} : y + t\theta) \Big| \times |\mathtt{v}_5^2 - \mathtt{v}_1^2 + h_{\mathtt{v}_1,2} y_2| \\
+ \Big| \frac{\partial^2 u}{\partial x_1 \partial x_3}(\mathtt{v}_1 - h_{\mathtt{v}_1} : y + t\theta) \Big| \times |\mathtt{v}_5^3 - \mathtt{v}_1^3 + h_{\mathtt{v}_1,3} y_3| \Big\} |\theta_1| \psi(y) dy.
\end{aligned}
$$

Now, for $|y| \le 1$ and $0 \le t \le 1$, we have

$$
|\theta| = |\theta_1| \le C h_{\mathtt{v}_1,1}\,, \qquad |\mathtt{v}_5^i - \mathtt{v}_1^i + h_{\mathtt{v}_1,i} y_i - \theta_1 t| \le C h_{\mathtt{v}_1,i}\,,
$$

and therefore, since $\operatorname{supp}(\psi) \subset B(0,1)$, we have

$$
\begin{aligned}
|I(t)| \le C \int \Big\{ \Big| \frac{\partial^2 u}{\partial x_1^2}(\mathtt{v}_1 - h_{\mathtt{v}_1} : y + \theta t) \Big| (h_{\mathtt{v}_1,1})^2 + \Big| \frac{\partial^2 u}{\partial x_1 \partial x_2}(\mathtt{v}_1 - h_{\mathtt{v}_1} : y + \theta t) \Big| h_{\mathtt{v}_1,1} h_{\mathtt{v}_1,2} \\
+ \Big| \frac{\partial^2 u}{\partial x_1 \partial x_3}(\mathtt{v}_1 - h_{\mathtt{v}_1} : y + \theta t) \Big| h_{\mathtt{v}_1,1} h_{\mathtt{v}_1,3} \Big\} \psi(y) dy.
\end{aligned}
$$

Now, making the change of variables $z = \mathtt{v}_1 - h_{\mathtt{v}_1} : y + \theta t$ and calling

$$
\phi(z) = \psi \left(- \frac{z_1 - [(1-t)\mathtt{v}_1^1 + t\mathtt{v}_2^1]}{(1-t) h_{\mathtt{v}_1,1} + t h_{\mathtt{v}_5,1}}, - \frac{z_2 - \mathtt{v}_1^2}{h_{\mathtt{v}_1,2}}, - \frac{z_3 - \mathtt{v}_1^3}{h_{\mathtt{v}_1,3}} \right)
$$

we obtain

$$
|I(t)| \le C \frac{1}{h_{\mathtt{v}_1,2} h_{\mathtt{v}_1,3}} \sum_{i=1}^3 h_{\mathtt{v}_1,i} \int \Big| \frac{\partial^2 u}{\partial x_1 \partial x_i}(z) \Big| \phi(z) dz
$$

where we have used that $h_{\mathtt{v}_1,1} \ge C((1-t) h_{\mathtt{v}_1,1} + t h_{\mathtt{v}_5,1})$. But, since $\operatorname{supp} \psi \subset B\left(0, \frac{1}{\sigma}\right)$, it follows that $\operatorname{supp} \phi \subset \tilde{R}$. Then, using the Schwarz inequality we obtain

$$
|I(t)| \le C \frac{1}{h_{\mathtt{v}_1,2} h_{\mathtt{v}_1,3}} \sum_{i=1}^3 h_{\mathtt{v}_1,i} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_1 \partial x_i} \right\| \left\| \frac{\phi}{\delta_{\tilde{R}}} \right\|
$$

and from Lemma 2.1 we know that

$$
\left\| \frac{\phi}{\delta_{\tilde{R}}} \right\|_{L^2(\tilde{R})} \le C (h_{\mathtt{v}_1,1} h_{\mathtt{v}_1,2} h_{\mathtt{v}_1,3})^{\frac{1}{2}}.
$$

Finally, using (2.19) we obtain

$$
|w(\mathtt{v}_1) - w(\mathtt{v}_5)| \left\| \frac{\partial \lambda_{\mathtt{v}_1}}{\partial x_1} \right\|_{L^2(R)} \le C \sum_{i=1}^3 h_{\mathtt{v}_1,i} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_1 \partial x_i} \right\|_{L^2(\tilde{R})}. \tag{2.22}
$$

Now, to estimate $|w(\mathtt{v}_2) - w(\mathtt{v}_6)|$ we write

$$
\begin{aligned}
w(\mathtt{v}_2) - w(\mathtt{v}_6) &= (u_{\mathtt{v}_1}(\mathtt{v}_2) - u_{\mathtt{v}_2}(\mathtt{v}_2)) - (u_{\mathtt{v}_1}(\mathtt{v}_6) - u_{\mathtt{v}_6}(\mathtt{v}_6)) \\
&= (u_{\mathtt{v}_1}(\mathtt{v}_2) - u_{\mathtt{v}_1}(\mathtt{v}_6)) - (u_{\mathtt{v}_2}(\mathtt{v}_2) - u_{\mathtt{v}_2}(\mathtt{v}_6)) - (u_{\mathtt{v}_2}(\mathtt{v}_6) - u_{\mathtt{v}_6}(\mathtt{v}_6)) \\
&=: I - II - III.
\end{aligned} \tag{2.23}
$$

Now we estimate $I - II$. We have

$$
I = \int \frac{\partial u}{\partial x_1}(x)(\mathtt{v}_2^1 - \mathtt{v}_6^1) \psi_{\mathtt{v}_1}(x) dx \qquad \text{and} \qquad II = \int \frac{\partial u}{\partial x_1}(x)(\mathtt{v}_2^1 - \mathtt{v}_6^1) \psi_{\mathtt{v}_2}(x) dx
$$

where we have used that $\mathtt{v}_2 - \mathtt{v}_6 = (\mathtt{v}_2^1 - \mathtt{v}_6^1, 0, 0)$. After a change of variables in both integrals we obtain

$$
I - II = \int \left[\frac{\partial u}{\partial x_1}(\mathtt{v}_1 - h_{\mathtt{v}_1} : y) - \frac{\partial u}{\partial x_1}(v_2 - h_{\mathtt{v}_2} : y) \right] (\mathtt{v}_2^1 - \mathtt{v}_6^1) \psi(y) dy
$$

and so, defining $\theta = (0, \theta_2, 0) := (0, \mathtt{v}_2^2 - \mathtt{v}_1^2 - (h_{\mathtt{v}_2,2} - h_{\mathtt{v}_1,2})y_2, 0)$ and

$$F_y(t) = \frac{\partial u}{\partial x_1}(\mathtt{v}_1 - h_{\mathtt{v}_1} : y + \theta t)$$

and taking into account that $h_{\mathtt{v}_1,1} = h_{\mathtt{v}_2,1}$ and $h_{\mathtt{v}_1,3} = h_{\mathtt{v}_2,3}$ we have

$$\begin{aligned} I - II &= -\int \int_0^1 F_y'(t)(\mathtt{v}_2^1 - \mathtt{v}_6^1)\psi(y) dt dy \\ &= -\int_0^1 \left\{ \int F_y'(t)(\mathtt{v}_2^1 - \mathtt{v}_6^1)\psi(y) dy \right\} dt \\ &=: \int_0^1 I(t) dt. \end{aligned}$$

Since

$$F_y'(t) = \frac{\partial^2 u}{\partial x_1 \partial x_2}(\mathtt{v}_1 - h_{\mathtt{v}_1} : y + \theta t)\theta_2$$

and for $y \in$ supp ψ, $|y| \leq 1$, we have

$$\begin{aligned} |I(t)| &\leq \int \left| \frac{\partial^2 u}{\partial x_1 \partial x_2}(\mathtt{v}_1 - h_{\mathtt{v}_1} : y + \theta t) \right| |\theta_2| |\mathtt{v}_2^1 - \mathtt{v}_6^1| |\psi(y)| dy \\ &\leq C h_{\mathtt{v}_2,1} h_{\mathtt{v}_2,2} \int \left| \frac{\partial^2 u}{\partial x_1 \partial x_2}(v_1 - h_{\mathtt{v}_1} : y + \theta t) \right| |\psi(y)| dy. \end{aligned}$$

Change now to the variable $z = \mathtt{v}_1 - h_{\mathtt{v}_1} : y + \theta t$ and define

$$\phi(z) = \psi \left(-\frac{z_1 - \mathtt{v}_1^1}{h_{\mathtt{v}_1,1}}, -\frac{z_2 - [(1-t)\mathtt{v}_1^2 + t\mathtt{v}_2^2]}{(1-t)h_{\mathtt{v}_1,2} + t h_{\mathtt{v}_2,2}}, -\frac{z_3 - \mathtt{v}_1^3}{h_{\mathtt{v}_1,3}} \right).$$

Then, since supp $\phi \subset \tilde{R}$ (because supp $\psi \subset B\left(0, \frac{1}{\sigma}\right)$), we can use Lemma 2.1 to obtain

$$\begin{aligned} |I(t)| &\leq C \frac{1}{h_{\mathtt{v}_1,3}} \int \left| \frac{\partial^2 u}{\partial x_1 \partial x_2}(z) \right| \phi(z) dz \\ &\leq C \frac{1}{h_{\mathtt{v}_1,3}} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})} \left\| \frac{\phi(z)}{\delta_{\tilde{R}}} \right\|_{L^2(\tilde{R})} \\ &\leq C \left(\frac{h_{\mathtt{v}_1,1} h_{\mathtt{v}_1,2}}{h_{\mathtt{v}_1,3}} \right)^{\frac{1}{2}} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})}. \end{aligned}$$

Therefore,

$$|I - II| \leq C \left(\frac{h_{\mathtt{v}_1,1} h_{\mathtt{v}_1,2}}{h_{\mathtt{v}_1,3}} \right)^{\frac{1}{2}} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})}.$$

The term III in equation (2.23) can be bounded by the same arguments used to obtain (2.22). Therefore we obtain

$$|w(\mathtt{v}_2) - w(\mathtt{v}_6)| \left\| \frac{\partial \lambda_{\mathtt{v}_1}}{\partial x_1} \right\|_{L^2(R)} \leq C \sum_{i=1}^{d} h_{\mathtt{v}_1,i} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_1 \partial x_i} \right\|_{L^2(\tilde{R})}. \tag{2.24}$$

The estimate of $w(\mathtt{v}_3) - w(\mathtt{v}_7)$ follows by the same arguments used to estimate $w(\mathtt{v}_2) - w(\mathtt{v}_6)$. Then, it remains to estimate $w(\mathtt{v}_4) - w(\mathtt{v}_8)$. We have

$$\begin{aligned} w(\mathtt{v}_4) - w(\mathtt{v}_8) &= (u_{\mathtt{v}_1}(\mathtt{v}_4) - u_{\mathtt{v}_4}(\mathtt{v}_4)) - (u_{\mathtt{v}_1}(\mathtt{v}_8) - u_{\mathtt{v}_8}(\mathtt{v}_8)) \\ &= [(u_{\mathtt{v}_1}(\mathtt{v}_4) - u_{\mathtt{v}_1}(\mathtt{v}_8)) - (u_{\mathtt{v}_3}(\mathtt{v}_4) - u_{\mathtt{v}_3}(\mathtt{v}_8))] \\ &\quad + [(u_{\mathtt{v}_3}(\mathtt{v}_4) - u_{\mathtt{v}_3}(\mathtt{v}_8)) - (u_{\mathtt{v}_4}(\mathtt{v}_4) - u_{\mathtt{v}_4}(\mathtt{v}_8))] + [u_{\mathtt{v}_8}(\mathtt{v}_8) - u_{\mathtt{v}_4}(v_8)] \\ &=: I + II + III. \end{aligned}$$

Now we deal with the term I. One can check that

$$I = \int \left[\frac{\partial u}{\partial x_1}(\mathtt{v}_1 - h_{\mathtt{v}_1} : y) - \frac{\partial u}{\partial x_1}(\mathtt{v}_3 - h_{\mathtt{v}_3} : y) \right] (\mathtt{v}_4^1 - \mathtt{v}_8^1)\psi(y) dy.$$

Defining now

$$F_y(t) := \frac{\partial u}{\partial x_1}(\mathtt{v}_3 - h_{\mathtt{v}_3} : y + t\theta)$$

where $\theta = (0, 0, \theta_3) := (0, 0, \mathtt{v}_1^3 - \mathtt{v}_3^3 - (h_{\mathtt{v}_1,3} - h_{\mathtt{v}_3,3})y_3)$ we have

$$\begin{aligned} I &= \int \int_0^1 F_y'(t)(\mathtt{v}_4^1 - \mathtt{v}_8^1)\psi(y)dydt \\ &= \int_0^1 \int F_y'(t)(\mathtt{v}_4^1 - \mathtt{v}_8^1)\psi(y)dtdy =: \int_0^1 I(t)dt. \end{aligned}$$

Since

$$F_y'(t) = \frac{\partial^2 u}{\partial x_1 \partial x_3}(\mathtt{v}_3 - h_{\mathtt{v}_3} : y + t\theta)\theta_3$$

and $|\theta_3| \le C h_{\mathtt{v}_1,3}$ if $|y| \le 1$ it follows that

$$|I(t)| \le h_{\mathtt{v}_1,1} h_{\mathtt{v}_1,3} \int \left| \frac{\partial^2 u}{\partial x_1 \partial x_3}(\mathtt{v}_3 - h_{\mathtt{v}_3} : y + t\theta) \right| \psi(y)dy$$

and so, changing variables and setting

$$\phi(z) = \psi\left(-\frac{z_1 - \mathtt{v}_3^1}{h_{\mathtt{v}_3,1}}, -\frac{z_2 - \mathtt{v}_3^2}{h_{\mathtt{v}_3,2}}, -\frac{z_3 - [(1-t)\mathtt{v}_3^3 + t\mathtt{v}_1^3]}{(1-t)h_{\mathtt{v}_3,3} + t h_{\mathtt{v}_1,3}} \right),$$

we obtain

$$|I(t)| \le C \frac{1}{h_{\mathtt{v}_1,2}} \int \left| \frac{\partial^2 u}{\partial x_1 \partial x_3}(z) \right| \phi(z)dz.$$

Now, taking into account that $\phi = 0$ on $\partial \tilde{R}$, it follows by the Schwarz inequality and Lemma 2.1 that

$$\begin{aligned} |I(t)| &\le C \frac{1}{h_{\mathtt{v}_1,2}} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_1 \partial x_3} \right\|_{L^2(\tilde{R})} \left\| \frac{\phi}{\delta_{\tilde{R}}} \right\|_{L^2(\tilde{R})} \\ &\le \left(\frac{h_{\mathtt{v}_1,1} h_{\mathtt{v}_1,3}}{h_{\mathtt{v}_1,2}} \right)^{\frac{1}{2}} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_1 \partial x_3} \right\|_{L^2(\tilde{R})}, \end{aligned}$$

and therefore,

$$|I| \left\| \frac{\partial \lambda_{\mathtt{v}_4}}{\partial x_1} \right\|_{L^2(\tilde{R})} \le h_{h_{\mathtt{v}_1,3}} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_1 \partial x_3} \right\|_{L^2(\tilde{R})}. \tag{2.25}$$

Finally, estimates for the terms II and III can be obtained with the arguments used for $(u_{\mathtt{v}_1}(\mathtt{v}_2) - u_{\mathtt{v}_1}(\mathtt{v}_6)) - (u_{\mathtt{v}_2}(\mathtt{v}_2) - u_{\mathtt{v}_2}(\mathtt{v}_6))$ in (2.23) and $u_{\mathtt{v}_5}(\mathtt{v}_5) - u_{\mathtt{v}_1}(\mathtt{v}_5)$ in (2.20) respectively. These estimates together with the inequalities (2.22),(2.24) and (2.25) conclude the proof. □

3. Error estimates for boundary elements

In this section we deal with the interpolation error on boundary elements for functions satisfying a homogeneous Dirichlet condition. For the sake of simplicity and because the proof is rather technical, we state and prove the main Theorem in the two dimensional case. However, analogous results can be obtained in three dimensions by using similar arguments.

We will use the notation of the previous section. Further, if $R = (a_1, b_1) \times (a_2, b_2)$ is a rectangle in $\mathcal{T}$, we set $R_{1i} = a_i$ and $l_{R,i} = (a_i, b_i)$. Also we define the function $\delta_{-,R}$ by

$$\delta_{-,R}(x) = \min\left\{ \frac{x_1 - a_1}{h_{R,1}}, \frac{x_2 - a_2}{h_{R,2}} \right\}.$$

We have $\delta_R(x) \le \delta_{-,R}(x)$ for all $x \in R$.

To estimate the error on a boundary element R we need to consider different cases according to the position of R. So, we decompose Ω into four regions (see Figure 3):

$$\Omega_1 = \bigcup \{R \in \mathcal{T} : R \cap \partial\Omega = \emptyset\}$$
$$\Omega_2 = \bigcup \{R \in \mathcal{T} : R \cap \{x : x_1 = 0\} = \emptyset \text{ and } R \cap \{x : x_2 = 0\} \neq \emptyset\}$$
$$\Omega_3 = \bigcup \{R \in \mathcal{T} : R \cap \{x : x_1 = 0\} \neq \emptyset \text{ and } R \cap \{x : x_2 = 0\} = \emptyset\}$$
$$\Omega_4 = R \in \mathcal{T} \text{ such that } (0,0) \in R.$$

Theorem 3.1. *There exists a constant C depending only on σ and ψ such that if $R \in \mathcal{T}$ for all $u \in H^2(\tilde{R})$ the following estimates hold,*

(i) *If $R \subset \Omega_2$ and $u \equiv 0$ on $\{x : x_2 = 0\}$*

$$\left\| \frac{\partial}{\partial x_1}(u - \Pi u) \right\|_{L^2(R)} \le C \left\{ h_{R,1} \left\| \delta_{-,\tilde{R}} \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2(\tilde{R})} + h_{R,2} \left\| \frac{x_1 - \tilde{R}_{11}}{h_{R,1}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})} \right\} \tag{3.1}$$

and

$$\left\| \frac{\partial}{\partial x_2}(u - \Pi u) \right\|_{L^2(R)} \le C \left\{ h_{R,1} \left\| \frac{x_1 - \tilde{R}_{11}}{h_{R,1}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})} + h_{R,2} \left\| \delta_{-,\tilde{R}} \frac{\partial^2 u}{\partial x_2^2} \right\|_{L^2(\tilde{R})} \right\}. \tag{3.2}$$

(ii) *If $R \subset \Omega_3$ and $u \equiv 0$ on $\{x : x_1 = 0\}$*

$$\left\| \frac{\partial}{\partial x_1}(u - \Pi u) \right\|_{L^2(R)} \le C \left\{ h_{R,1} \left\| \delta_{-,\tilde{R}} \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2(\tilde{R})} + h_{R,2} \left\| \frac{x_2 - \tilde{R}_{12}}{h_{R,2}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})} \right\} \tag{3.3}$$

and

$$\left\| \frac{\partial}{\partial x_2}(u - \Pi u) \right\|_{L^2(R)} \le C \left\{ h_{R,1} \left\| \frac{x_2 - \tilde{R}_{12}}{h_{R,2}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})} + h_{R,2} \left\| \delta_{-,\tilde{R}} \frac{\partial^2 u}{\partial x_2^2} \right\|_{L^2(\tilde{R})} \right\}. \tag{3.4}$$

(iii) *If $R \subset \Omega_4$ and $u \equiv 0$ on $\{x : x_1 = 0$ or $x_2 = 0\}$*

$$\left\| \frac{\partial}{\partial x_1}(u - \Pi u) \right\|_{L^2(R)} \le C \left\{ h_{R,1} \left\| \delta_{-,\tilde{R}} \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2(\tilde{R})} + h_{R,2} \left\| \left\{ \frac{x_1}{h_{R,1}} + \frac{x_2}{h_{R,2}} \right\} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})} \right\} \tag{3.5}$$

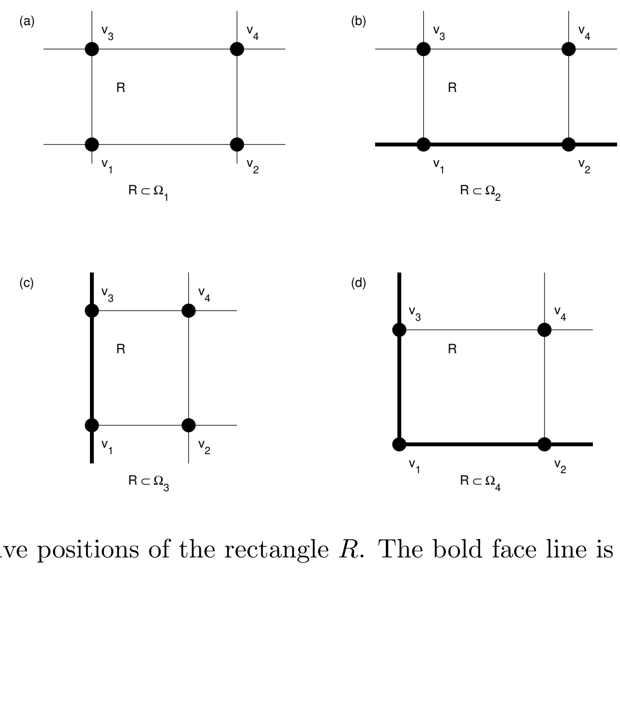

FIGURE 3. Relative positions of the rectangle R. The bold face line is the boundary of Ω.

and

$$\left\|\frac{\partial}{\partial x_2}(u-\Pi u)\right\|_{L^2(R)} \le C\left\{h_{R,2}\left\|\delta_{-,\tilde R}\frac{\partial^2 u}{\partial x_2^2}\right\|_{L^2(\tilde R)}+h_{R,1}\left\|\left\{\frac{x_1}{h_{R,1}}+\frac{x_2}{h_{R,2}}\right\}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(\tilde R)}\right\}. \tag{3.6}$$

Proof of Part (i). We now use the notation of Figure 3(b). We have

$$\Pi u_{|R} = u_{\mathtt{v}_3}(\mathtt{v}_3)\lambda_{\mathtt{v}_3} + u_{\mathtt{v}_4}(\mathtt{v}_4)\lambda_{\mathtt{v}_4}.$$

From (2.10) we know that $\|\frac{\partial}{\partial x_1}(u-u_{\mathtt{v}_3})\|_{L^2(R)}$ is bounded by the right hand side of (3.1). So, to prove (3.1), it is enough to estimate $\|\frac{\partial}{\partial x_1}(u_{\mathtt{v}_3}-\Pi u)\|_{L^2(R)}$.

Since $(u_{\mathtt{v}_3}-\Pi u)_{|R}\in\mathcal{Q}_1$ we have (see for example [18])

$$\begin{aligned}\frac{\partial}{\partial x_1}(u_{\mathtt{v}_3}-\Pi u) &= ((u_{\mathtt{v}_3}-\Pi u)(\mathtt{v}_2)-(u_{\mathtt{v}_3}-\Pi u)(\mathtt{v}_1))\frac{\partial\lambda_{\mathtt{v}_2}}{\partial x_1}\\ &\quad + ((u_{\mathtt{v}_3}-\Pi u)(\mathtt{v}_4)-(u_{\mathtt{v}_3}-\Pi u)(\mathtt{v}_3))\frac{\partial\lambda_{\mathtt{v}_4}}{\partial x_1}\\ &= (u_{\mathtt{v}_3}(\mathtt{v}_2)-u_{\mathtt{v}_3}(\mathtt{v}_1))\frac{\partial\lambda_{\mathtt{v}_2}}{\partial x_1}+(u_{\mathtt{v}_3}(\mathtt{v}_4)-u_{\mathtt{v}_4}(\mathtt{v}_4))\frac{\partial\lambda_{\mathtt{v}_4}}{\partial x_1}.\end{aligned} \tag{3.7}$$

Taking into account that $\frac{\partial u}{\partial x_1}\equiv 0$ on $(x_1,0)$ it is easy to see that

$$u_{\mathtt{v}_3}(\mathtt{v}_2)-u_{\mathtt{v}_3}(\mathtt{v}_1) = (\mathtt{v}_2^1-\mathtt{v}_1^1)\int_{l_{\tilde R,2}}\int_{l_{\tilde R,1}}\int_0^{x_2}\frac{\partial^2 u}{\partial x_1\partial x_2}(x_1,t)\psi_{\mathtt{v}_3}(x)\,dt\,dx_1\,dx_2$$

and then,

$$\begin{aligned}|u_{\mathtt{v}_3}(\mathtt{v}_2)-u_{\mathtt{v}_3}(\mathtt{v}_1)| &\le C h_{\mathtt{v}_3,1}\int_{l_{\tilde R,2}}\int_{l_{\tilde R,1}}\int_{l_{\tilde R,2}}\left|\frac{\partial^2 u}{\partial x_1\partial x_2}(x_1,t)\right|\psi_{\mathtt{v}_3}(x)\,dt\,dx_1\,dx_2\\ &\le C h_{\mathtt{v}_3,1}\int_{l_{\tilde R,2}}\left\|\frac{x_1-\tilde R_{11}}{h_{\mathtt{v}_3,1}}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(\tilde R)}\left\|\psi_{\mathtt{v}_3}(x)\frac{h_{\mathtt{v}_3,1}}{x_1-\tilde R_{11}}\right\|_{L^2(\tilde R)}dx_2.\end{aligned}$$

Using the one dimensional Hardy inequality (2.3) we have

$$\begin{aligned}\int_{l_{\tilde R,1}}\left|\frac{\psi_{\mathtt{v}_3}(x)}{x_1-\tilde R_{11}}\right|^2 dx_1 &\le \frac{C}{h_{\mathtt{v}_3,1}^4 h_{\mathtt{v}_3,2}^2}\int_{l_{\tilde R,1}}\left|\frac{\partial\psi}{\partial x_1}\left(\frac{\mathtt{v}_3^1-x_1}{h_{\mathtt{v}_3,1}},\frac{\mathtt{v}_1^2-x_2}{h_{\mathtt{v}_3,2}}\right)\right|^2 dx_1\\ &\le C\frac{1}{h_{\mathtt{v}_3,1}^3 h_{\mathtt{v}_3,2}^2}\end{aligned} \tag{3.8}$$

and then it follows that

$$|u_{\mathtt{v}_3}(\mathtt{v}_2)-u_{\mathtt{v}_3}(\mathtt{v}_1)| \le C(h_{\mathtt{v}_3,1}h_{\mathtt{v}_3,2})^{\frac12}\left\|\frac{x_1-\tilde R_{11}}{h_{\mathtt{v}_3,1}}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(\tilde R)}$$

and so

$$|u_{\mathtt{v}_3}(\mathtt{v}_2)-u_{\mathtt{v}_3}(\mathtt{v}_1)|\left\|\frac{\partial\lambda_{\mathtt{v}_2}}{\partial x_1}\right\|_{L^2(\tilde R)} \le C h_{\mathtt{v}_3,2}\left\|\frac{x_1-\tilde R_{11}}{h_{\mathtt{v}_3,1}}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(\tilde R)}. \tag{3.9}$$

On the other hand, with the same argument that we have used to obtain (2.22) in the proof of Theorem 2.5 we can show that

$$|(u_{\mathtt{v}_3}(\mathtt{v}_4)-u_{\mathtt{v}_4}(\mathtt{v}_4))|\left\|\frac{\partial\lambda_{\mathtt{v}_4}}{\partial x_1}\right\|_{L^2(R)} \le C\sum_{i=1}^{2}\left(h_{\mathtt{v}_3,i}\left\|\delta_{\tilde R}\frac{\partial^2 u}{\partial x_i\partial x_j}\right\|_{L^2(\tilde R)}\right)$$

which together with (3.7) and (3.9) concludes the proof of (3.1).

Now, to prove (3.2), using again Lemma 2.3, we have to estimate $\|\frac{\partial}{\partial x_2}(u_{\mathtt{v}_3}-\Pi u)\|_{L^2(R)}$. Using again the expression for the derivative of a $\mathcal{Q}_1$ function we have

$$
\begin{aligned}
\frac{\partial}{\partial x_2}(u_{\mathtt{v}_3}-\Pi u) &= -u_{\mathtt{v}_3}(\mathtt{v}_1)\frac{\partial\lambda_{\mathtt{v}_3}}{\partial x_2}+(u_{\mathtt{v}_3}(\mathtt{v}_4)-u_{\mathtt{v}_4}(\mathtt{v}_4))\frac{\partial\lambda_{\mathtt{v}_4}}{\partial x_2}-u_{\mathtt{v}_3}(\mathtt{v}_2)\frac{\partial\lambda_{\mathtt{v}_4}}{\partial x_2}\\
&= -u_{\mathtt{v}_3}(\mathtt{v}_1)\frac{\partial\lambda_{\mathtt{v}_3}}{\partial x_2}+(u_{\mathtt{v}_3}(\mathtt{v}_4)-u_{\mathtt{v}_3}(\mathtt{v}_2))\frac{\partial\lambda_{\mathtt{v}_4}}{\partial x_2}-\\
&\quad (u_{\mathtt{v}_4}(\mathtt{v}_4)-u_{\mathtt{v}_4}(\mathtt{v}_2))\frac{\partial\lambda_{\mathtt{v}_4}}{\partial x_2}-u_{\mathtt{v}_4}(\mathtt{v}_2)\frac{\partial\lambda_{\mathtt{v}_4}}{\partial x_2}\\
&=: -u_{\mathtt{v}_3}(\mathtt{v}_1)\frac{\partial\lambda_{\mathtt{v}_3}}{\partial x_2}+(I-II)\frac{\partial\lambda_{\mathtt{v}_4}}{\partial x_2}-u_{\mathtt{v}_4}(\mathtt{v}_2)\frac{\partial\lambda_{\mathtt{v}_4}}{\partial x_2}.
\end{aligned}
\tag{3.10}
$$

Defining now

$$\theta=(\theta_1,0):=(\mathtt{v}_4^1-\mathtt{v}_3^1-(h_{\mathtt{v}_4,1}-h_{\mathtt{v}_3,1})y_1,0)$$

and

$$F_y(t)=\frac{\partial u}{\partial x_2}(\mathtt{v}_3-h_{\mathtt{v}_3}:y+\theta t)$$

we have

$$
\begin{aligned}
I-II &= (\mathtt{v}_4^2-\mathtt{v}_2^2)\int\left[\frac{\partial u}{\partial x_2}(\mathtt{v}_3-h_{\mathtt{v}_3}:y)-\frac{\partial u}{\partial x_2}(\mathtt{v}_4-h_{\mathtt{v}_4}:y)\right]\psi(y)dy\\
&= (\mathtt{v}_4^2-\mathtt{v}_2^2)\int(F_y(0)-F_y(1))\psi(y)dy\\
&= -(\mathtt{v}_4^2-\mathtt{v}_2^2)\int\int_0^1 F_y'(t)dt\psi(y)dy
\end{aligned}
$$

but,

$$F_y'(t)=\frac{\partial^2 u}{\partial x_1\partial x_2}(\mathtt{v}_3-h_{\mathtt{v}_3}:y+\theta t)\theta_1$$

and so,

$$
\begin{aligned}
I-II &= -(\mathtt{v}_4^2-\mathtt{v}_2^2)\int_0^1\int\frac{\partial^2 u}{\partial x_1\partial x_2}(\mathtt{v}_3-h_{\mathtt{v}_3}:y+\theta t)\theta_1\psi(y)dydt\\
&=: -(\mathtt{v}_4^2-\mathtt{v}_2^2)\int_0^1 I(t)dt.
\end{aligned}
$$

We will estimate $I(t)$. Since supp $\psi\subset B(0,1)$ we have

$$|I(t)|\le Ch_{\mathtt{v}_3,1}\int\left|\frac{\partial^2 u}{\partial x_1\partial x_2}(\mathtt{v}_3-h_{\mathtt{v}_3}:y+\theta t)\right|\psi(y)dy.$$

Now, setting $z=\mathtt{v}_3-h_{\mathtt{v}_3}:y+\theta t$, taking into account that $Ch_{\mathtt{v}_3,1}\le(1-t)h_{\mathtt{v}_3,1}+th_{\mathtt{v}_4,1}(0\le t\le 1)$, and defining

$$\phi(z)=\psi\left(\frac{(1-t)\mathtt{v}_3^1+t\mathtt{v}_4^1-z_1}{(1-t)h_{\mathtt{v}_3,1}+th_{\mathtt{v}_4,1}},\frac{\mathtt{v}_3^2-z_2}{h_{\mathtt{v}_3,2}}\right)$$

we obtain

$$|I(t)|\le C\frac{1}{h_{\mathtt{v}_3,2}}\int\left|\frac{\partial^2 u}{\partial x_1\partial x_2}(z)\right|\phi(z)dz$$

and, since $\phi\equiv 0$ on $\partial\tilde{R}$ we can use Lemma 2.1 to obtain

$$\begin{aligned}|I(t)| &\le C\frac{1}{h_{\mathtt{v}_3,2}}\left\|\frac{\phi}{\delta_{\tilde{R}}}\right\|_{L^2(\tilde{R})}\left\|\delta_{\tilde{R}}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(\tilde{R})}\\ &\le C\frac{1}{h_{\mathtt{v}_3,2}}\left(h_{R,1}\left\|\frac{\partial\phi}{\partial z_1}\right\|_{L^2(\tilde{R})}+h_{R,2}\left\|\frac{\partial\phi}{\partial z_2}\right\|_{L^2(\tilde{R})}\right)\left\|\delta_{\tilde{R}}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(\tilde{R})}\\ &\le C\left(\frac{h_{R,1}}{h_{R,2}}\right)^{\frac12}\left\|\delta_{\tilde{R}}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(\tilde{R})}.\end{aligned}$$

Therefore,

$$|I-II|\le C(h_{R,1}h_{R,2})^{\frac12}\left\|\delta_{\tilde{R}}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(\tilde{R})},$$

so

$$\left\|(I-II)\frac{\partial\lambda_{\mathtt{v}_3}}{\partial x_2}\right\|_{L^2(R)}\le Ch_{R,1}\left\|\delta_{\tilde{R}}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(\tilde{R})}. \tag{3.11}$$

Now, to estimate the first term of formula (3.10), $u_{\mathtt{v}_3}(\mathtt{v}_1)\frac{\partial\lambda_{\mathtt{v}_3}}{\partial x_2}$, we observe that, since $u(x_1,0)\equiv 0$ then one can check that

$$\begin{aligned}u_{\mathtt{v}_3}(\mathtt{v}_1) &= -\int\int_{t=0}^{x_2}\frac{\partial^2 u}{\partial x_2^2}(x_1,t)(t-\mathtt{v}_1^2)\psi_{\mathtt{v}_3}(x)dtdx+\int(\mathtt{v}_1^1-x_1)\frac{\partial u}{\partial x_1}(x)\psi_{\mathtt{v}_3}(x)dx\\ &=: A+B.\end{aligned} \tag{3.12}$$

We will estimate A and B. Since $\mathtt{v}_1^2=0$ we have

$$\begin{aligned}|A| &\le Ch_{\mathtt{v}_3,2}\int\int_{t=0}^{x_2}\frac{x_1-\tilde{R}_{11}}{h_{\mathtt{v}_3,1}}\frac{t}{h_{\mathtt{v}_3,2}}\left|\frac{\partial^2 u}{\partial x_2^2}(x_1,t)\right|\psi_{\mathtt{v}_3}(x)\frac{h_{\mathtt{v}_3,1}}{x_1-\tilde{R}_{11}}dtdx\\ &\le Ch_{\mathtt{v}_3,2}\int_{l_{\tilde{R},2}}\int_{l_{\tilde{R},1}}\int_{l_{\tilde{R},2}}\delta_{-,\tilde{R}}(x_1,t)\left|\frac{\partial^2 u}{\partial x_2^2}(x_1,t)\right|\psi_{\mathtt{v}_3}(x)\frac{h_{\mathtt{v}_3,1}}{x_1-\tilde{R}_{11}}dtdx_1dx_2.\end{aligned}$$

Therefore, using the Schwarz inequality and (3.8) we obtain

$$|A|\le C\frac{(h_{\mathtt{v}_3,2})^{\frac32}}{(h_{\mathtt{v}_3,1})^{\frac12}}\left\|\delta_{-,\tilde{R}}\frac{\partial^2 u}{\partial x_2^2}\right\|_{L^2(\tilde{R})}$$

and then,

$$|A|\left\|\frac{\partial\lambda_{\mathtt{v}_3}}{\partial x_2}\right\|_{L^2(R)}\le Ch_{\mathtt{v}_3,2}\left\|\delta_{-,\tilde{R}}\frac{\partial^2 u}{\partial x_2^2}\right\|_{L^2(\tilde{R})}. \tag{3.13}$$

In order to estimate B we note that, since $\frac{\partial u}{\partial x_1}(x_1,0)\equiv 0$ then,

$$\begin{aligned}B &= \int_{l_{\tilde{R},1}}(\mathtt{v}_1^1-x_1)\int_{l_{\tilde{R},2}}\frac{\partial u}{\partial x_1}(x)\psi_{\mathtt{v}_3}(x)dx_2dx_1\\ &= \int_{l_{\tilde{R},1}}(\mathtt{v}_1^1-x_1)\int_{l_{\tilde{R},2}}\int_{t=0}^{x_2}\frac{\partial^2 u}{\partial x_2\partial x_1}(x_1,t)\psi_{\mathtt{v}_3}(x)dtdx_2dx_1.\end{aligned}$$

Then,

$$\begin{aligned}|B| &\le Ch_{\mathtt{v}_3,1}\int_{l_{\tilde{R},2}}\int_{l_{\tilde{R},1}}\int_{l_{\tilde{R},2}}\left|\frac{\partial^2 u}{\partial x_1\partial x_2}(x_1,t)\right|\psi_{\mathtt{v}_3}(x)dtdx_1dx_2\\ &\le C(h_{\mathtt{v}_3,1}h_{\mathtt{v}_3,2})^{\frac12}\left\|\frac{x_1-\tilde{R}_{11}}{h_{\mathtt{v}_3,1}}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(\tilde{R})}\end{aligned}$$

where we have used the Schwarz inequality and the same argument used to obtain (3.9). Consequently we obtain

$$|B|\left\|\frac{\partial\lambda_{\mathtt{v}_3}}{\partial x_2}\right\|_{L^2(R)}\leq Ch_{\mathtt{v}_3,1}\left\|\frac{x_1-\tilde{R}_{11}}{h_{\mathtt{v}_3,1}}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(\tilde{R})}$$

which together with (3.12) and (3.13) implies

$$\left\|u_{\mathtt{v}_3}(\mathtt{v}_1)\frac{\partial\lambda_{\mathtt{v}_3}}{\partial x_2}\right\|_{L^2(R)}\leq C\left\{h_{\mathtt{v}_3,2}\left\|\delta_{-,\tilde{R}}\frac{\partial^2 u}{\partial x_2^2}\right\|_{L^2(\tilde{R})}+h_{\mathtt{v}_3,1}\left\|\frac{x_1-\tilde{R}_{11}}{h_{\mathtt{v}_3,1}}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(\tilde{R})}\right\}.$$

Clearly an analogous estimate follows for $\|u_{\mathtt{v}_4}(\mathtt{v}_2)\frac{\partial\lambda_{\mathtt{v}_4}}{\partial x_2}\|_{L^2(R)}$, and then, in view of (3.10) and (3.11) we conclude the proof of inequality (3.2).

The proof of Part (ii) is, of course, analogous to that of Part (i).

Proof of Part (iii): We will use the notation of the Figure 3(d). Then

$$\Pi u|_R=u_{\mathtt{v}_4}(\mathtt{v}_4)\lambda_{\mathtt{v}_4}.$$

In this case the error can be split as

$$(u-\Pi u)_{|R}=(u-u_{\mathtt{v}_4})+(u_{\mathtt{v}_4}-\Pi u)$$

and it is enough to bound $u_{\mathtt{v}_4}-\Pi u$, which is piecewise $\mathcal{Q}_1$. Then we have

$$\begin{aligned}\frac{\partial}{\partial x_1}(u_{\mathtt{v}_4}-\Pi u)&=((u_{\mathtt{v}_4}-\Pi u)(\mathtt{v}_4)-(u_{\mathtt{v}_4}-\Pi u)(\mathtt{v}_3))\frac{\partial\lambda_{\mathtt{v}_4}}{\partial x_1}\\&\quad+((u_{\mathtt{v}_4}-\Pi u)(\mathtt{v}_2)-(u_{\mathtt{v}_4}-\Pi u)(\mathtt{v}_1))\frac{\partial\lambda_{\mathtt{v}_2}}{\partial x_1}\\&=-u_{\mathtt{v}_4}(\mathtt{v}_3)\frac{\partial\lambda_{\mathtt{v}_4}}{\partial x_1}+(u_{\mathtt{v}_4}(\mathtt{v}_2)-u_{\mathtt{v}_4}(\mathtt{v}_1))\frac{\partial\lambda_{\mathtt{v}_2}}{\partial x_1}.\end{aligned}\tag{3.14}$$

First we estimate $|u_{\mathtt{v}_4}(\mathtt{v}_2)-u_{\mathtt{v}_4}(\mathtt{v}_1)|$. Using that $\frac{\partial u}{\partial x_1}(x_1,0)\equiv 0$ we have

$$\begin{aligned}u_{\mathtt{v}_4}(\mathtt{v}_2)-u_{\mathtt{v}_4}(\mathtt{v}_1)&=\int(P(x,\mathtt{v}_2)-P(x,\mathtt{v}_1))\psi_{\mathtt{v}_4}(x)dx\\&=(\mathtt{v}_2^1-\mathtt{v}_1^1)\int\frac{\partial u}{\partial x_1}(x)\psi_{\mathtt{v}_4}(x)dx\\&=(\mathtt{v}_2^1-\mathtt{v}_1^1)\int\int_{t=0}^{x_2}\frac{\partial^2 u}{\partial x_1\partial x_2}(x_1,t)\psi_{\mathtt{v}_4}(x)dtdx.\end{aligned}$$

It follows that

$$\begin{aligned}|u_{\mathtt{v}_4}(\mathtt{v}_2)-u_{\mathtt{v}_4}(\mathtt{v}_1)|&\leq Ch_{\mathtt{v}_4,1}\int_{l_{\tilde{R},2}}\int_{l_{\tilde{R},1}}\int_{l_{\tilde{R},2}}\left|\frac{\partial^2 u}{\partial x_1\partial x_2}(x_1,t)\right|\psi_{\mathtt{v}_4}(x)dtdx_1dx_2\\&\leq Ch_{\mathtt{v}_4,1}\int_{l_{\tilde{R},2}}\int_{l_{\tilde{R},2}}\int_{l_{\tilde{R},1}}\frac{x_1}{h_{\mathtt{v}_4,1}}\left|\frac{\partial^2 u}{\partial x_1\partial x_2}(x_1,t)\right|\psi_{\mathtt{v}_4}(x)\frac{h_{\mathtt{v}_4,1}}{x_1}dx_1dtdx_2,\end{aligned}$$

and an argument similar to that used to obtain (3.9) gives

$$|u_{\mathtt{v}_4}(\mathtt{v}_2)-u_{\mathtt{v}_4}(\mathtt{v}_1)|\leq C(h_{\mathtt{v}_4,1}h_{\mathtt{v}_4,2})^{\frac{1}{2}}\left\|\frac{x_1}{h_{\mathtt{v}_4,1}}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(\tilde{R})}.$$

Therefore,

$$|u_{\mathtt{v}_4}(\mathtt{v}_2)-u_{\mathtt{v}_4}(\mathtt{v}_1)|\left\|\frac{\partial\lambda_{\mathtt{v}_2}}{\partial x_1}\right\|\leq Ch_{\mathtt{v}_4,2}\left\|\frac{x_1}{h_{\mathtt{v}_4,1}}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(\tilde{R})}.\tag{3.15}$$

Now we consider the other term in (3.14). We have to estimate $|u_{\mathtt{v}_4}(\mathtt{v}_3)|$. Using that $u(0,x_2)\equiv 0$ and $\mathtt{v}_3=(0,\mathtt{v}_3^2)$ we obtain

$$\begin{aligned} u_{\mathtt{v}_4}(\mathtt{v}_3) &= -\int\int_{t=0}^{x_1} t\frac{\partial^2 u}{\partial x_1^2}(t,x_2)\psi_{\mathtt{v}_4}(x)dtdx + \int(\mathtt{v}_3^2 - x_2)\frac{\partial u}{\partial x_2}(x)\psi_{\mathtt{v}_4}(x)dx \\ &=: A + B \end{aligned}$$

and we have to estimate A and B. We have

$$\begin{aligned} |A| &\le h_{\mathtt{v}_4,1}\int\int_{t=0}^{x_1}\frac{t}{h_{\mathtt{v}_4,1}}\frac{x_2}{h_{\mathtt{v}_4,2}}\left|\frac{\partial^2 u}{\partial x_1^2}(t,x_2)\right|\psi_{\mathtt{v}_4}(x)\frac{h_{\mathtt{v}_4,2}}{x_2}dtdx \\ &\le Ch_{\mathtt{v}_4,1}\int_{l_{\tilde R,1}}\int_{l_{\tilde R,2}}\int_{l_{\tilde R,1}}\delta_{-,\tilde R}(t,x_2)\left|\frac{\partial^2 u}{\partial x_1^2}(t,x_2)\right|\psi_{\mathtt{v}_4}(x)\frac{h_{\mathtt{v}_4,2}}{x_2}dtdx_2dx_1. \end{aligned}$$

But again, by an argument similar to that used in the proof of (3.9), we obtain

$$|A| \le C\frac{(h_{\mathtt{v}_4,1})^{\frac32}}{(h_{\mathtt{v}_4,2})^{\frac12}}\left\|\delta_{-,\tilde R}\frac{\partial^2 u}{\partial x_1^2}\right\|_{L^2(\tilde R)}.$$

Therefore,

$$|A|\left\|\frac{\partial\lambda_{\mathtt{v}_4}}{\partial x_1}\right\|_{L^2(R)} \le Ch_{\mathtt{v}_4,1}\left\|\delta_{-,\tilde R}\frac{\partial^2 u}{\partial x_1^2}\right\|_{L^2(\tilde R)}. \tag{3.16}$$

On the other hand, using now that $\frac{\partial u}{\partial x_2}(0,x_2)\equiv 0$, we have

$$\begin{aligned} B &= \int(\mathtt{v}_3^2 - x_2)\frac{\partial u}{\partial x_2}(x)\psi_{\mathtt{v}_4}(x)dx \\ &= \int_{l_{\tilde R,1}}\int_{l_{\tilde R,2}}(\mathtt{v}_3^2 - x_2)\int_{t=0}^{x_1}\frac{\partial^2 u}{\partial x_1\partial x_2}(t,x_2)\psi_{\mathtt{v}_4}(x)dtdx_2dx_1 \end{aligned}$$

and then

$$|B| \le Ch_{\mathtt{v}_4,2}\int_{l_{\tilde R,1}}\int_{l_{\tilde R,2}}\int_{l_{\tilde R,1}}\frac{x_2}{h_{\mathtt{v}_4,2}}\frac{\partial^2 u}{\partial x_1\partial x_2}(t,x_2)\psi_{\mathtt{v}_4}(x)\frac{h_{\mathtt{v}_4,2}}{x_2}dtdx_2dx_1.$$

Hence

$$|B|\left\|\frac{\partial\lambda_{\mathtt{v}_4}}{\partial x_1}\right\|_{L^2(R)} \le Ch_{\mathtt{v}_4,2}\left\|\frac{x_2}{h_{\mathtt{v}_4,2}}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(\tilde R)}. \tag{3.17}$$

Now, inequality (3.5) follows from (3.14), (3.15), (3.16) and (3.17).

Since (3.6) is analogous to (3.5) the proof is concluded. □

4. Application to a reaction-diffusion problem

As an example of application of our results we consider in this section the singular perturbation model problem

$$\begin{aligned} -\varepsilon^2\Delta u + u &= f \qquad \text{in } (0,2)\times(0,2) \\ u &= 0 \qquad \text{on } \partial\{(0,2)\times(0,2)\}. \end{aligned} \tag{4.1}$$

Compatibility conditions are assumed in order to have the regularity results proved in [15] and [16]. As we will show, appropriate graded anisotropic meshes can be defined in order to obtain almost optimal order error estimates in the energy norm valid uniformly in the parameter ε. These estimates follow from our results of Sections 2 and 3.

The meshes that we construct are very different from the Shishkin type meshes that have been used in other papers for this problem (see for example [2, 16]). In particular, our almost optimal error estimate in the energy norm is obtained with meshes independent of ε.

Given a partition $\mathcal{T}_h$ of $(0,2)\times(0,2)$ into rectangles, we call u_h the $\mathcal{Q}_1$ finite element approximation of the solution of problem (4.1). Since u_h is the orthogonal projection in the scalar product associated with the energy norm

$$\|v\|_\varepsilon = \left\{\varepsilon^2\|\nabla v\|^2_{L^2((0,2)^2)} + \|v\|^2_{L^2((0,2)^2)}\right\}^{\frac{1}{2}}$$

we know that, for any v_h in the finite element space,

$$\|u-u_h\|_\varepsilon \le \|u-v_h\|_\varepsilon.$$

In particular, if Π is the average interpolation operator associated with the partition $\mathcal{T}_h$ introduced in Section 2, we have

$$\|u-u_h\|_\varepsilon \le \|u-\Pi u\|_\varepsilon. \tag{4.2}$$

Therefore, we will construct the meshes in order to have a good estimate for the right hand side of (4.2).

We will obtain our estimates in $\Omega = (0,1)\times(0,1)$. Clearly, analogous arguments can be applied for the rest of the domain. The constant C will be always independent of ε.

In order to bound the part of the error which contains the first derivatives we will make use of the estimates obtained in the previous sections together with the fact that the solution of (4.1) satisfies some weighted a priori estimates which are valid uniformly in the parameter ε. We state these a priori estimates in the next two lemmas but postpone the proofs until the end of the section.

Lemma 4.1. *There exists a constant C such that if $\alpha \ge \frac{1}{2}$ then*

$$\left\|x_1^\alpha \frac{\partial u}{\partial x_1}\right\|_{L^2((0,\frac{3}{2})\times(0,\frac{3}{2}))} \le C \qquad \text{and} \qquad \left\|x_2^\alpha \frac{\partial u}{\partial x_2}\right\|_{L^2((0,\frac{3}{2})\times(0,\frac{3}{2}))} \le C. \tag{4.3}$$

Lemma 4.2. *There exists a constant C such that if $\alpha \ge \frac{1}{2}$ then*

$$\varepsilon\left\|x_1^\alpha \frac{\partial^2 u}{\partial x_1^2}\right\|_{L^2((0,\frac{3}{2})\times(0,\frac{3}{2}))} \le C, \qquad \varepsilon\left\|x_2^\alpha \frac{\partial^2 u}{\partial x_2^2}\right\|_{L^2((0,\frac{3}{2})\times(0,\frac{3}{2}))} \le C, \tag{4.4}$$

$$\varepsilon\left\|x_1^\alpha \frac{\partial u}{\partial x_1\partial x_2}\right\|_{L^2((0,\frac{3}{2})\times(0,\frac{3}{2}))} \le C \qquad \text{and} \qquad \varepsilon\left\|x_2^\alpha \frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2((0,\frac{3}{2})\times(0,\frac{3}{2}))} \le C. \tag{4.5}$$

To estimate the error in the L^2 norm we will use a priori estimates in the following norms. For $v: R \to \mathbb{R}$, where R is the rectangle $R = l_1 \times l_2$, define

$$\|v\|_{\infty\times\underline{1},R} := \Big\|\|v(x_1,\cdot)\|_{L^1(l_2)}\Big\|_{L^\infty(l_1)} \quad \text{and} \quad \|v\|_{\underline{1}\times\infty,R} := \Big\|\|v(\cdot,x_2)\|_{L^1(l_1)}\Big\|_{L^\infty(l_2)}. \tag{4.6}$$

Then we have the following lemma, which also will be proved at the end of the section.

Lemma 4.3. *There exists a constant C such that*

$$\left\|\frac{\partial u}{\partial x_1}\right\|_{\underline{1}\times\infty,(0,\frac{3}{2})\times(0,\frac{3}{2})} \le C \quad \text{and} \quad \left\|\frac{\partial u}{\partial x_2}\right\|_{\infty\times\underline{1},(0,\frac{3}{2})\times(0,\frac{3}{2})} \le C.$$

Let us now define the graded meshes. Given a parameter $h>0$ and $\alpha\in(0,1)$ we introduce the partition $\{\xi_i\}_{i=0}^N$ of the interval $[0,1]$ given by $\xi_0 = 0$, $\xi_1 = h^{\frac{1}{1-\alpha}}$, $\xi_{i+1} = \xi_i + h\xi_i^\alpha$ for $i = 2,\cdots,N-2$, where N is such that $\xi_{N-1} < 1$ and $\xi_{N-1} + h\xi_{N-1}^\alpha \ge 1$, and $\xi_N = 1$. We assume that the last interval $(\xi_{N-1},1)$ is not too small in comparison with the previous one (ξ_{N-2},ξ_{N-1}) (if this is not the case we just eliminate the node ξ_{N-1}).

We define the partitions $\mathcal{T}_{h,\alpha}$ such that they are symmetric with respect to the lines $x_1 = 1$ and $x_2 = 1$ and in the subdomain $\Omega = (0,1)\times(0,1)$ are given by

$$\{R\subset\Omega : R = (\xi_{i-1},\xi_i)\times(\xi_{j-1},\xi_j) \text{ for } 1\le i,j\le N\}.$$

Observe that the family of meshes $\mathcal{T}_{h,\alpha}$ satisfies our local regularity condition (2.1) with $\sigma = 2^\alpha$, that is, if $S, T \in \mathcal{T}_{h,\alpha}$ are neighboring elements then

$$\frac{h_{T,i}}{h_{S,i}} \le 2^\alpha.$$

For these meshes we have the following error estimates. We set $\tilde{\Omega} = \cup\{\tilde{R} : R \subset \Omega\}$ where we are using the notations of the previous sections.

Theorem 4.4. *If $u \in H^2(\Omega)$ and $u \equiv 0$ on $\{x : x_1 = 0 \text{ or } x_2 = 0\}$ then there exists a constant C such that*

$$\begin{aligned}\|u - \Pi u\|_{L^2(\Omega)} \le Ch &\left\{ \left\| x_1^\alpha \frac{\partial u}{\partial x_1} \right\|_{L^2(\tilde{\Omega})} + \left\| x_2^\alpha \frac{\partial u}{\partial x_2} \right\|_{L^2(\tilde{\Omega})} \right\} \\ &+ Ch^{\frac{1}{2-2\alpha}} \left\{ \left\| \frac{\partial u}{\partial x_2} \right\|_{\infty\times\underline{1},\tilde{\Omega}} + \left\| \frac{\partial u}{\partial x_1} \right\|_{\underline{1}\times\infty,\tilde{\Omega}} \right\},\end{aligned} \tag{4.7}$$

$$\left\| \frac{\partial(u - \Pi u)}{\partial x_1} \right\|_{L^2(\Omega)} \le Ch \left(\left\| x_1^\alpha \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2(\tilde{\Omega})} + \left\| (x_1^\alpha + x_2^\alpha) \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{\Omega})} \right), \tag{4.8}$$

and

$$\left\| \frac{\partial(u - \Pi u)}{\partial x_2} \right\|_{L^2(\Omega)} \le Ch \left(\left\| (x_1^\alpha + x_2^\alpha) \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{\Omega})} + \left\| x_2^\alpha \frac{\partial^2 u}{\partial x_2^2} \right\|_{L^2(\tilde{\Omega})} \right). \tag{4.9}$$

Proof. We will estimate the error on each element according to its position. So, we decompose the domain Ω into four parts, Ω_i, $i = 1, \dots, 4$ defined as

$$\begin{aligned}\Omega_1 &= [\xi_1, \xi_N]^2 & \Omega_2 &= [\xi_1, \xi_N] \times [0, \xi_1] \\ \Omega_3 &= [0, \xi_1] \times [\xi_1, \xi_N] & \Omega_4 &= [0, \xi_1]^2,\end{aligned}$$

and we set $\tilde{\Omega}_i = \cup\{\tilde{R} : R \subset \Omega_i\}$, $i = 1, \dots, 4$.

In order to prove (4.7) we split the error as follows

$$\|u - \Pi u\|^2_{L^2(\Omega)} = \sum_{i=1}^4 \|u - \Pi u\|^2_{L^2(\Omega_i)} =: S_1 + S_2 + S_3 + S_4. \tag{4.10}$$

First we estimate S_1. If $\tilde{R} \cap \{x : x_1 = 0 \text{ or } x_2 = 0\} = \emptyset$ we have that, for each $S \subset \tilde{R}$, $h_{S,1} \le h x_1^\alpha$ and $h_{S,2} \le h x_2^\alpha$ for all $(x_1, x_2) \in S$ and then, Theorem 2.4 gives

$$\begin{aligned}\|u - \Pi u\|^2_{L^2(R)} &\le C \left\{ h_{R,1}^2 \int_{\tilde{R}} \left| \frac{\partial u}{\partial x_1} \right|^2 dx + h_{R,2}^2 \int_{\tilde{R}} \left| \frac{\partial u}{\partial x_2} \right|^2 dx \right\} \\ &\le C \sum_{S \subset \tilde{R}} \left\{ h_{S,1}^2 \int_S \left| \frac{\partial u}{\partial x_1} \right|^2 dx + h_{S,2}^2 \int_S \left| \frac{\partial u}{\partial x_2} \right|^2 dx \right\} \\ &\le C \sum_{S \subset \tilde{S}} \left\{ h^2 \int_S x_1^{2\alpha} \left| \frac{\partial u}{\partial x_1} \right|^2 dx + h^2 \int_S x_2^{2\alpha} \left| \frac{\partial u}{\partial x_2} \right|^2 dx \right\} \\ &= C \left\{ h^2 \int_{\tilde{R}} x_1^{2\alpha} \left| \frac{\partial u}{\partial x_1} \right|^2 dx + h^2 \int_{\tilde{R}} x_2^{2\alpha} \left| \frac{\partial u}{\partial x_2} \right|^2 dx \right\}.\end{aligned}$$

Now, suppose that $R \subset \Omega_1$, $\tilde{R} \cap \{x : x_2 = 0\} \ne \emptyset$ and $\tilde{R} \cap \{x : x_1 = 0\} = \emptyset$. Then $\tilde{R}_{12} = 0$ and, if $S \subset \tilde{R}$, we have $h_{S,1} \le h x_1^\alpha$ for $(x_1, x_2) \in S$ and $h_{S,2} \le Ch^{\frac{1}{1-\alpha}}$. Therefore, using Theorem 2.4 we obtain

$$\begin{aligned}\|u-\Pi u\|_{L^2(R)}^2 \leq & Ch_{R,1}^2 \int_{\tilde R} \delta_{\tilde R}^{2\alpha}(x) \left|\frac{\partial u}{\partial x_1}\right|^2 dx + Ch_{R,2}^2 \int_{\tilde R} \delta_{\tilde R}^{2\alpha}(x) \left|\frac{\partial u}{\partial x_2}\right| dx \\ \leq & C \sum_{S\subset \tilde R} \left\{ h_{S,1}^2 \int_S \left|\frac{\partial u}{\partial x_1}\right|^2 dx + Ch_{S,2}^{2-2\alpha} \int_S x_2^{2\alpha} \left|\frac{\partial u}{\partial x_2}\right|^2 dx \right\} \\ \leq & C \sum_{S\subset \tilde R} \left\{ h^2 \int_S x_1^{2\alpha} \left|\frac{\partial u}{\partial x_1}\right|^2 dx + Ch^2 \int_S x_2^{2\alpha} \left|\frac{\partial u}{\partial x_2}\right|^2 dx \right\} \\ = & C \left\{ h^2 \int_{\tilde R} x_1^{2\alpha} \left|\frac{\partial u}{\partial x_1}\right|^2 dx + h^2 \int_{\tilde R} x_2^{2\alpha} \left|\frac{\partial u}{\partial x_2}\right|^2 dx \right\}.\end{aligned}$$

Now, if $0 \in \tilde R$, that is $\tilde R \cap \{x : x_1 = 0\} \neq \emptyset$ and $\tilde R \cap \{x : x_2 = 0\} \neq \emptyset$, then, $\tilde R_{11} = \tilde R_{12} = 0$ and $h_{R,1} \leq Ch^{\frac{1}{1-\alpha}}, h_{R,2} \leq Ch^{\frac{1}{1-\alpha}}$. Then, from Theorem 2.4 we have

$$\begin{aligned}\|u-\Pi u\|_{L^2(R)}^2 \leq & Ch_{R,1}^2 \int_{\tilde R} \delta_{\tilde R}^{2\alpha}(x) \left|\frac{\partial u}{\partial x_1}\right|^2 dx + Ch_{R,2}^2 \int_{\tilde R} \delta_{\tilde R}^{2\alpha}(x) \left|\frac{\partial u}{\partial x_2}\right| dx \\ \leq & Ch_{R,1}^{2-2\alpha} \int_{\tilde R} x_1^{2\alpha} \left|\frac{\partial u}{\partial x_1}\right|^2 dx + Ch_{R,2}^{2-2\alpha} \int_{\tilde R} x_2^{2\alpha} \left|\frac{\partial u}{\partial x_2}\right|^2 dx \\ \leq & C \left\{ h^2 \int_{\tilde R} x_1^{2\alpha} \left|\frac{\partial u}{\partial x_1}\right|^2 dx + h^2 \int_{\tilde R} x_2^{2\alpha} \left|\frac{\partial u}{\partial x_2}\right|^2 dx \right\}.\end{aligned}$$

A similar estimate can be obtained for $\|u-\Pi u\|_{L^2(R)}$ when $\tilde R \cap \{x : x_1 = 0\} \neq \emptyset$ and $\tilde R \cap \{x : x_2 = 0\} = \emptyset$. Therefore, we have

$$\begin{aligned}S_1 \leq & C \sum_{R\subset\Omega_1} \left\{ h^2 \int_{\tilde R} x_1^{2\alpha} \left|\frac{\partial u}{\partial x_1}\right|^2 dx + h^2 \int_{\tilde R} x_2^{2\alpha} \left|\frac{\partial u}{\partial x_2}\right|^2 dx \right\} \\ \leq & Ch^2 \int_{\tilde\Omega} x_1^{2\alpha} \left|\frac{\partial u}{\partial x_1}\right|^2 dx + Ch^2 \int_{\tilde\Omega} x_2^{2\alpha} \left|\frac{\partial u}{\partial x_2}\right|^2 dx.\end{aligned} \tag{4.11}$$

Now, we estimate S_2. From Theorem 2.4 we know that $\|\Pi u\|_{L^2(R)} \leq C\|u\|_{L^2(\tilde R)}$ for all $R \in \mathcal{T}_{h,\alpha}$ and therefore

$$S_2 = \sum_{R\subset\Omega_2} \|u-\Pi u\|_{L^2(R)}^2 \leq C \sum_{R\subset\Omega_2} \|u\|_{L^2(\tilde R)}^2 \leq C\|u\|_{L^2(\tilde\Omega_2)}^2. \tag{4.12}$$

So, we have to estimate $\|u\|_{L^2(\tilde\Omega_2)}$. We have $\tilde\Omega_2 = l_{\tilde\Omega_2,1} \times l_{\tilde\Omega_2,2}$ with $|l_{\tilde\Omega_2,1}| \leq C$ and $|l_{\tilde\Omega_2,2}| \leq Ch^{\frac{1}{1-\alpha}}$. Using that $u(x_1, 0) \equiv 0$ we have

$$\begin{aligned}\|u\|_{L^2(\tilde\Omega_2)}^2 = & \int_{l_{\tilde\Omega_2,1}} \int_{l_{\tilde\Omega_2,2}} u^2(x)dx \\ = & \int_{l_{\tilde\Omega_2,1}} \int_{l_{\tilde\Omega_2,2}} \left\{ \int_0^{x_2} \frac{\partial u}{\partial x_2}(x_1,t)dt \right\}^2 dx_2 dx_1 \\ \leq & C \int_{l_{\tilde\Omega_2,2}} \left\| \left\| \frac{\partial u}{\partial x_2}(x_1,\cdot) \right\|_{L^1(l_{\tilde\Omega_2,2})}^2 \right\|_{L^\infty(l_{\tilde\Omega_2,1})} dx_2 \\ \leq & Ch^{\frac{1}{1-\alpha}} \left\| \frac{\partial u}{\partial x_2} \right\|_{\infty\times\underline{1},\tilde\Omega_2}^2\end{aligned} \tag{4.13}$$

and so, it follows from (4.12) and (4.13) that

$$S_2 \leq Ch^{\frac{1}{1-\alpha}} \left\| \frac{\partial u}{\partial x_2} \right\|_{\infty\times\underline{1},\tilde\Omega_2}^2. \tag{4.14}$$

Analogously we can prove that

$$S_3 \le Ch^{\frac{1}{1-\alpha}} \left\| \frac{\partial u}{\partial x_1} \right\|^2_{\underline{1}\times\infty,\tilde{\Omega}_3}, \tag{4.15}$$

$$S_4 \le Ch^{\frac{2}{1-\alpha}} \left\| \frac{\partial u}{\partial x_2} \right\|^2_{\infty\times\underline{1},\tilde{\Omega}_4} \quad \text{and} \quad S_4 \le Ch^{\frac{2}{1-\alpha}} \left\| \frac{\partial u}{\partial x_1} \right\|^2_{\underline{1}\times\infty,\tilde{\Omega}_4} \tag{4.16}$$

and inserting inequalities (4.11), (4.14), (4.15) and (4.16) in (4.10) we obtain (4.7) (note that $\tilde{\Omega}_4 \subset \tilde{\Omega}_2$ and $\tilde{\Omega}_4 \subset \tilde{\Omega}_3$).

Let us now prove (4.8). Inequality (4.9) follows in a similar way. Again we use the decomposition of Ω into the four subsets $\Omega_i, i = 1, \dots, 4$ defined above. Then we have

$$\left\| \frac{\partial}{\partial x_1}(u - \Pi u) \right\|^2_{L^2(\Omega)} = \sum_{i=1}^{4} \left\| \frac{\partial}{\partial x_1}(u - \Pi u) \right\|^2_{L^2(\Omega_i)} =: S_1 + S_2 + S_3 + S_4 \tag{4.17}$$

and we have to estimate S_i, $i = 1, \dots, 4$.

For S_1, Theorem 2.5 gives

$$\begin{aligned} S_1 &= \sum_{R\subset\Omega_1} \left\| \frac{\partial}{\partial x_1}(u - \Pi u) \right\|^2_{L^2(R)} \\ &\le \sum_{R\subset\Omega_1} \left\{ h^2_{R,1} \int_{\tilde{R}} \delta^{2\alpha}_{\tilde{R}}(x) \left| \frac{\partial^2 u}{\partial x_1^2}(x) \right|^2 dx + h^2_{R,2} \int_{\tilde{R}} \delta^{2\alpha}_{\tilde{R}}(x) \left| \frac{\partial^2 u}{\partial x_1 \partial x_2}(x) \right|^2 dx \right\} \\ &=: \sum_{R\subset\Omega_1} I_R. \end{aligned}$$

Now, if $\tilde{R} \cap \{x : x_1 = 0 \text{ or } x_2 = 0\} = \emptyset$ we have

$$|I_R| \le C \sum_{T\subset\tilde{R}} \left\{ h^2_{T,1} \int_T \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx + h^2_{T,2} \int_T \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx \right\}$$

but, for $T \subset \Omega_1$, we have that

$$h_{T,1} \le Chx_1^\alpha, \qquad h_{T,2} \le Chx_2^\alpha \qquad \forall (x_1, x_2) \in T, \tag{4.18}$$

and therefore,

$$|I_R| \le C \left\{ h^2 \int_{\tilde{R}} x_1^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx + h^2 \int_{\tilde{R}} x_2^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx \right\}.$$

On the other hand, if $\tilde{R} \cap \{x : x_2 = 0\} \ne \emptyset$ and $\tilde{R} \cap \{x : x_1 = 0\} = \emptyset$, there are some elements $T \subset \tilde{R}$ that not verify condition (4.18). For a such elements T we have $h_{T,2} \le h^{\frac{1}{1-\alpha}}$ while the condition on $h_{T,1}$ in (4.18) remains valid. So we obtain

$$\begin{aligned} |I_R| &\le C \sum_{T\subset\tilde{R}} \left\{ h^2 \int_T x_1^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx + h^{2-2\alpha}_{T,2} \int_T x_2^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx \right\} \\ &\le C \sum_{T\subset\tilde{R}} \left\{ h^2 \int_T x_1^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx + h^2 \int_T x_2^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx \right\}. \end{aligned}$$

Now, if $(0,0) \in \tilde{R}$ we have $h_{T,1} \le Ch^{\frac{1}{1-\alpha}}$ and $h_{T,2} \le Ch^{\frac{1}{1-\alpha}}$ for all $T \subset \tilde{R}$ and therefore,

$$|I_R| \leq C \sum_{T\subset\tilde{R}} \left\{ h_{T,1}^{2-2\alpha} \int_T x_1^{2\alpha} \left|\frac{\partial^2 u}{\partial x_1^2}\right|^2 dx + h_{T,2}^{2-2\alpha} \int_T x_2^{2\alpha} \left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|^2 dx \right\}$$
$$\leq C \sum_{T\subset\tilde{R}} \left\{ h^2 \int_T x_1^{2\alpha} \left|\frac{\partial^2 u}{\partial x_1^2}\right|^2 dx + h^2 \int_T x_2^{2\alpha} \left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|^2 dx \right\}.$$

If $\tilde{R} \cap \{x : x_1 = 0\} \neq \emptyset$ and $\tilde{R} \cap \{x : x_2 = 0\} = \emptyset$ we can estimate I_R analogously and so we obtain

$$S_1 \leq C \left\{ h^2 \int_{\tilde{\Omega}} x_1^{2\alpha} \left|\frac{\partial^2 u}{\partial x_1^2}\right|^2 dx + h^2 \int_{\tilde{\Omega}} x_2^{2\alpha} \left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|^2 dx \right\}. \tag{4.19}$$

Let us now estimate S_2. From Theorem 3.1(i) we have

$$\begin{aligned} S_2 &\leq \sum_{R\in\Omega_2} \left\{ h_{R,1}^2 \int_{\tilde{R}} \delta_{-,\tilde{R}}^{2\alpha}(x) \left|\frac{\partial^2 u}{\partial x_1^2}\right|^2 dx + h_{R,2}^2 \int_{\tilde{R}} \left(\frac{x_1 - \tilde{R}_{11}}{h_{R,1}}\right)^{2\alpha} \left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|^2 dx \right\} \\ &=: \sum_{R\in\Omega_2} I_R. \end{aligned} \tag{4.20}$$

Now, if $R \subset \Omega_2$ is such that $\tilde{R} \cap \{x : x_1 = 0\} = \emptyset$ then, we have

$$|I_R| \leq C \left\{ \sum_{T\subset\tilde{R}} h_{R,1}^2 \int_T \left|\frac{\partial^2 u}{\partial x_1^2}\right|^2 dx + \sum_{T\subset\tilde{R}} h_{R,2}^2 \int_T \left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|^2 dx \right\}$$

but, in this case, for $T \subset \tilde{R}$,

$$h_{T,2} \leq C h_{T,1} \leq C h x_1^{\alpha} \qquad \forall x = (x_1, x_2) \in T$$

and therefore,

$$|I_R| \leq C \left\{ h^2 \int_{\tilde{R}} x_1^{2\alpha} \left|\frac{\partial^2 u}{\partial x_1^2}\right|^2 dx + h^2 \int_{\tilde{R}} x_1^{2\alpha} \left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|^2 dx \right\}. \tag{4.21}$$

On the other hand, if $R \subset \Omega_2$ is such that $\tilde{R} \cap \{x : x_1 = 0\} \neq \emptyset$, $\tilde{R}_{11} = 0$ and so, it follows from (4.20) that (note that $h_{R,2} \leq h_{R,1}$)

$$|I_R| \leq C \left\{ \sum_{T\subset\tilde{R}} h_{R,1}^{2-2\alpha} \int_T x_1^{2\alpha} \left|\frac{\partial^2 u}{\partial x_1^2}\right|^2 dx + \sum_{T\subset\tilde{R}} h_{R,1}^{2-2\alpha} \int_T x_1^{2\alpha} \left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|^2 dx \right\}$$

but in this case, for $T \subset \tilde{R}$, $h_{T,1} \leq C h^{\frac{1}{1-\alpha}}$ and then

$$|I_R| \leq h^2 \int_{\tilde{R}} x_1^{2\alpha} \left|\frac{\partial^2 u}{\partial x_1^2}\right|^2 dx + h^2 \int_{\tilde{R}} x_1^{2\alpha} \left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|^2 dx. \tag{4.22}$$

Therefore, inserting inequalities (4.21) and (4.22) in (4.20) we obtain

$$S_2 \leq C \left\{ h^2 \int_{\tilde{\Omega}_2} x_1^{2\alpha} \left|\frac{\partial^2 u}{\partial x_1^2}\right|^2 dx + h^2 \int_{\tilde{\Omega}_2} x_1^{2\alpha} \left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|^2 dx \right\}. \tag{4.23}$$

Let us now estimate S_3. Using Theorem 3.1(ii) we have

$$\begin{aligned} S_3 &\leq C \sum_{R\in\Omega_3} \left\{ h_{R,1}^2 \int_{\tilde{R}} \delta_{-,\tilde{R}}^{2\alpha}(x) \left|\frac{\partial^2 u}{\partial x_1^2}\right|^2 dx + h_{R,2}^2 \int_{\tilde{R}} \left(\frac{x_2 - \tilde{R}_{12}}{h_{R,2}}\right)^{2\alpha} \left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|^2 dx \right\} \\ &=: \sum_{R\in\Omega_3} I_R. \end{aligned}$$

If $R \subset \Omega_3$ is such that $\tilde{R} \cap \{x : x_2 = 0\} = \emptyset$ then, for $T \subset \tilde{R}$,

$$h_{T,1} \le C h^{\frac{1}{1-\alpha}} \qquad h_{T,2} \le C h x_2^{\alpha} \qquad \forall (x_1, x_2) \in T,$$

and so

$$(4.24)\qquad \begin{aligned} |I_R| &\le C \sum_{T\subset\tilde{R}} \left\{ h_{T,1}^{2-2\alpha} \int_T x_1^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx + h_{T,2}^2 \int_T \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx \right\} \\ &\le \sum_{T\subset\tilde{R}} \left\{ h^2 \int_T x_1^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx + h^2 \int_T x_2^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx \right\}. \end{aligned}$$

If $\tilde{R} \cap \{x : x_2 = 0\} \neq \emptyset$ then $\tilde{R}_{12} = 0$ and so (4.24) can be obtained also for this case using similar arguments. Therefore, we have

$$(4.25)\qquad S_3 \le C \left\{ h^2 \int_{\tilde{\Omega}_3} x_1^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx + h^2 \int_{\tilde{\Omega}_3} x_2^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx \right\}.$$

Finally, to estimate S_4, note that Ω_4 contains only one element R. Now, using Theorem 3.1(iii) and the fact that for this element $h_{R,1} = h_{R,2} = h^{\frac{1}{1-\alpha}}$ we obtain

$$(4.26)\qquad S_4 \le C h^2 \left\{ \left\| x_1^{\alpha} \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2(\tilde{R})}^2 + \left\| x_1^{\alpha} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})}^2 + \left\| x_2^{\alpha} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})}^2 \right\}.$$

Collecting the inequalities (4.19), (4.23), (4.25) and (4.26) we obtain (4.8) concluding the proof. □

As a consequence of Theorem 4.4 and the a priori estimates for the solution of problem (4.1) we obtain the following error estimates for the finite element approximations obtained using the family of meshes $\mathcal{T}_{h,\alpha}$. To simplify notation we omit the subscript α in the approximate solution.

Corollary 4.5. *Let u be the solution of (4.1) and u_h its $\mathcal{Q}_1$ finite element approximation obtained using the mesh $\mathcal{T}_{h,\alpha}$ with $\frac{1}{2} \le \alpha < 1$. If N is the number of nodes of $\mathcal{T}_{h,\alpha}$ then, there exists a constant C independent of ε and N such that*

$$(4.27)\qquad \|u - u_h\|_{\varepsilon} \le C \frac{1}{1-\alpha} \frac{1}{\sqrt{N}} \log N.$$

Proof. From (4.2), Lemmas 4.1, 4.2 and 4.3, and Theorem 4.4 (and its extension to the rest of $(0,2)\times(0,2)$) it follows that if h is small enough ($h < \frac{1}{2}$ is sufficient) and $\alpha \ge \frac{1}{2}$, then

$$\|u - u_h\|_{\varepsilon} \le Ch.$$

So we have to estimate h in terms of N. If we denote with M the number of nodes in each direction in the subdomain Ω, we have $N \sim M^2$ and we will estimate M. Let $f(\xi) = \xi + h\xi^{\alpha}$. Then, $\xi_0 = 0$, $\xi_1 = h^{\frac{1}{1-\alpha}}$ and $\xi_{i+1} = f(\xi_i), i = 1, \dots, M_f - 1$ where $M_f (= M)$ is the first number i such that $\xi_i \ge 1$. Since $\alpha < 1$ we have that

$$f(\xi) > \xi + h\xi =: g(\xi), \qquad \forall \xi \in (0,1).$$

Now, consider the sequence $\{\eta_i\}_{i=0}^{M_g}$ given by $\eta_1 = \xi_1$, and $\eta_{i+1} = g(\eta_i), i = 2, \dots M_g$ where M_g is defined analogously to M_f. Then, it is easy to see that $M_f < M_g$ and therefore, it is enough to estimate M_g. But, $M_g = [m]$ where m solves $(1+h)^{m-1}\xi_1 = 1$. Since $\xi_1 = h^{\frac{1}{1-\alpha}}$, for $0 \le h \le 1$, we obtain

$$(4.28)\qquad \frac{1}{1-\alpha} \frac{1}{h} \log \frac{1}{h} \le m - 1 \le C \frac{1}{1-\alpha} \frac{1}{h} \log \frac{1}{h}.$$

Now, from inequalities (4.28) we easily arrive at

$$h \le C \frac{1}{1-\alpha} \frac{1}{M} \log M$$

for all h small enough. □

The Lemmas 4.1, 4.2 and 4.3 are straightforward consequences of the following estimates

$$\left|\frac{\partial^k u}{\partial x_1^k}(x_1,x_2)\right| \leq C\left\{1+\varepsilon^{-k}e^{-\frac{x_1}{\varepsilon}}+\varepsilon^{-k}e^{-\frac{2-x_1}{\varepsilon}}\right\} \tag{4.29}$$

$$\left|\frac{\partial^k u}{\partial x_2^k}(x_1,x_2)\right| \leq C\left\{1+\varepsilon^{-k}e^{-\frac{x_2}{\varepsilon}}+\varepsilon^{-k}e^{-\frac{2-x_2}{\varepsilon}}\right\} \tag{4.30}$$

provided that $0 \leq k \leq 4$ and $(x_1,x_2) \in [0,2]\times[0,2]$, which are proved in [16]. As an example we prove the first inequality in (4.5). Observe that, for $r = 0,1,2$, $\frac{\partial^r u}{\partial x_i^r}(x_1,x_2) \equiv 0$ when $x_2 = 0$ or $x_2 = 2$ for $i = 1$ and when $x_1 = 0$ or $x_1 = 2$ for $i = 2$. Then we have

$$\begin{aligned}
\int_0^2\int_0^{\frac{3}{2}} x_1^{2\alpha}\left|\frac{\partial^2 u}{\partial x_1\partial x_2}\right|^2 dx_1dx_2 &= -\int_0^{\frac{3}{2}}\int_0^2 x_1^{2\alpha}\frac{\partial u}{\partial x_1}\frac{\partial^3 u}{\partial x_1\partial x_2^2}dx_2dx_1\\
&= -\int_0^2\left\{x_1^{2\alpha}\frac{\partial u}{\partial x_1}\frac{\partial^2 u}{\partial x_2^2}\bigg|_{x_1=0}^{\frac{3}{2}} - \int_0^{\frac{3}{2}}\frac{\partial}{\partial x_1}\left(x_1^{2\alpha}\frac{\partial u}{\partial x_1}\right)\frac{\partial^2 u}{\partial x_2^2}dx_1\right\}dx_2\\
&= -(\frac{3}{2})^{2\alpha}\int_0^2\frac{\partial u}{\partial x_1}(\frac{3}{2},x_2)\frac{\partial^2 u}{\partial x_2^2}(\frac{3}{2},x_2)dx_2 + \int_0^2\int_0^{\frac{3}{2}} 2\alpha x_1^{2\alpha-1}\frac{\partial u}{\partial x_1}\frac{\partial^2 u}{\partial x_2^2}dx_1dx_2\\
&\quad + \int_0^2\int_0^{\frac{3}{2}} x_1^{2\alpha}\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 u}{\partial x_2^2}dx_1dx_2\\
&=: I + II + III.
\end{aligned} \tag{4.31}$$

Now, since

$$\begin{aligned}
&\left|\frac{\partial u}{\partial x_1}(\frac{3}{2},x_2)\right| \leq C\\
&\left|\frac{\partial^2 u}{\partial x_2^2}(\frac{3}{2},x_2)\right| \leq C(1+\varepsilon^{-2})\\
&\left|\frac{\partial u}{\partial x_1}(x_1,x_2)\right| \leq C(1+\varepsilon^{-1}e^{-\frac{x_1}{\varepsilon}}) \quad (0 \leq x_1 \leq 3/2)\\
&\left|\frac{\partial^2 u}{\partial x_1^2}(x_1,x_2)\right| \leq C(1+\varepsilon^{-2}e^{-\frac{x_1}{\varepsilon}}) \quad (0 \leq x_1 \leq 3/2)\\
&\left|\frac{\partial^2 u}{\partial x_2^2}(x_1,x_2)\right| \leq C(1+\varepsilon^{-2})
\end{aligned}$$

we easily obtain

$$|I| \leq C(1+\varepsilon^{-2}) \tag{4.32}$$

$$|II| \leq C(\varepsilon^{-2}+\varepsilon^{2\alpha-3}) \tag{4.33}$$

$$|III| \leq C(\varepsilon^{-2}+\varepsilon^{2\alpha-3}). \tag{4.34}$$

Now, using inequalities (4.32), (4.33) and (4.34) in (4.31) we conclude the proof.

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