Approximation of the vibration modes of a plate by Reissner-Mindlin equations

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Abstract

This paper deals with the approximation of the vibration modes of a plate modelled by the Reissner-Mindlin equations. It is well known that, in order to avoid locking, some kind of reduced integration or mixed interpolation has to be used when solving this equations by finite element methods. In particular one of the most widely used procedures is the mixed interpolation tensorial components based on the family of elements called MITC. We use the lowest order method of this family.

Applying a general approximation theory for spectral problems, we obtain optimal order error estimates for the eigenvectors and the eigenvalues. Under mild assumptions, these estimates are valid with constants independent of the plate thickness. The optimal double order for the eigenvalues is derived from a corresponding L^2-estimate for a load problem which is proven here. This optimal order L^2-estimate is of interest in itself.

Finally, we present several numerical examples showing the very good behavior of the numerical procedure even in some cases not covered by our theory.

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1 Introduction

Finite element discretization of Reisnner-Mindlin equations is, to date, the usual way to approximate the bending of an elastic plate of moderate or small thickness (see for instance [4, 13, 19]).

For load problems, because of the ellipticity of these equations, the classical theory ensures the convergence of standard finite element approximations. However these elements lead to wrong results when the thickness is small with respect to the other dimensions of the plate. This is because of the so-called *locking* phenomenon, which is now well understood (see for instance [7]). In order to avoid this drawback, reduced integration or mixed methods are usually applied. To perform their mathematical analysis, a family of problems, one for each thickness $t > 0$, is considered and approximation results valid uniformly on t are sought (see for instance [2, 7, 10]).

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Among these methods, the so-called *MITC* ones, introduced by Bathe and Dvorkin in [6], are very likely the most used in practice. Their application to load problems has also deserved much attention from the mathematical point of view ([5, 8, 10, 15, 16]). The aim of this paper is to analyze one of these methods, that of lowest order, when applied to approximate the free vibration bending modes of a plate. Being non conforming, the spectral theory based on *minimum-maximum* principles (see Section 8 of [3]) cannot be directly applied to this method. Instead, our analysis will be based on the abstract spectral theory for compact operators as stated in Section 7 of [3].

Optimal order of convergence in H^1 norm is known for the application of the lowest order MITC method to load problems ([8, 10, 16]). This, combined with known regularity results ([2, 7]), allow us to prove analogous estimates for the approximation of the eigenfunctions in vibration problems. Let us remark that this would not be the case for higher order methods, since the eigenfunctions are in general not regular enough as to attain similar results.

For conforming methods (which is not our case) the convergence of the eigenfunctions in H^1 directly yields a double order of convergence for the eigenvalues (Lemma 9.1 in [3]). Alternatively, when such double order holds for the L^2 convergence in the corresponding load problem without further assumptions on the regularity of the solution, the abstract spectral theory can be used to prove a similar result even for non conforming methods (see for instance Section 7 of [3]).

We prove such optimal (in order and regularity) L^2 error estimates for the lowest order MITC method, valid uniformly on the thickness t. This kind of estimates have been proved before for higher order MITC methods in [8, 15], but the arguments therein do not apply to our case. Thus, our results complete the analysis of the MITC elements.

In Section 2 we present the mathematical setting and state the spectral approximation results. The analysis carried out yield t-independent optimal error estimates for the approximation of eigenfunctions and eigenvalues. The proofs are valid for eigenvalues remaining uniformly separated of the rest of the spectrum as the thickness becomes small. These results rely on properties of the associated load problems which are proved in Section 3. Finally, in Section 4, we present numerical experiments confirming the theoretical results and showing the good performance of the method. We also exhibit the potential applicability of this method to problems not covered by the theoretical analysis.

2 Approximation of the eigenvalue problem

Consider an elastic plate of thickness t with reference configuration $\Omega \times (-\frac{t}{2}, \frac{t}{2})$, where $\Omega \in \mathbb{R}^2$ is a convex polygonal domain. The deformation of the plate is described by means of Reissner-Mindlin model in terms of the rotations $\beta_t = (\beta_t^1, \beta_t^2)$ of its midplane Ω and the transverse displacement w_t (see for example [7]).

Assuming that the plate is clamped, its free vibration modes are solutions of the following problem:

Find non trivial $(\beta_t, w_t) \in H_0^1(\Omega)^2 \times H_0^1(\Omega)$ and $\omega_t > 0$ such that:

$$t^3 a(\beta_t, \eta) + \kappa t \int_\Omega (\nabla w_t - \beta_t) \cdot (\nabla v - \eta) = \omega_t^2 \left[t \int_\Omega \rho\, w_t v + \frac{t^3}{12} \int_\Omega \rho\, \beta_t \cdot \eta \right],$$
$$\forall (\eta, v) \in H_0^1(\Omega)^2 \times H_0^1(\Omega), \tag{2.1}$$

where ω_t is the angular vibration frequency, a is the $H_0^1(\Omega)^2$-elliptic bilinear form defined by

$$a(\beta, \eta) := \frac{E}{12(1-\nu^2)} \int_\Omega \left[\sum_{i,j=1}^{2} (1-\nu) \varepsilon_{ij}(\beta) \varepsilon_{ij}(\eta) + \nu \operatorname{div} \beta \operatorname{div} \eta \right],$$

$\varepsilon_{ij}(\beta)$ being the linear strain tensor, E the Young modulus and ν the Poisson ratio, $\kappa := Ek/2(1+\nu)$ is the shear modulus, with k being a correction factor usually taken as $5/6$, and ρ is the density.

The lowest vibration frequencies ω_t correspond to bending modes of the plate and for each of them $\lambda_t := \rho\,\omega_t^2/t^2$ attains a finite limit as t goes to zero as it is shown below. Therefore it is convenient for the mathematical analysis to consider the following rescaled eigenvalue problem:

$$a(\beta_t,\eta) + \frac{\kappa}{t^2}(\nabla w_t - \beta_t, \nabla v - \eta) = \lambda_t \left[(w_t, v) + \frac{t^2}{12}(\beta_t,\eta) \right], \quad \forall (\eta, v) \in H_0^1(\Omega)^2 \times H_0^1(\Omega), \tag{2.2}$$

where $(\cdot,\cdot)$ denotes the standard L^2 inner product. Note that all the eigenvalues in (2.2) are strictly positive because of the symmetry of the involved bilinear forms and the ellipticity of its left hand side (see [7]).

Introducing the shear strain $\gamma_t := \frac{\kappa}{t^2}(\nabla w_t - \beta_t)$, problem (2.2) can be written as

$$\begin{cases} a(\beta_t,\eta) + (\gamma_t, \nabla v - \eta) = \lambda_t \left[(w_t, v) + \frac{t^2}{12}(\beta_t,\eta) \right], & \forall (\eta, v) \in H_0^1(\Omega)^2 \times H_0^1(\Omega), \\ \gamma_t = \frac{\kappa}{t^2}(\nabla w_t - \beta_t). \end{cases} \tag{2.3}$$

In order to analyze the approximation of this eigenvalue problem it is convenient to introduce the operator

$$T_t : L^2(\Omega)^2 \times L^2(\Omega) \to L^2(\Omega)^2 \times L^2(\Omega),$$

defined by $T_t(\theta, f) := (\beta_t, w_t)$, where $(\beta_t, w_t) \in H_0^1(\Omega)^2 \times H_0^1(\Omega)$ is the solution of

$$\begin{cases} a(\beta_t,\eta) + (\gamma_t, \nabla v - \eta) = (f, v) + \frac{t^2}{12}(\theta,\eta), & \forall (\eta, v) \in H_0^1(\Omega)^2 \times H_0^1(\Omega), \\ \gamma_t = \frac{\kappa}{t^2}(\nabla w_t - \beta_t). \end{cases} \tag{2.4}$$

We denote by $(\cdot,\cdot)_t$ the weighted $L^2(\Omega)^2 \times L^2(\Omega)$ inner product in the right hand side of the first equation of (2.4) and by $|\cdot|_t$ its corresponding norm. Clearly T_t is compact and selfadjoint with respect to $(\cdot,\cdot)_t$. Then, apart from $\mu = 0$, its spectrum consists of a sequence of finite multiplicity real eigenvalues converging to zero. Note that λ_t is an eigenvalue of (2.3) if and only if $\mu_t := 1/\lambda_t$ is an eigenvalue of T_t with the same multiplicity and corresponding eigenfunctions.

It is known (see [7]) that, when $t \to 0$, the solution of problem (2.4) converges to $(\beta_0, w_0) \in H_0^1(\Omega)^2 \times H_0^1(\Omega)$ such that there exists $\gamma_0 \in H_0(\mathrm{rot},\Omega)'$ satisfying

$$\begin{cases} a(\beta_0,\eta) + \langle \gamma_0, \nabla v - \eta \rangle = (f, v), & \forall (\eta, v) \in H_0^1(\Omega)^2 \times H_0^1(\Omega), \\ \nabla w_0 - \beta_0 = 0, \end{cases} \tag{2.5}$$

where $\langle\cdot,\cdot\rangle$ stands for the duality pairing in $H_0(\mathrm{rot},\Omega) := \{\psi \in L^2(\Omega)^2 : \ \mathrm{rot}\,\psi \in L^2(\Omega) \text{ and } \psi\cdot\tau = 0 \text{ on } \partial\Omega\}$ (τ being a unit vector tangent to $\partial\Omega$). This is a mixed formulation of the Kirchhoff model for the deflection of clamped thin plates:

$$w_0 \in H_0^2(\Omega) \qquad \text{and} \qquad \frac{E}{12(1-\nu^2)}\Delta^2 w_0 = f, \quad \text{in } \Omega.$$

Moreover, defining $T_0(\theta, f) := (\beta_0, w_0)$ for $(\theta, f) \in L^2(\Omega)^2 \times L^2(\Omega)$, we will show in the next section that for $0 \le t \le t_{\max}$

$$\|(T_t - T_0)(\theta, f)\|_{H^1(\Omega)^2 \times H^1(\Omega)} \le C t\, |(\theta, f)|_t. \tag{2.6}$$

In particular, since for $t \in (0, t_{\max})$, $|\cdot|_t \le t_{\max}\|\cdot\|_{H^1(\Omega)^2\times H^1(\Omega)}$, it follows that $T_t|_{H^1(\Omega)^2\times H^1(\Omega)}$ converges to $T_0|_{H^1(\Omega)^2\times H^1(\Omega)}$ in norm. Then, standard properties of separation of isolated parts of the spectrum (see for instance [14]) yield the following result:

Lemma 2.1 *Let $\mu_0 > 0$ be an eigenvalue of T_0 of multiplicity m. Let D be any disc in the complex plane centered at μ_0 and containing no other element of the spectrum of T_0. Then, for t small enough, D contains exactly m eigenvalues of T_t (repeated according to their respective multiplicities). Consequently, each eigenvalue $\mu_0 > 0$ of T_0 is a limit of eigenvalues μ_t of T_t, as t goes to zero.*

Mixed finite elements for the load problem (2.4) have been introduced and analyzed in several papers (see for example [6, 8, 10, 15, 16]). The method that we will use here can be seen as the lowest degree member of the so-called MITC elements, which are based on relaxing the second equation of (2.3). In order to recall this method let us introduce some notation.

Assume that we have a family of triangulations $\{\mathcal{T}_h\}$ satisfying the usual minimum angle condition. The finite element space for the rotations consists of piecewise linear functions augmented in such a way that they have quadratic tangential components on the boundary of each element. Namely, for each $K \in \mathcal{T}_h$, let n be a unit normal on ∂K and define

$$\mathcal{Q}(K) := \{\eta \in \mathcal{P}_2(K)^2 : \eta \cdot n|_\ell \in \mathcal{P}_1(\ell), \text{ for each edge } \ell \text{ of } K\};$$

then, the finite element space for the rotations is defined by

$$H_h := \{\eta \in H_0^1(\Omega)^2 : \eta|_K \in \mathcal{Q}(K), \ \forall K \in \mathcal{T}_h\}.$$

For the transverse displacements we take standard piecewise linear elements, namely,

$$W_h := \{v \in H_0^1(\Omega) : v|_K \in \mathcal{P}_1(K), \ \forall K \in \mathcal{T}_h\}.$$

In order to define the method, we also need the *reduction operator*

$$R : H^1(\Omega)^2 \cap H_0(\mathrm{rot}, \Omega) \longrightarrow \Gamma_h,$$

where Γ_h is the lowest order rotated Raviart-Thomas space, namely,

$$\Gamma_h := \{\psi \in H_0(\mathrm{rot}, \Omega) : \psi|_K \in \mathcal{P}_0^2 \oplus (-x_2, x_1)\mathcal{P}_0, \ \forall K \in \mathcal{T}_h\},$$

and R is the operator locally defined for each $\psi \in H^1(\Omega)^2$ by (see [7, 17])

$$\int_\ell R\psi \cdot \tau = \int_\ell \psi \cdot \tau, \tag{2.7}$$

for every edge ℓ of the triangulation (τ being a unit tangent vector along ℓ). It is easy to see that the operator R satisfies

$$\int_K \mathrm{rot}\,(\psi - R\psi) = 0, \qquad \forall \psi \in H^1(\Omega)^2, \tag{2.8}$$

for any element $K \in \mathcal{T}_h$, and it is also known ([7, 17]) that

$$\|\psi - R\psi\|_0 \le Ch\|\psi\|_1; \tag{2.9}$$

here and thereafter $\|\cdot\|_k$ denotes the standard norm of $H^k(\Omega)$ or $H^k(\Omega)^2$, as corresponds.

Now, the finite element approximate solution $(\beta_{th}, w_{th}) \in H_h \times W_h$ of the load problem (2.4) is defined by

$$\begin{cases} a(\beta_{th}, \eta) + (\gamma_{th}, \nabla v - R\eta) = (f, v) + \dfrac{t^2}{12}(\theta, \eta), & \forall (\eta, v) \in H_h \times W_h, \\ \gamma_{th} = \dfrac{\kappa}{t^2}(\nabla w_{th} - R\beta_{th}). \end{cases} \tag{2.10}$$

The method is non conforming since consistency terms arise because of the reduction operator. Existence and uniqueness of the solution of (2.10) follows easily (see [10]). As in the continuous case, we introduce the operator

$$T_{th} : L^2(\Omega)^2 \times L^2(\Omega) \longrightarrow L^2(\Omega)^2 \times L^2(\Omega)$$

given by $T_{th}(\theta, f) := (\beta_{th}, w_{th})$. T_{th} is also selfadjoint with respect to $(\cdot,\cdot)_t$.

The corresponding eigenvalue problem reads

$$\begin{cases} a(\beta_{th}, \eta) + (\gamma_{th}, \nabla v - R\eta) = \lambda_{th}\left[(w_{th}, v) + \dfrac{t^2}{12}(\beta_{th}, \eta)\right], & \forall (\eta, v) \in H_h \times W_h, \\ \gamma_{th} = \dfrac{\kappa}{t^2}(\nabla w_{th} - R\beta_{th}). & \end{cases}$$

Once more λ_{th} is an eigenvalue of this problem if and only if $\mu_{th} := 1/\lambda_{th}$ is a strictly positive eigenvalue of T_{th} with the same multiplicity and corresponding eigenfunctions.

For $t > 0$ fixed, the spectral theory for compact operators in [3] can be directly applied to prove convergence of the eigenpairs of T_{th} to those of T_t. However, further considerations are needed to show that the error estimates do not deteriorate as t becomes small.

To this goal we will make use of the following result, which means that optimal error estimates in the H^1 norm for the rotations and the transverse displacement hold for the approximations given by (2.10):

$$\|(T_t - T_{th})(\theta, f)\|_{H^1(\Omega)^2 \times H^1(\Omega)} \leq Ch\,|(\theta, f)|_t, \tag{2.11}$$

with a constant C independent of t and h. This has been proved for instance in [10] for pure transversal loads (i.e., $\theta = 0$), but the proofs therein extend trivially to our case.

As a consequence of (2.11), if μ_t is an eigenvalue of T_t with multiplicity m, then exactly m eigenvalues $\mu_{th}^{(1)}, \ldots, \mu_{th}^{(m)}$ of T_{th} (repeated according to their respective multiplicities) converge to μ_t as h goes to zero (see [14]). The following theorem shows that, under mild assumptions, optimal t-independent error estimates in the H^1 norm are valid for the eigenfunctions:

Theorem 2.1 *Let μ_t be an eigenvalue of T_t converging to a simple eigenvalue μ_0 of T_0, as t goes to zero. Let μ_{th} be the eigenvalue of T_{th} that converges to μ_t, as h goes to zero. Let (β_t, w_t) and (β_{th}, w_{th}) be the corresponding eigenfunctions normalized in the same manner. Then, for t and h small enough, it holds*

$$\|(\beta_t, w_t) - (\beta_{th}, w_{th})\|_{H^1(\Omega)^2 \times H^1(\Omega)} \leq Ch, \tag{2.12}$$

with a constant C independent of t and h.

PROOF. For any fixed $t \in (0, t_{\max})$, $|\cdot|_t \leq t_{\max}\|\cdot\|_{H^1(\Omega)^2\times H^1(\Omega)}$. Hence, because of (2.11), $T_{th}|_{H^1(\Omega)^2\times H^1(\Omega)}$ converges to $T_t|_{H^1(\Omega)^2\times H^1(\Omega)}$ in norm. Then (2.12) is a direct consequence of the estimate (2.11) and Theorem 7.1 in [3], with a constant C depending on the constant in (2.11) (which is independent of t) and on the inverse of the distance of μ_t to the rest of the spectrum of T_t. Now, according to Lemma 2.1, (2.6) implies that for t small enough this distance can be bounded below in terms of the distance of μ_0 to the rest of the spectrum of T_0, which obviously does not depend on t. $\square$

In the next section we will prove the following higher order L^2 error estimate for the aproximate solution of the load problem (2.4):

$$\|(T_t - T_{th})(\theta, f)\|_{L^2(\Omega)^2 \times L^2(\Omega)} \leq Ch^2|(\theta, f)|_t, \tag{2.13}$$

with a constant C independent of t and h. By using it we are able to prove a double order of convergence for the eigenvalues:

Theorem 2.2 *Let μ_t and μ_{th} be as in Theorem 2.1. Then, for t and h small enough, it holds*

$$|\mu_t - \mu_{th}| \leq Ch^2,$$

with a constant C independent of t and h.

PROOF. Let (β_t, w_t) be an eigenfunction corresponding to μ_t normalized in $|\cdot|_t$. Since T_t and T_{th} are selfadjoint with respect to $(\cdot,\cdot)_t$, we may apply Remark 7.5 in [3] which in our case reads

$$|\mu_t - \mu_{th}| \leq C\left[\Big((T_t - T_{th})(\beta_t, w_t), (\beta_t, w_t)\Big)_t + |(T_t - T_{th})(\beta_t, w_t)|_t^2\right], \tag{2.14}$$

with a constant C depending only on the inverse of the distance of μ_t to the rest of the spectrum of T_t. By repeating the arguments in the proof of Theorem 2.1 we observe that, for t small enough, this constant can be chosen independent of t. Thus, since $|\cdot|_t \leq t_{\max}\|\cdot\|_{L^2(\Omega)^2\times L^2(\Omega)}$, by using estimate (2.13) in (2.14) we conclude the proof. $\square$

Another consequence of estimate (2.13) is a double order of convergence in the L^2 norm for the eigenfunctions:

Theorem 2.3 *Let μ_t, μ_{th}, (β_t, w_t) and (β_{th}, w_{th}) be as in Theorem 2.1. Then, for t and h small enough, it holds*

$$\|(\beta_t, w_t) - (\beta_{th}, w_{th})\|_{L^2(\Omega)^2\times L^2(\Omega)} \leq Ch^2,$$

with a constant C independent of t and h.

PROOF. Since $|\cdot|_t \leq t_{\max}\|\cdot\|_{L^2(\Omega)^2\times L^2(\Omega)}$, the arguments in the proof of Theorem 2.1 can be repeated using $\|\cdot\|_{L^2(\Omega)^2\times L^2(\Omega)}$ instead of $\|\cdot\|_{H^1(\Omega)^2\times H^1(\Omega)}$ and estimate (2.13) instead of estimate (2.11). $\square$

The three theorems of this section are stated for those eigenvalues of Reissner-Mindlin equations converging to simple eigenvalues of Kirchhoff model. A multiple eigenvalue of the latter arises usually because of symmetries of the geometry of the plate; in such a case, the eigenvalue of the former converging to it has the same multiplicity. The proofs of these theorems extend trivially to cover this case.

Instead, if the Kirchhoff equations had a multiple eigenvalue not due to symmetry reasons, it could split into different eigenvalues in the Reissner-Mindlin model. In this case, the proofs of the theorems above do not provide estimates independent of the thickness. In fact, the constants therein blow up as the distance between the Reissner-Mindlin eigenvalues becomes smaller.

For conforming methods, the *minimum-maximum principles* yield estimates not involving this distance (see, for instance, Section 8 of [3]). However, to the best of our knowledge, estimates of this kind have not been proved for non conforming methods as ours. Nevertheless, the numerical experiments in Section 4 show that such estimates also hold in this case for our method.

3 Proofs

The optimal spectral convergence results in the previous section rely on properties (2.6), (2.11) and (2.13). The proof of the first one is standard, but we include it for the sake of completeness. The second one is an H^1 norm estimate for the load problem including shear loads and its proof is an immediate extension of those in [8, 10, 16]. Instead, property (2.13) regarding the approximation with optimal order in the L^2 norm was not previously known. Similar L^2 estimates have been proved for higher order methods (see [8, 15]), but with proofs relying on arguments which do not apply to the lowest order case we are considering.

In what follows we will make use of the known *a priori* estimates for the solutions of problems (2.4) and (2.5) (see for instance [2]):

$$\begin{aligned} \|\beta_t\|_2 + \|w_t\|_2 + \|\gamma_t\|_0 + t\,\|\gamma_t\|_1 &\leq C\left(t^2\|\theta\|_0 + \|f\|_0\right) \\ &\leq C\,|(\theta, f)|_t \end{aligned} \tag{3.1}$$

valid for $0 \leq t \leq t_{\max}$, with a constant C independent of t.

We begin with the proof of property (2.6).

Lemma 3.1 *There exists a constant C, independent of t, such that*

$$\|(T_t - T_0)(\theta, f)\|_{H^1(\Omega)^2 \times H^1(\Omega)} \leq Ct\,|(\theta, f)|_t,$$

for all $(\theta, f) \in L^2(\Omega)^2 \times L^2(\Omega)$.

Proof. Subtracting (2.5) from (2.4) we have

$$\begin{cases} a(\beta_t - \beta_0, \eta) + (\gamma_t - \gamma_0, \nabla v - \eta) = \dfrac{t^2}{12}(\theta, \eta), \qquad \forall(\eta, v) \in H_0^1(\Omega)^2 \times H_0^1(\Omega), \\ \gamma_t = \dfrac{\kappa}{t^2}\Big[\nabla(w_t - w_0) - (\beta_t - \beta_0)\Big], \end{cases}$$

and taking $\eta = \beta_t - \beta_0$ and $v = w_t - w_0$ we obtain

$$a(\beta_t - \beta_0, \beta_t - \beta_0) = \frac{t^2}{12}(\theta, \beta_t - \beta_0) - \frac{t^2}{\kappa}(\gamma_t - \gamma_0, \gamma_t).$$

Now, using the coerciveness of a and the *a priori* estimate (3.1) for $\|\gamma_t\|_0$ and $\|\gamma_0\|_0$ we have

$$\begin{aligned} \|\beta_t - \beta_0\|_1^2 &\leq Ct^2\|\theta\|_0\,\|\beta_t - \beta_0\|_0 + Ct^2\left(\|\gamma_t\|_0 + \|\gamma_0\|_0\right)\|\gamma_t\|_0 \\ &\leq Ct\,|(\theta, f)|_t\,\|\beta_t - \beta_0\|_0 + Ct^2|(\theta, f)|_t^2 \end{aligned}$$

and therefore

$$\|\beta_t - \beta_0\|_1 \leq Ct\,|(\theta, f)|_t. \tag{3.2}$$

Finally, observe that

$$\nabla(w_t - w_0) = (\beta_t - \beta_0) + \frac{t^2}{\kappa}\gamma_t$$

and so, using again the a priori estimate (3.1) for $\|\gamma_t\|_0$, we obtain

$$\|w_t - w_0\|_1 \leq C\left(\|\beta_t - \beta_0\|_0 + t^2|(\theta, f)|_t\right)$$

which together with (3.2) allow us to conclude the proof. $\square$

The arguments to prove the remaining properties are based on the fact that there exists an operator

$$\Pi : H_0^1(\Omega)^2 \times \left[H_0^1(\Omega) \cap H^2(\Omega)\right] \longrightarrow H_h \times W_h$$

such that, if $(\hat{\eta}, \hat{w}) := \Pi(\eta, w)$, then

$$R(\nabla w - \eta) = \nabla\hat{w} - R\hat{\eta}, \tag{3.3}$$

(R being the reduction operator satisfying (2.7), (2.8) and (2.9)) and

$$\|\eta - \hat{\eta}\|_1 \le Ch\,\|\eta\|_2. \tag{3.4}$$

The construction of such operator is based on known properties of the spaces H_h, W_h and Γ_h (see [10] for details). It is worth observing that (3.3) corresponds to a commutative diagram property, usual in the analysis of mixed methods. In fact, introducing the operators B and B_h such that $B(\eta, v) := \nabla v - \eta$, for $(\eta, v) \in H_0^1(\Omega)^2 \times H_0^1(\Omega)$, and $B_h(\eta, v) := \nabla v - R\eta$, for $(\eta, v) \in H_h \times W_h$, that property can be summarized in the following commutative diagram:

$$\begin{array}{ccc} H_0^1(\Omega)^2 \times [H_0^1(\Omega) \cap H^2(\Omega)] & \xrightarrow{B} & H^1(\Omega)^2 \cap H_0(\mathrm{rot}, \Omega) \\ \Pi \big\downarrow & & \big\downarrow R \\ H_h \times W_h & \xrightarrow{B_h} & \Gamma_h \end{array}$$

Note that, if R were an L^2 projection, this commutative diagram would correspond to the well known Fortin's property (which in its turn implies that an inf-sup condition, analogous to that proved in [7] for the continuous problem, holds for the discrete case). Of course, R is not an L^2 projection and so the commutative diagram property given above can be seen as a generalization of Fortin's one. In fact, optimal error estimates in H^1 for the rotations and the transverse displacement yielding (2.11) follow from this generalized property:

Lemma 3.2 *There exists a constant C, independent of t and h, such that*

$$\|(T_t - T_{th})(\theta, f)\|_{H^1(\Omega)^2 \times H^1(\Omega)} \le Ch\,|(\theta, f)|_t,$$

for all $(\theta, f) \in L^2(\Omega)^2 \times L^2(\Omega)$.

Proof. Arguments identical to those in [10], combined with the *a priori* estimate (3.1), yield in our case

$$\begin{aligned} \|\beta_t - \beta_{th}\|_1 + t\,\|\gamma_t - \gamma_{th}\|_0 &\le Ch\Big(\|\beta_t\|_2 + t\,\|\gamma_t\|_1 + \|\gamma_t\|_0\Big) \\ &\le Ch\,|(\theta, f)|_t \end{aligned} \tag{3.5}$$

and, as a consequence,

$$\begin{aligned} \|w_t - w_{th}\|_1 &\le Ch\Big(\|\beta_t\|_2 + t\,\|\gamma_t\|_1 + \|\gamma_t\|_0\Big) \\ &\le Ch\,|(\theta, f)|_t, \end{aligned}$$

therefore concluding the proof. □

We will use a duality argument to show that, under the same conditions, optimal L^2 error estimates for the rotations and the transverse displacement also hold. First we will prove a lemma which will be useful to bound the consistency terms arising from the reduced integration of the shear strain.

For $\psi \in H_0^1(\Omega)^2$, we denote by $\psi_I \in H_0^1(\Omega)^2$ a piecewise linear average interpolant as defined in [9, 18], satisfying

$$\|\psi_I\|_1 \le C\|\psi\|_1 \tag{3.6}$$

and

$$\|\psi - \psi_I\|_1 \le Ch\|\psi\|_2. \tag{3.7}$$

The following estimate holds:

Lemma 3.3 *Given $\zeta \in H(\mathrm{div}, \Omega)$ and $\psi \in H_0^1(\Omega)^2$ it holds:*

$$|(\zeta, \psi_I - R\psi_I)| \le Ch^2 \| \operatorname{div} \zeta \|_0 \, \|\psi\|_1.$$

Proof. From property (2.8) it follows that

$$\operatorname{rot} R\psi_I = P(\operatorname{rot} \psi_I),$$

where P is the L^2 projection onto the piecewise constant functions. Now, since $(\operatorname{rot} \psi_I)|_K \in \mathcal{P}_0$, then $(\operatorname{rot} \psi_I) = P(\operatorname{rot} \psi_I)$. Hence $\operatorname{rot}(\psi_I - R\psi_I) = 0$ and so there exists $r \in H^1(\Omega)$ such that

$$\nabla r = \psi_I - R\psi_I; \tag{3.8}$$

actually, we can take $r \in H_0^1(\Omega)$ because the tangential component of $(\psi_I - R\psi_I)$ vanishes on $\partial\Omega$. Then, we have

$$|(\zeta, \psi_I - R\psi_I)| = |(\zeta, \nabla r)| = |(\operatorname{div} \zeta, r)| \le \| \operatorname{div} \zeta \|_0 \, \|r\|_0. \tag{3.9}$$

On the other hand, from property (2.7) defining R it follows that, for every edge ℓ with end points A and B, we have

$$r(A) - r(B) = \int_\ell \nabla r \cdot \tau = \int_\ell (\psi_I - R\psi_I) \cdot \tau = 0.$$

Therefore, since r vanishes on $\partial\Omega$ we conclude that r vanishes at all the nodes of the triangulation. Hence, since $r|_K \in \mathcal{P}_2$, a standard scaling argument on each element K yields

$$\|r\|_0 \le Ch \|\nabla r\|_0,$$

which together with (2.9), (3.6), (3.8) and (3.9) allow us to conclude the proof. □

Finally, we will prove the main result of this section, property (2.13), concerning L^2 error estimates optimal in order and regularity. Let us remark that this result completes the analysis of the MITC elements carried out in [8, 15] for the higher order methods. (To simplify the notation we will drop in this Lemma the subscript t from β_t, w_t, γ_t and from their discrete approximations.)

Lemma 3.4 *Given $(\theta, f) \in L^2(\Omega)^2 \times L^2(\Omega)$, let $(\beta, w) = T_t(\theta, f)$ and $(\beta_h, w_h) = T_{th}(\theta, f)$. Then there exists a constant C, independent of t and h, such that*

$$\|\beta - \beta_h\|_0 + \|w - w_h\|_0 \le Ch^2 |(\theta, f)|_t$$

or, equivalently,

$$\|(T_t - T_{th})(\theta, f)\|_{L^2(\Omega)^2 \times L^2(\Omega)} \le Ch^2 |(\theta, f)|_t.$$

Proof. Subtracting (2.10) from (2.4) we obtain the error equation

$$a(\beta - \beta_h, \eta) + (\gamma - \gamma_h, \nabla v - R\eta) = (\gamma, \eta - R\eta), \qquad \forall (\eta, v) \in H_h \times W_h, \tag{3.10}$$

with $\gamma = \dfrac{\kappa}{t^2}(\nabla w - \beta)$ and $\gamma_h = \dfrac{\kappa}{t^2}(\nabla w_h - R\beta_h)$.

We will use a duality argument: let $(\varphi, u) \in H_0^1(\Omega)^2 \times H_0^1(\Omega)$ be the solution of

$$\begin{cases} a(\eta, \varphi) + (\nabla v - \eta, \delta) = (v, w - w_h) + (\eta, \beta - \beta_h), \qquad \forall (\eta, v) \in H_0^1(\Omega)^2 \times H_0^1(\Omega), \\ \delta = \dfrac{\kappa}{t^2}(\nabla u - \varphi). \end{cases} \tag{3.11}$$

An *a priori* estimate analogous to (3.1) is valid for this problem, namely,

$$\|\varphi\|_2 + \|u\|_2 + \|\delta\|_0 + t\,\|\delta\|_1 \le C\Big(\|\beta - \beta_h\|_0 + \|w - w_h\|_0\Big), \tag{3.12}$$

and by taking $\eta = 0$ in (3.11) we have

$$-\operatorname{div}\delta = w - w_h. \tag{3.13}$$

By taking $v = w - w_h$ and $\eta = \beta - \beta_h$ in the dual problem (3.11) we have

$$\begin{aligned}\|w - w_h\|_0^2 + \|\beta - \beta_h\|_0^2 &= a(\beta - \beta_h, \varphi) + \Big(\nabla(w - w_h) - (\beta - R\beta_h), \delta\Big) + (\beta_h - R\beta_h, \delta)\\ &= a(\beta - \beta_h, \varphi) + \frac{t^2}{\kappa}(\gamma - \gamma_h, \delta) + (\beta_h - R\beta_h, \delta),\end{aligned}$$

and using the error equation (3.10) with $(\eta, v) = (\hat\varphi, \hat u) := \Pi(\varphi, u)$ we obtain

$$\begin{aligned}\|w - w_h\|_0^2 + \|\beta - \beta_h\|_0^2 &= a(\beta - \beta_h, \varphi - \hat\varphi) + \frac{t^2}{\kappa}(\gamma - \gamma_h, \delta) - (\gamma - \gamma_h, \nabla\hat u - R\hat\varphi)\\ &\quad + (\beta_h - R\beta_h, \delta) + (\gamma, \hat\varphi - R\hat\varphi).\\ &= a(\beta - \beta_h, \varphi - \hat\varphi) + \frac{t^2}{\kappa}(\gamma - \gamma_h, \delta - R\delta)\\ &\quad + (\beta_h - R\beta_h, \delta) + (\gamma, \hat\varphi - R\hat\varphi),\end{aligned} \tag{3.14}$$

where we have used the commutative diagram property (3.3), $R\delta = \dfrac{\kappa}{t^2}(\nabla\hat u - R\hat\varphi)$, for the last equality. Now it only remains to estimate the four terms in the last expression.

The first two can be easily bounded. In fact, using the error estimates (3.4) and (3.5) and the *a priori* estimate (3.12) we have

$$\begin{aligned}|a(\beta - \beta_h, \varphi - \hat\varphi)| &\le C\|\beta - \beta_h\|_1\,\|\varphi - \hat\varphi\|_1 \le C\|\beta - \beta_h\|_1\,h\,\|\varphi\|_2\\ &\le Ch^2|(\theta, f)|_t\Big(\|\beta - \beta_h\|_0 + \|w - w_h\|_0\Big)\end{aligned} \tag{3.15}$$

and, from (2.9), (3.5) and (3.12),

$$\begin{aligned}\frac{t^2}{\kappa}\Big|(\gamma - \gamma_h, \delta - R\delta)\Big| &\le Ct^2\|\gamma - \gamma_h\|_0\,\|\delta - R\delta\|_0 \le Ct^2\|\gamma - \gamma_h\|_0\,h\,\|\delta\|_1\\ &\le Ch^2|(\theta, f)|_t\Big(\|\beta - \beta_h\|_0 + \|w - w_h\|_0\Big).\end{aligned} \tag{3.16}$$

For the third term in (3.14) we have,

$$(\beta_h - R\beta_h, \delta) = \Big((\beta_h - \beta_I) - R(\beta_h - \beta_I), \delta\Big) + (\beta_I - R\beta_I, \delta).$$

Now, using successively (2.9), (3.5), (3.7), (3.12) and (3.1) we obtain

$$\begin{aligned}\Big|\Big((\beta_h - \beta_I) - R(\beta_h - \beta_I), \delta\Big)\Big| &\le C\|\delta\|_0\,h\,\|\beta_h - \beta_I\|_1 \le C\|\delta\|_0\,h\Big(\|\beta_h - \beta\|_1 + \|\beta - \beta_I\|_1\Big)\\ &\le Ch^2\Big(\|\beta - \beta_h\|_0 + \|w - w_h\|_0\Big)|(\theta, f)|_t,\end{aligned}$$

whereas by using Lemma 3.3, (3.13) and (3.1) we have

$$\begin{aligned}|(\beta_I - R\beta_I, \delta)| &\le Ch^2\|\beta\|_1\,\|\operatorname{div}\delta\|_0\\ &\le Ch^2|(\theta, f)|_t\,\|w - w_h\|_0\end{aligned}$$

Therefore

$$|(\beta_h - R\beta_h, \delta)| \leq Ch^2 |(\theta, f)|_t \Big(\|\beta - \beta_h\|_0 + \|w - w_h\|_0 \Big). \tag{3.17}$$

The last term in (3.14) can be bounded in an almost identical way, by using (3.4) to estimate $\|\hat{\varphi} - \varphi\|_1$ and the fact that

$$-\operatorname{div} \gamma_t = f, \qquad \text{in } \Omega$$

which follows by taking $\eta = 0$ in problem (2.4) and (2.5).

By so doing we obtain

$$|(\gamma, \hat{\varphi} - R\hat{\varphi})| \leq Ch^2 |(\theta, f)|_t \Big(\|\beta - \beta_h\|_0 + \|w - w_h\|_0 \Big). \tag{3.18}$$

Finally, from (3.14), (3.15), (3.16), (3.17) and (3.18), simple algebra allows us to conclude the proof. □

4 Numerical experiments

In this section we summarize the numerical experimentation carried out with our method. The aim of these computations was two-fold: to study the performance of the method and to discuss the pertinence of the assumptions made in Section 2 to prove error estimates.

To validate our codes and to test the effectiveness of the method to deal with different boundary conditions, we have first considered a typical benchmark problem: the computation of the lowest frequency vibration modes of a square plate. We have applied our codes to thin and moderately thick plates and compared the obtained results with those reported in [1]. In this reference the same problems have been treated by means of a method introduced by Huang and Hinton in [12] which is based on biquadratic rectangular finite elements with enhanced shear interpolation.

We have considered a square plate of side length L and two thicknes-to-span (t/L) ratios of 0.1 (moderately thick) and 0.01 (thin). In each case we have also considered four different types of boundary conditions:

- a clamped plate (as described in Section 2);
- a hard simply supported plate (i.e., with transversal displacements and rotations tangential to the boundary both vanishing on each edge);
- a plate with mixed boundary conditions (with two opposite edges being hard simply supported and the other two clamped);
- a plate with a free edge (with three clamped edges and the fourth being free, i.e., no constraints either on the transversal displacements or on the rotations along this edge).

We denote each case by C-C-C-C, S-S-S-S, S-C-S-C and C-C-C-F, respectively (C stands for clamped, S for hard simply supported and F for free edges).

We have used succesive refinements of a uniform mesh as that in Figure 1, the refinement parameter N being the number of element edges on each side of the square (hence, $h = \sqrt{2}L/N$).

We have applied our codes to the original unscaled problems analogous to (2.1). Thus we have computed approximations of the free vibration angular frequencies $\omega_t = t\sqrt{\lambda_t/\rho}$. In order to compare the obtained results with those in [1] we present the computed frequencies ω_{mn}^h in the following non-dimensional form:

$$\widehat{\omega}_{mn} := \omega_{mn}^h L \left(\frac{2(1+\nu)\rho}{E} \right)^{1/2},$$

m and n being the numbers of half-waves occurring in the modes shapes in the x and y directions, respectively.

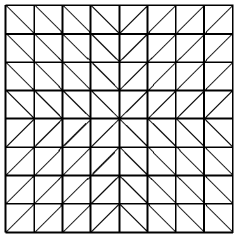

Figure 1: Finite element mesh for the square plate: $N = 8$.

Tables I and II show the four lowest vibration frequencies computed by our method with three different meshes ($N = 10, 20, 40$) and each set of boundary conditions for each thickness-to-span ratio ($t/L = 0.1$ and $t/L = 0.01$, respectively). Each table includes the results obtained by Huang and Hinton's method in [12] (column H-H) and also by an analytical approximation obtained by Dawe and Roufaeil in [11] (column D-R), both as reported in [1]. In every case we have used a Poisson ratio $\nu = 0.3$ and different correction factors depending on the boundary conditions, but always the same as those used in [1] to allow for the comparison: $k = 0.8601$ for C-C-C-C and C-C-C-F, $k = 0.8333$ for S-S-S-S and $k = 0.822$ for S-C-S-C. The reported non-dimensional frequencies are independent of the remaining geometrical and physical parameters, except for the thickness-to-span ratio.

Table I: Lowest non-dimensional vibration frequencies for *moderately thick* square plates: $t/L = 0.1$.

Bound. cond.	Mode	$N = 10$	$N = 20$	$N = 40$	H-H	D-R
C-C-C-C	$\widehat{\omega}_{11}$	1.5947	1.5921	1.5913	1.591	1.594
	$\widehat{\omega}_{21}$	3.1181	3.0595	3.0441	3.039	3.046
	$\widehat{\omega}_{12}$	3.1181	3.0595	3.0441	3.039	3.046
	$\widehat{\omega}_{22}$	4.4477	4.3106	4.2746	4.263	4.285
S-S-S-S	$\widehat{\omega}_{11}$	0.9384	0.9323	0.9308	0.930	0.930
	$\widehat{\omega}_{21}$	2.2893	2.2366	2.2236	2.219	2.219
	$\widehat{\omega}_{12}$	2.2893	2.2366	2.2236	2.219	2.219
	$\widehat{\omega}_{22}$	3.5657	3.4450	3.4153	3.405	3.406
S-C-S-C	$\widehat{\omega}_{11}$	1.3060	1.3016	1.3005	1.300	1.302
	$\widehat{\omega}_{21}$	2.4664	2.4120	2.3984	2.394	2.398
	$\widehat{\omega}_{12}$	2.9617	2.9043	2.8895	2.885	2.888
	$\widehat{\omega}_{22}$	4.0126	3.8830	3.8500	3.839	3.852
C-C-C-F	$\widehat{\omega}_{\frac{1}{2}1}$	1.0959	1.0848	1.0814	1.081	1.089
	$\widehat{\omega}_{\frac{3}{2}1}$	1.7759	1.7525	1.7454	1.744	1.758
	$\widehat{\omega}_{\frac{1}{2}2}$	2.7413	2.6787	2.6612	2.657	2.673
	$\widehat{\omega}_{\frac{5}{2}1}$	3.3186	3.2282	3.2035	3.197	3.216

Table II: Lowest non-dimensional vibration frequencies for *thin* square plates: $t/L = 0.01$.

Bound. cond.	Mode	$N=10$	$N=20$	$N=40$	H-H	D-R
C-C-C-C	$\widehat{\omega}_{11}$	0.1754	0.1754	0.1754	0.1754	0.1754
	$\widehat{\omega}_{21}$	0.3668	0.3599	0.3580	0.3574	0.3576
	$\widehat{\omega}_{12}$	0.3668	0.3599	0.3580	0.3574	0.3576
	$\widehat{\omega}_{22}$	0.5487	0.5323	0.5279	0.5264	0.5274
S-S-S-S	$\widehat{\omega}_{11}$	0.0972	0.0965	0.0963	0.0963	0.0963
	$\widehat{\omega}_{21}$	0.2486	0.2426	0.2411	0.2406	0.2406
	$\widehat{\omega}_{12}$	0.2486	0.2426	0.2411	0.2406	0.2406
	$\widehat{\omega}_{22}$	0.4035	0.3893	0.3859	0.3847	0.3848
S-C-S-C	$\widehat{\omega}_{11}$	0.1417	0.1413	0.1412	0.1411	0.1411
	$\widehat{\omega}_{21}$	0.2748	0.2688	0.2673	0.2668	0.2668
	$\widehat{\omega}_{12}$	0.3474	0.3402	0.3383	0.3377	0.3377
	$\widehat{\omega}_{22}$	0.4814	0.4657	0.4617	0.4604	0.4608
C-C-C-F	$\widehat{\omega}_{\frac{1}{2}1}$	0.1182	0.1170	0.1167	0.1166	0.1171
	$\widehat{\omega}_{\frac{3}{2}1}$	0.1977	0.1956	0.1950	0.1949	0.1951
	$\widehat{\omega}_{\frac{1}{2}2}$	0.3193	0.3109	0.3086	0.3080	0.3093
	$\widehat{\omega}_{\frac{5}{2}1}$	0.3874	0.3771	0.3744	0.3736	0.3740

Both tables show that the method can be safely used for any of these boundary conditions and for thin as well as moderately thick plates.

The goal of our second experiment was to test if the hypothesis on the uniform separation of the spectrum assumed throughout Section 2 is actually necessary. For any convex clamped plate, according to Theorem 2.2, the convergence for each eigenvalue is quadratic. However, the constants in the error estimates arising in its proof blow up with the inverse of the distance of the approximated eigenvalue to the rest of the spectrum.

In order to test if this assumption is essential or if it only arises because of the techniques used to prove the theorems, we have considered a clamped rectangular steel plate with two very close vibration frequencies, but not exactly coincident. We have chosen the following values of the physical and geometric parameters: side lengths: 2 m and 1 m; Young modulus: $E = 1.43 \times 10^{11}$ Pa; Poisson ratio: $\nu = 0.35$; density: $\rho = 7.7 \times 10^3$ Kg, and correction factor $k = 5/6$.

We have considered again two plates of different thickness: one moderately thick ($t = 0.1$ m) and the other thin ($t = 0.01$ m). For each of them we have computed the lowest frequency vibration modes with different successively refined uniform meshes analogous to that in Figure 1, the parameter N being now the number of edge elements on the shortest side of the plate.

We have observed that the relative error of the approximated angular frequencies ω^h_{mn} roughly behaves like

$$\frac{\omega^h_{mn} - \omega_{mn}}{\omega_{mn}} \approx C_{mn} h^{\alpha},$$

with an order of convergence α very close to 2 and constants C_{mn} which depend on the numbers of half-waves m and n, but are almost independent of the thickness of the plate. We have also observed that these constants remain remarkably stable even for very close eigenvalues.

Then, for each mode, we have estimated the exact vibration frequencies ω_{mn}, the value of the constants C_{mn} and the order of convergence α by means of a least square fitting of the model

$$\omega_{mn}^h \approx \omega_{mn}\left(1 + C_{mn}h^\alpha\right)$$

to the approximate frequencies computed on highly refined meshes ($N = 16$, 32, 48, 64, 80).

We summarize the obtained results in Table III for the moderately thick plate and in Table IV for the thin one.

Table III: Lowest vibration frequencies (in rad/s) of a clamped *moderately thick* rectangular plate: $2\,\mathrm{m}\times 1\,\mathrm{m}$, $t = 0.1\,\mathrm{m}$.

Mode	$N = 8$	$N = 16$	$N = 32$	α	ω_{mn}	C_{mn}
ω_{11}^h	3035.212	3025.827	3023.214	1.97	3022.317	0.071
ω_{21}^h	3918.755	3875.817	3864.488	1.98	3860.631	0.242
ω_{31}^h	5441.309	5351.958	5328.053	1.97	5319.841	0.365
ω_{12}^h	7536.399	7333.836	7280.257	1.98	7262.144	0.616
ω_{41}^h	7630.812	7399.422	7338.489	1.98	7317.795	0.689
ω_{22}^h	8340.381	8076.061	8006.128	1.98	7982.469	0.731
ω_{32}^h	9673.715	9326.597	9233.320	1.97	9201.449	0.830

Table IV: Lowest vibration frequencies (in rad/s) of a clamped *thin* rectangular plate: $2\,\mathrm{m}\times 1\,\mathrm{m}$, $t = 0.01\,\mathrm{m}$.

Mode	$N = 8$	$N = 16$	$N = 32$	α	ω_{mn}	C_{mn}
ω_{11}^h	328.760	327.663	327.347	1.99	327.240	0.086
ω_{21}^h	429.627	425.210	424.050	2.00	423.665	0.239
ω_{31}^h	607.831	598.992	596.605	1.98	595.794	0.336
ω_{41}^h	878.302	851.895	844.852	1.98	842.468	0.702
ω_{12}^h	888.126	860.941	853.510	1.98	850.980	0.727
ω_{22}^h	991.486	957.454	948.250	1.98	945.136	0.819
ω_{32}^h	1162.198	1121.967	1110.801	1.97	1106.970	0.828

It can be observed that the fourth and the fifth eigenvalues in both tables are very close (approximately 1% of difference). However the corresponding constants C_{mn} are not larger than those for other eigenvalues. This fact suggests that the assumption on the separation of the spectrum made in Section 2 is not really necessary.

Finally, we have made a stringent test of this hypothesis and, at the same time, of the stability of the method as t goes to zero. The eigenvalues of Kirchhoff equations for a rectangular plate with simply supported boundary conditions are exactly known:

$$\lambda_0^{mn} := \frac{E\pi^4}{12(1-\nu^2)}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2, \qquad m, n = 1, 2, 3, \ldots,$$

where a and b are the side lengths of the plate. If we consider the same lengths as in the previous experiment ($a = 2\,\mathrm{m}$, $b = 1\,\mathrm{m}$) the fifth and the sixth eigenvalues exactly coincide: $\lambda_0^{41} = \lambda_0^{22}$.

In Reissner-Mindlin equations, this double eigenvalue splits into two different ones, both converging to this common value and hence getting closer and closer to each other as t goes to zero.

This phenomenon occurs for both hard and soft simply supported plates. We have chosen the latter (i.e., with vanishing transversal displacements but no constraint imposed on the rotations along the edges) in order to test our method also for these boundary conditions, but similar results are valid for hard simply supported plates too.

We have considered the same values of the physical parameters and successively refined uniform meshes as those of the previous experiment. We have computed approximations λ_{th}^{41} and λ_{th}^{22} on the meshes corresponding to $N = 8, 16, 32, 64$, for decreasing values of the thickness $t = 0.1, 0.01, 0.001, 0.0001\,\mathrm{m}$.

We have observed that the relative error of the computed eigenvalues roughly behaves again like

$$\frac{\lambda_{th}^{mn} - \lambda_t^{mn}}{\lambda_t^{mn}} \approx C_{mn} h^{\alpha}$$

with constants C_{mn} depending neither on the thickness t nor on the meshsize h and orders of convergence α close to 2.

For each thickness t and both vibration modes, we have estimated the orders of convergence α, the constants C_{mn} and the exact eigenvalues λ_t^{mn} by a least square fitting of the computed approximate eigenvalues λ_{th}^{mn}, similar to that of the previous experiment. Finally we have also estimated by extrapolation the limit values $\lambda_{0h}^{mn} = \lim_{t\to 0} \lambda_{th}^{mn}$.

We summarize these results for λ_{th}^{41} in Table V and for λ_{th}^{22} in Table VI.

Table V: Fifth eigenvalue λ_{th}^{41} (multiplied by 10^{-10}) of Reissner-Mindlin equations for a soft simply supported rectangular plate: $2\,\mathrm{m}\times 1\,\mathrm{m}$.

Thickness	$N=8$	$N=16$	$N=32$	$N=64$	α	λ_t^{41}	C_{41}
$t = 0.1$	3032.768	2801.861	2736.523	2718.535	1.83	2711.216	2.817
$t = 0.01$	3670.259	3407.412	3341.052	3322.356	1.96	3316.899	3.180
$t = 0.001$	3678.556	3416.164	3351.528	3335.415	2.02	3330.245	3.459
$t = 0.0001$	3678.639	3416.253	3351.638	3335.572	2.02	3330.405	3.464
$t = 0$ (extrap.)	3678.640	3416.254	3351.640	3335.574	2.02	3330.406	3.464

Table VI: Sixth eigenvalue λ_{th}^{22} (multiplied by 10^{-10}) of Reissner-Mindlin equations for a soft simply supported rectangular plate: $2\,\mathrm{m}\times 1\,\mathrm{m}$.

Thickness	$N=8$	$N=16$	$N=32$	$N=64$	α	λ_t^{22}	C_{22}
$t = 0.1$	3039.576	2802.288	2736.243	2718.268	1.85	2711.174	2.994
$t = 0.01$	3676.006	3407.927	3341.083	3322.346	1.98	3317.021	3.322
$t = 0.001$	3684.276	3416.686	3351.582	3335.421	2.04	3330.376	3.613
$t = 0.0001$	3684.359	3416.775	3351.693	3335.579	2.04	3330.536	3.618
$t = 0$ (extrap.)	3684.359	3416.776	3351.694	3335.581	2.04	3330.537	3.618

Two conclusions arise immediately from the previous tables. First, the constants do not deteriorate as the thickness becomes small (and consequently the eigenvalues of Reissner-Mindlin equations get closer). Secondly, the method is locking free and provides very accurate approximations of the eigenvalues of the Kirchhoff equations, in this case, $\lambda_0^{41} = \lambda_0^{22} = 3330.225 \times 10^{10}$.

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