ERROR ESTIMATES FOR LOW-ORDER ISOPARAMETRIC QUADRILATERAL FINITE ELEMENTS FOR PLATES

RICARDO G. DURÁN*, ERWIN HERNÁNDEZ†, LUIS HERVELLA-NIETO‡, ELSA LIBERMAN§, AND RODOLFO RODRíGUEZ¶

Abstract. This paper deals with the numerical approximation of the bending of a plate modeled by Reissner-Mindlin equations. It is well known that, in order to avoid locking, some kind of reduced integration or mixed interpolation has to be used when solving these equations by finite element methods. In particular, one of the most widely used procedures is based on the family of elements called *MITC* (mixed interpolation of tensorial components). We consider two lowest-order methods of this family on quadrilateral meshes.

Under mild assumptions we obtain optimal H^1 and L^2 error estimates for both methods. These estimates are valid with constants independent of the plate thickness. We also obtain error estimates for the approximation of the plate vibration problem. Finally, we report some numerical experiments showing the very good behavior of the methods, even in some cases not covered by our theory.

Key words. Reissner-Mindlin, *MITC* methods, isoparametric quadrilaterals.

AMS subject classifications. 65N30, 74S05, 74K20

1. Introduction.

Reissner-Mindlin model is the most widely used for the analysis of thin or moderately thick elastic plates. It is now very well understood that standard finite element methods applied to this model produce very unsatisfactory results due to the so-called *locking* phenomenon. Therefore, some special method based on reduced integration or mixed interpolation has to be used. Among them, the *MITC* methods introduced by Bathe and Dvorkin in [7] or variants of them are very likely the most used in practice.

A great number of papers dealing with the mathematical analysis of this kind of methods have been published (see for example [2, 6, 10, 12, 13, 18, 20, 23]). In those papers, optimal order error estimates, valid uniformly on the plate thickness, have been obtained for several methods. However, although one of the most commonly used elements in engineering applications are the isoparametric quadrilaterals (indeed, the original Bathe and Dvorkin's paper deals with these elements), no available result seems to exist for this case.

On the other hand, it has been recently noted that the extension to general quadrilaterals of convergence results valid for rectangular elements is not straightforward and, even more, the order of convergence can deteriorate when non-standard finite elements are used in distorted quadrilaterals, even if they satisfy the usual shape regularity assumption (see [3, 4]).

*Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, 1428 – Buenos Aires, Argentina. Supported by ANPCyT under grant PICT 03-05009 and by CONICET under grant PIP 0660/98. Member of CONICET (Argentina).

†Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile. Supported by FONDECYT 2.000.114 (Chile).

‡Departamento de Matemáticas, Universidade da Coruña, 15071 – A Coruña, Spain. Partially supported by FONDAP in Applied Mathematics.

§Comisión de Investigaciones Científicas de la Provincia de Buenos Aires and Departamento de Matemática, Facultad de Ciencias Exactas, Universidad Nacional de La Plata, C.C. 172, 1900 – La Plata, Argentina.

¶GI²MA, Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile. Partially supported by FONDAP in Applied Mathematics (Chile).

The aim of this paper is to analyze two low-order methods based on quadrilateral meshes. One is the original *MITC4* introduced in [7], while the other one is an extension to the quadrilateral case of a method introduced in [12] for triangular elements (from now on the latter will be called *DL4*). We are interested not only in load problems but also in the determination of the free vibration modes of the plate.

For nested uniform meshes of rectangles, an optimal order error estimate in H^1 norm has been proved in [6] for *MITC4*. However, the regularity assumptions on the exact solution required in that paper are not optimal. These assumptions have been weakened in [12], but they are still not optimal. Let us remark that to obtain approximation results for the plate vibration spectral problem, it is important to remove this extra regularity assumption.

On the other hand, for low-order elements as those considered here, an optimal error estimate in L^2 norm is difficult to obtain because of the consistency term arising in the error equation. For triangular elements such estimate has been only recently proved in [13]. However, the proof given in that paper can not be extended straightforwardly, even for the case of rectangular elements.

In this paper we prove optimal in order and regularity H^1 and L^2 error estimates for both methods, *MITC4* and *DL4*, under appropriate assumptions on the family of meshes. As a consequence, following the arguments in [13], we obtain also optimal error estimates for the approximation of the corresponding plate vibration spectral problem.

In order to prove the H^1 error estimate for *MITC4* we require an additional assumption on the meshes (which is satisfied, for instance, by uniform refinements of any starting mesh). Instead, no assumption other than the usual shape regularity is needed for *DL4*.

On the other hand, a further assumption on the meshes is made to prove the L^2 error estimates: the meshes must be formed by higher order perturbations of parallelograms. This restriction is related with approximation properties of the Raviart-Thomas elements which are used in our arguments and do not hold for general quadrilateral elements. However, this assumption is only needed for extremely refined meshes. Indeed, the L^2 estimate holds for any regular mesh as long as the mesh-size is comparable with the plate thickness. Moreover, we believe that this quasi-parallelogram assumption is of a technical character. In fact, the numerical experiments reported here seem to show that it is not necessary.

The rest of the paper is organized as follows. In section 2 we recall Reissner-Mindlin equations and introduce the two discrete methods. We prove optimal order error estimates for both methods in H^1 and L^2 norms in sections 3 and 4, respectively. In section 5 we prove error estimates for the spectral plate vibration problem. Finally, in section 6, we report some numerical experiments.

Throughout the paper we denote by C a positive constant not necessarily the same at each occurrence, but always independent of the mesh-size and the plate thickness.

2. Statement of the problem.

2.1. Reissner-Mindlin model. Let $\Omega \times (-\frac{t}{2}, \frac{t}{2})$ be the region occupied by an undeformed elastic plate of thickness t, where Ω is a convex polygonal domain of $\mathbb{R}^2$. In order to describe the deformation of the plate, we consider the Reissner-Mindlin model, which is written in terms of the rotations $\beta = (\beta^1, \beta^2)$ of the fibers initially normal to the plate mid-surface and the transverse displacement w. The following equations describe the plate response to conveniently scaled transversal and shear loads $f \in \mathrm{L}^2(\Omega)$ and $\theta \in \mathrm{L}^2(\Omega)^2$, respectively (see, for instance, [9, 13]):

PROBLEM 2.1. *Find* $(\beta, w) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$ *such that:*

$$\left\{\begin{array}{ll} a(\beta,\eta) + (\gamma, \nabla v - \eta) = (f, v) + \dfrac{t^2}{12}(\theta,\eta) & \forall(\eta, v) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega), \\ \gamma = \dfrac{\kappa}{t^2}(\nabla w - \beta). & \end{array}\right. \tag{2.1}$$

In this expression, $\kappa := Ek/2(1+\nu)$ is the shear modulus, with E being the Young modulus, ν the Poisson ratio, and k a correction factor. We have also introduced the shear stress γ and denoted by $(\cdot,\cdot)$ the standard L^2 inner product. Finally, a is the $\mathrm{H}_0^1(\Omega)^2$ elliptic bilinear form defined by

$$a(\beta,\eta) := \frac{E}{12(1-\nu^2)} \int_\Omega \left[\sum_{i,j=1}^{2} (1-\nu)\varepsilon_{ij}(\beta)\varepsilon_{ij}(\eta) + \nu \operatorname{div}\beta \operatorname{div}\eta \right],$$

with $\varepsilon_{ij}(\beta) = \frac{1}{2}(\partial\beta_i/\partial x_j + \partial\beta_j/\partial x_i)$ being the components of the linear strain tensor.

Let us remark that we have included in our formulation the shear load term $\frac{t^2}{12}(\theta,\eta)$ since it arises naturally when considering the free vibration plate problem. In fact, it is simple to see that the free vibration modes of the plate are determined by

$$t^3 a(\beta,\eta) + \kappa t \int_\Omega (\nabla w - \beta)\cdot(\nabla v - \eta) = \omega^2 \left(t \int_\Omega \rho\, w v + \frac{t^3}{12}\int_\Omega \rho\, \beta\cdot\eta \right) \qquad \forall(\eta, v) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega),$$

where ω denotes the angular vibration frequency, β and w the rotation and transversal displacement amplitudes, respectively, and ρ the plate density (see [13] for further details). Thus, rescaling the problem with $\lambda := \rho\omega^2/t^2$, we obtain the following, which is the spectral problem associated to Problem 2.1:

PROBLEM 2.2. *Find* $\lambda \in \mathbb{R}$ *and* $0 \neq (\beta, w) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$ *such that:*

$$\left\{\begin{array}{ll} a(\beta,\eta) + (\gamma, \nabla v - \eta) = \lambda \left[(w,v) + \dfrac{t^2}{12}(\beta,\eta) \right] & \forall(\eta, v) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega), \\ \gamma = \dfrac{\kappa}{t^2}(\nabla w - \beta). & \end{array}\right.$$

This paper deals with the finite element approximation of Problems 2.1 and 2.2. It is well known that both are well-posed (see [9] and [13]). Furthermore, we will use the following regularity result for the solution of equations (2.1) (see [2]):

$$\|\beta\|_{2,\Omega} + \|w\|_{2,\Omega} + \|\gamma\|_{0,\Omega} + t\,\|\gamma\|_{1,\Omega} \leq C\left(t^2\|\theta\|_{0,\Omega} + \|f\|_{0,\Omega}\right) \leq C\,|(\theta,f)|_t\,, \tag{2.2}$$

where, for any open subset O of Ω and any integer k, $\|\cdot\|_{k,O}$ denotes the standard norm of $\mathrm{H}^k(O)$ or $\mathrm{H}^k(O)^2$, as corresponds, and $|(\cdot,\cdot)|_t$ is the norm in $\mathrm{L}^2(\Omega)^2 \times \mathrm{L}^2(\Omega)$ induced by the weighted inner product in the right hand side of the first equation in (2.1) (see [13]).

2.2. Discrete problems. In what follows we consider two lowest-degree methods on isoparametric quadrilateral meshes for the approximation of Problem 2.1: the so-called *MITC4* (see [7]) and an extension to quadrilaterals of a method introduced in [12] that we call *DL4*. Both methods are based on relaxing the shear terms in equation (2.1) by introducing an interpolation operator, called *reduction operator*.

Let $\{\mathcal{T}_h\}$ be a family of decompositions of Ω into convex quadrilaterals, satisfying the usual condition of regularity (see for instance [19]); i.e., there exist constants $\sigma > 1$ and $0 < \varrho < 1$ independent of h such that

$$h_K \leq \sigma \rho_K, \qquad |\cos \vartheta_{iK}| \leq \varrho, \quad i = 1,2,3,4, \qquad \forall K \in \mathcal{T}_h,$$

where h_K is the diameter of K, ρ_K the diameter of the largest circle contained in K, and ϑ_{iK}, $i = 1,2,3,4$, the four angles of K.

Let $\widehat{K} := [0,1]^2$ be the reference element. We denote by $Q_{i,j}(\widehat{K})$ the space of polynomials of degree less than or equal to i in the first variable and to j in the second one. Also, we set $Q_k(\widehat{K}) = Q_{k,k}(\widehat{K})$.

Let $K \in \mathcal{T}_h$. We denote by F_K a bilinear mapping of $\widehat{K}$ onto K, with Jacobian matrix and determinant denoted by DF_K and J_{F_K}, respectively. The regularity assumptions above lead to

$$ch_K^2 \leq J_{F_K} \leq Ch_K^2,$$

with c and C only depending on σ and ϱ (see [19]). In particular, $J_{F_K} > 0$ and, hence, F_K is a one-to-one map. Let ℓ_i, $i = 1,2,3,4$, be the edges of K; then $\ell_i = F_K(\widehat{\ell}_i)$, with $\widehat{\ell}_i$ being the edges of $\widehat{K}$. Let $\widehat{\tau}_i$ be a unit vector tangent to $\widehat{\ell}_i$ on the reference element; then $\tau_i := DF_K \widehat{\tau}_i / \|DF_K \widehat{\tau}_i\|$ is a unit vector tangent to ℓ_i on K (see Figure 2.1).

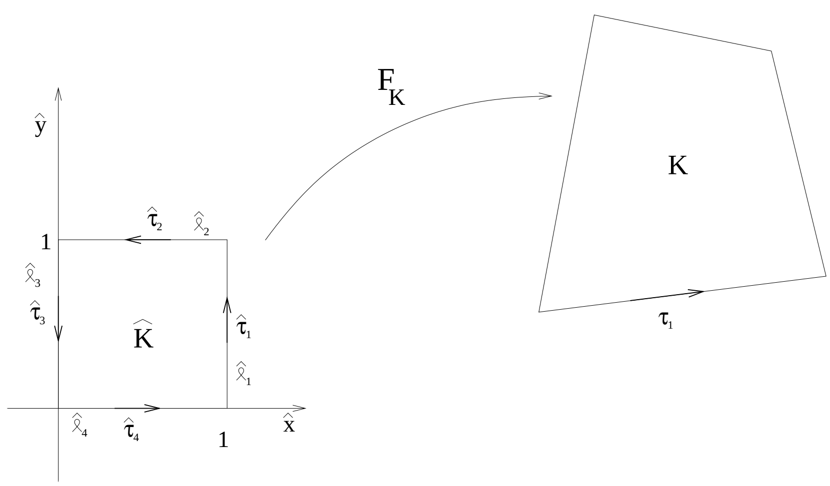

FIG. 2.1. *Bilinear mapping onto an element $K \in \mathcal{T}_h$.*

Let

$$\mathcal{N}(\widehat{K}) := \left\{ \widehat{p} : \ \widehat{p} \in Q_{0,1}(\widehat{K}) \times Q_{1,0}(\widehat{K}) \right\},$$

and, from this space, we define through covariant transformation

$$\mathcal{N}(K) := \left\{ p : \ p \circ F_K = DF_K^{-\mathrm{t}} \widehat{p}, \ \widehat{p} \in \mathcal{N}(\widehat{K}) \right\}.$$

Let us remark that the mapping between $\mathcal{N}(K)$ and $\mathcal{N}(\widehat{K})$ is a kind of "Piola transformation" for the "rot" operator, $\operatorname{rot} p := \partial p_1/\partial x_2 - \partial p_2/\partial x_1$ (the Piola transformation is defined for the "div" operator in, for example, [9]). Then we have the following results which are easily established (see [23, 24]):

$$\int_{\ell_i} p \cdot \tau_i = \int_{\widehat{\ell}_i} \widehat{p} \cdot \widehat{\tau}_i, \qquad i = 1, 2, 3, 4, \tag{2.3}$$

and

$$(\operatorname{rot} p) \circ F_K = J_{F_K}^{-1} \,\widehat{\operatorname{rot}}\, \widehat{p} \qquad \text{in } \widehat{K}. \tag{2.4}$$

We define the lowest-order rotated Raviart-Thomas space (see [21, 24])

$$\Gamma_h := \{\psi \in \mathrm{H}_0(\operatorname{rot}, \Omega) : \ \psi|_K \in \mathcal{N}(K) \ \forall K \in \mathcal{T}_h\},$$

which will be used to approximate the shear stress γ. We remark that, since $\Gamma_h \subset \mathrm{H}_0(\operatorname{rot}, \Omega)$, the tangential component of a function in Γ_h must be continuous along inter-element boundaries and vanish on $\partial\Omega$. In fact, the integrals (2.3) of these tangential components are the degrees of freedom defining an element of Γ_h.

We consider the "interpolation" operator

$$R : \mathrm{H}^1(\Omega)^2 \cap \mathrm{H}_0(\operatorname{rot}, \Omega) \longrightarrow \Gamma_h, \tag{2.5}$$

defined by (see [21])

$$\int_\ell R\psi \cdot \tau_\ell = \int_\ell \psi \cdot \tau_\ell \qquad \forall \text{ edge } \ell \text{ of } \mathcal{T}_h, \tag{2.6}$$

where, from now on, τ_ℓ denotes a unit vector tangent to ℓ. Clearly, the operator R satisfies $\forall \psi \in \mathrm{H}^1(\Omega)^2$,

$$\int_K \operatorname{rot}(\psi - R\psi) = 0 \qquad \forall K \in \mathcal{T}_h. \tag{2.7}$$

Taking into account the rotation mentioned above, it is proved in Theorem III.4.4 of [14] that

$$\|\operatorname{rot} R\psi\|_{0,\Omega} \le C \|\psi\|_{1,\Omega} \tag{2.8}$$

and

$$\|\psi - R\psi\|_{0,\Omega} \le Ch \|\psi\|_{1,\Omega}. \tag{2.9}$$

To approximate the transverse displacements we will use the space of standard bilinear isoparametric elements

$$W_h := \left\{v \in \mathrm{H}_0^1(\Omega) : \ v|_K \in Q(K) \ \forall K \in \mathcal{T}_h\right\},$$

where, $\forall K \in \mathcal{T}_h$, $Q(K) := \left\{p \in \mathrm{L}^2(K) : \ p \circ F_K \in Q_1(\widehat{K})\right\}$.

The following lemma establishes some relations between the spaces Γ_h and W_h:

Lemma 2.1. *The following properties hold:*

$$\nabla W_h = \{\mu \in \Gamma_h : \ \operatorname{rot} \mu = 0\}$$

and

$$R(\nabla w) = \nabla(w^{\mathrm{I}}) \qquad \forall w \in \mathrm{H}^2(\Omega),$$

where w^{I} *is the Lagrange interpolant of* w *on* W_h*.*

Proof. For $\mu \in \Gamma_h$ and $K \in \mathcal{T}_h$, let $\widehat{\mu} \in \mathcal{N}(\widehat{K})$ be such that $\mu|_K \circ F_K = DF_K^{-\mathrm{t}}\widehat{\mu}$. Then, according to (2.4), we have $\operatorname{rot}\mu|_K \circ F_K = J_{F_K}^{-1}\,\widehat{\operatorname{rot}}\,\widehat{\mu}$. Hence, since $J_{F_K} > 0$, $\operatorname{rot}\mu = 0$ if and only if $\widehat{\operatorname{rot}}\,\widehat{\mu} = 0$.

On the other hand, note that if $\widehat{\mu} \in \mathcal{N}(\widehat{K})$, then $\widehat{\mu} = (a+b\widehat{y}, c+d\widehat{x})$, with $a, b, c, d \in \mathbb{R}$, and $\widehat{\operatorname{rot}}\,\widehat{\mu} = d - b$. Therefore, $\widehat{\operatorname{rot}}\,\widehat{\mu} = 0$ if and only if $\widehat{\mu} = (a + d\widehat{y}, c + d\widehat{x}) = \widehat{\nabla}\widehat{v}$, for $\widehat{v} = a\widehat{x} + c\widehat{y} + d\widehat{x}\widehat{y} \in Q_1(\widehat{K})$.

Thus, $\operatorname{rot}\mu|_K = 0$ if and only if $\mu|_K = (DF_K^{-\mathrm{t}}\widehat{\mu}) \circ F_K^{-1} = \nabla v$, with $v = \widehat{v} \circ F_K^{-1} \in Q(K)$.

To prove the second property, since we have already proved that $\nabla w^{\mathrm{I}} \in \Gamma_h$, it is enough to show that the degrees of freedom defining $R(\nabla w)$ and ∇w^{I} coincide. Indeed, consider an edge ℓ with end points A and B as in Figure 2.2. Then,

$$\int_\ell R(\nabla w) \cdot \tau_\ell = \int_\ell \nabla w \cdot \tau_\ell = w(B) - w(A) = w^{\mathrm{I}}(B) - w^{\mathrm{I}}(A) = \int_\ell \nabla w^{\mathrm{I}} \cdot \tau_\ell,$$

and we conclude the proof. $\square$

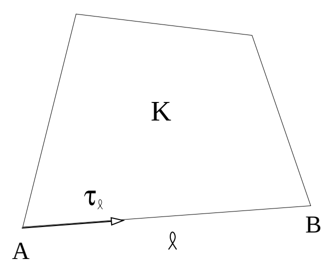

FIG. 2.2. *Geometry of* K*.*

The two methods that we analyze in this paper only differ in the space used to approximate the rotations. Let us now specify them:

MITC4: The spaces W_h and Γ_h are the ones defined above, whereas the space of standard isoparametric bilinear functions is used for the rotations; namely:

$$H_h^1 := \left\{ \eta \in \mathrm{H}_0^1(\Omega)^2 : \ \eta|_K \in Q(K)^2 \ \forall K \in \mathcal{T}_h \right\}.$$

DL4: While for this method W_h and Γ_h are the same as for *MITC4*, the space for the rotations is enriched by using a rotation of a space used for the approximation of the Stokes problem in [14].

In fact, for each edge $\widehat{\ell}_i$ of $\widehat{K}$, $i = 1, 2, 3, 4$, let $\widehat{p}_i$ be cubic functions vanishing on $\widehat{\ell}_j$ for $j \neq i$. Namely, $\widehat{p}_1 = \widehat{x}\widehat{y}(1-\widehat{y})$, $\widehat{p}_2 = \widehat{x}\widehat{y}(1-\widehat{x})$, $\widehat{p}_3 = \widehat{y}(1-\widehat{x})(1-\widehat{y})$, and $\widehat{p}_4 = \widehat{x}(1-\widehat{x})(1-\widehat{y})$ (see Figure 2.1). Then we define $p_i := (\widehat{p}_i \circ F_K^{-1})\tau_i$ and we set

$$H_h^2 := \left\{ \eta \in \mathrm{H}_0^1(\Omega)^2 : \ \eta|_K \in Q(K)^2 \oplus \langle p_1, p_2, p_3, p_4 \rangle \ \forall K \in \mathcal{T}_h \right\}.$$

From now on we use H_h to denote any of the two spaces H_h^1 or H_h^2. In both methods we use R defined by (2.5)-(2.6) as *reduction operator*. Then, the discretization of Problem 2.1 can be written in both cases as follows:

PROBLEM 2.3. *Find* $(\beta_h, w_h) \in H_h \times W_h$ *such that:*

$$\begin{cases} a(\beta_h,\eta) + (\gamma_h, \nabla v - R\eta) = (f,v) + \dfrac{t^2}{12}(\theta,\eta) & \forall(\eta,v) \in H_h \times W_h, \\ \gamma_h = \dfrac{\kappa}{t^2}(\nabla w_h - R\beta_h). \end{cases} \tag{2.10}$$

On the other hand, the discretization of Problem 2.2 is as follows:

PROBLEM 2.4. *Find* $\lambda_h \in \mathbb{R}$ *and* $0 \neq (\beta_h, w_h) \in H_h \times W_h$ *such that:*

$$\begin{cases} a(\beta_h,\eta) + (\gamma_h, \nabla v - R\eta) = \lambda_h \left[(w_h,v) + \dfrac{t^2}{12}(\beta_h,\eta)\right] & \forall(\eta,v) \in H_h \times W_h, \\ \gamma_h = \dfrac{\kappa}{t^2}(\nabla w_h - R\beta_h). \end{cases}$$

Existence and uniqueness of solution for Problem 2.3 follow easily (see [12]). Regarding Problem 2.4, it leads to a well posed generalized matrix eigenvalue problem, since the bilinear form in the right hand side of the first equation is an inner product.

3. H^1 **error estimates.** To prove optimal error estimates in H^1 norm we will use the abstract theory developed in [12]. In particular, sufficient conditions to obtain such estimates have been settled in Theorem 3.1 of this reference. By virtue of Lemma 2.1 this theorem reads in our case:

THEOREM 3.1. *Let* H_h, W_h, Γ_h, *and the operator* R *be defined as above. Let* (β, w, γ) *and* (β_h, w_h, γ_h) *be the solutions of equations (2.1) and (2.10), respectively. If there exist* $\widetilde{\beta} \in H_h$ *and an operator* $\Pi : \mathrm{H}_0(\mathrm{rot},\Omega) \cap \mathrm{H}^1(\Omega)^2 \longrightarrow \Gamma_h$ *satisfying*

$$\|\beta - \widetilde{\beta}\|_{1,\Omega} \leq Ch\|\beta\|_{2,\Omega}, \tag{3.1}$$

$$\|\eta - \Pi\eta\|_{0,\Omega} \leq Ch\|\eta\|_{1,\Omega} \qquad \forall \eta \in \mathrm{H}^1(\Omega)^2 \cap \mathrm{H}_0(\mathrm{rot},\Omega), \tag{3.2}$$

and

$$\mathrm{rot}\left(\frac{t^2}{\kappa}\Pi\gamma + R\widetilde{\beta}\right) = 0, \tag{3.3}$$

then, the following error estimate holds true:

$$\|\beta - \beta_h\|_{1,\Omega} + t\|\gamma - \gamma_h\|_{0,\Omega} + \|w - w_h\|_{1,\Omega} \leq Ch\left(\|\beta\|_{2,\Omega} + t\|\gamma\|_{1,\Omega} + \|\gamma\|_{0,\Omega}\right).$$

Then, our next step is to construct an approximation $\widetilde{\beta}$ of β and an operator Π satisfying the hypotheses of the previous theorem for each one of the methods, *MITC4* and *DL4*.

3.1. *MITC4***.** Several studies have been carried out for this method in, for example, [6], [12], and [17]. Since the variational equations for plates have a certain similitude with those of the Stokes problem, the main results are based on properties already known for the latter. An order h of convergence is obtained in those references only for uniform meshes of square elements. Moreover, more regularity of the solutions is also required. Although these results can be adapted for parallelogram meshes, they cannot be extended to general quadrilateral ones.

In what follows we obtain error estimates optimal in order and regularity for this method on somewhat more general meshes. We assume specifically the following condition:

ASSUMPTION 3.1. *The mesh $\mathcal{T}_h$ is a refinement of a coarser partition $\mathcal{T}_{2h}$, obtained by joining the midpoints of each opposite edge in each $M \in \mathcal{T}_{2h}$ (called macro-element). In addition, $\mathcal{T}_{2h}$ is a similar refinement of a still coarser regular partition $\mathcal{T}_{4h}$.*

Let

$$Q_h := \left\{ q_h \in \mathrm{L}^2_0(\Omega) : \ q_h|_K = c_K, \ c_K \in \mathbb{R}, \ \forall K \in \mathcal{T}_h \right\},$$

where $\mathrm{L}^2_0(\Omega) := \left\{ q \in \mathrm{L}^2(\Omega) : \ \int_\Omega q = 0 \right\}$. Note that, for parallelogram meshes, we have $Q_h = \operatorname{rot} \Gamma_h$, but this does not hold for general quadrilateral meshes.

For each macro-element $M \in \mathcal{T}_{2h}$ we introduce four functions q_i, $i = 1, 2, 3, 4$, taking the values 1 and -1 according to the pattern of Figure 3.1.

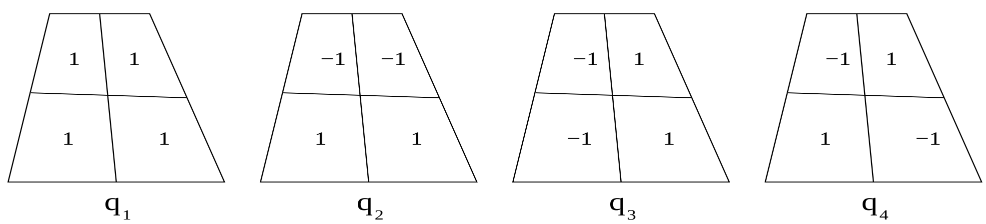

FIG. 3.1. *Bases for the macro-elements.*

Let

$$Q_{h4} := \{ q_h \in Q_h : \ q_h|_M = c_M q_4, \ c_M \in \mathbb{R}, \ \forall M \in \mathcal{T}_{2h} \}$$

and $\widetilde{Q}_h$ be its $\mathrm{L}^2(\Omega)$ orthogonal complement on Q_h; then,

$$\widetilde{Q}_h := \{ q_h \in Q_h : \ q_h|_M \in \langle q_1, q_2, q_3 \rangle \ \forall M \in \mathcal{T}_{2h} \} .$$

We associate to these spaces the subspace of H^1_h defined by

$$\widetilde{H}^1_h := \left\{ \eta_h \in H^1_h : \ \int_\Omega \operatorname{rot} \eta_h \, q_h = 0 \quad \forall q_h \in Q_{h4} \right\}.$$

The following lemma provides the approximation $\widetilde{\beta}$ required by Theorem 3.1. Moreover, this $\widetilde{\beta} \in \widetilde{H}^1_h$, and this fact will be used below to define the operator Π required by the same theorem.

LEMMA 3.2. *Let $\beta \in \mathrm{H}^1_0(\Omega)$. Then, there exists $\widetilde{\beta} \in \widetilde{H}^1_h$ such that*

$$\int_\Omega \operatorname{rot}(\widetilde{\beta} - \beta) q_h = 0 \qquad \forall q_h \in \widetilde{Q}_h$$

and the estimate (3.1) holds true.

Proof. It follows from the results in section VI.5.4 of [9] by changing "div" by "rot" and rotating 90° the fields, which in its turn are based on the results for isoparametric elements in [22] (see also [19]). ▯

Our next step is to define the operator Π satisfying the requirements of Theorem 3.1. To do this, we will use a particular projector $\widetilde{P}$ onto $\operatorname{rot}\Gamma_h$.

We have already mentioned that, in general, $Q_h \neq \operatorname{rot}\Gamma_h$. In fact, it is simple to show that

$$\operatorname{rot}\Gamma_h = \left\{ \sum_{K\in\mathcal{T}_h} \frac{c_K}{J_{F_K}}\chi_K : \ c_K \in \mathbb{R}\ \forall K \in \mathcal{T}_h \right\} \cap \mathrm{L}^2_0(\Omega), \tag{3.4}$$

where χ_K denotes the characteristic function of K.

For each macro-element $M \in \mathcal{T}_{2h}$, we consider the bilinear mapping F_M as shown in Figure 3.2. Therefore, for any $\eta_h \in \Gamma_h$ we have

$$\operatorname{rot}\eta_h|_M = \frac{1}{J_{F_M}} \sum_{i=1}^{4} c_i \chi_{K_i},$$

where K_i are the four elements in M (see Figure 3.2).

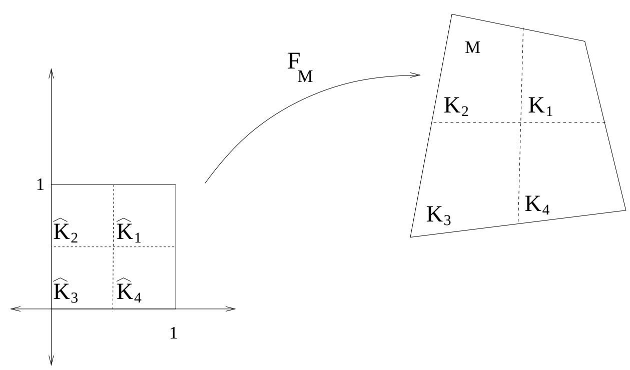

FIG. 3.2. *Bilinear mapping on macro-elements.*

We define $\widetilde{P} : \mathrm{L}^2_0(\Omega) \longrightarrow \operatorname{rot}\Gamma_h$ as follows: given $p \in \mathrm{L}^2_0(\Omega)$,

$$\forall M = \bigcup_{i=1}^{4} K_i \in \mathcal{T}_{2h}, \qquad \widetilde{P}p|_M = \sum_{i=1}^{4} \frac{c_i}{J_{F_M}} \chi_{K_i},$$

with c_i chosen such that

$$\int_M \widetilde{P}p\, q_i = \int_M p q_i, \quad i = 1,2,3, \qquad \text{and} \qquad \int_M \widetilde{P}p\, q_4 = 0.$$

Straightforward computations show that $\widetilde{P}$ is well defined by the equations above, and that they can be equivalently written

$$\int_\Omega \widetilde{P}p\, q_h = \int_\Omega p q_h \quad \forall q_h \in \widetilde{Q}_h \qquad \text{and} \qquad \int_\Omega \widetilde{P}p\, q_h = 0 \quad \forall q_h \in Q_{h4}. \tag{3.5}$$

The following properties of this operator will be used in the sequel:

LEMMA 3.3. *The following estimates hold* $\forall p \in \mathrm{L}^2(\Omega)$*:*

$$\|p - \widetilde{P}p\|_{0,\Omega} \leq C\|p\|_{0,\Omega}, \tag{3.6}$$

$$\|p - \widetilde{P}p\|_{-1,\Omega} \leq Ch\|p\|_{0,\Omega}. \tag{3.7}$$

Proof. To verify (3.6) it is enough to prove that $\|\widetilde{P}p\|_{0,\Omega} \leq C\|p\|_{0,\Omega}$. From the definition of $\widetilde{P}$ we have

$$\int_M (\widetilde{P}p)^2 = \int_M \widetilde{P}p \left(\sum_{i=1}^4 \frac{c_i}{J_{F_M}} \chi_{K_i} \right) \leq \frac{1}{\min\limits_M J_{F_M}} \int_M \widetilde{P}p \left(\sum_{i=1}^4 c_i \chi_{K_i} \right).$$

On the other hand, if we write $\sum_{i=1}^4 c_i \chi_{K_i}$ in terms of the basis functions q_i, we obtain $\sum_{i=1}^4 c_i \chi_{K_i} = \sum_{i=1}^4 d_i q_i$, with d_i related to c_i by

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}.$$

Hence,

$$|d_i| \leq 2 \max_{1 \leq j \leq 4} |c_j|, \qquad i = 1, 2, 3, 4.$$

Therefore, from the definition of $\widetilde{P}$ we have

$$\begin{aligned} \int_M \widetilde{P}p \left(\sum_{i=1}^4 c_i \chi_{K_i} \right) &= \int_M \widetilde{P}p \left(\sum_{i=1}^4 d_i q_i \right) = \sum_{i=1}^3 d_i \left(\int_M p q_i \right) \\ &\leq \|p\|_{0,M} \left(\sum_{i=1}^3 |d_i| \, \|q_i\|_{0,M} \right) \leq C|M|^{1/2} \|p\|_{0,M} \left(\max_{1 \leq j \leq 4} |c_j| \right) \\ &\leq C \max_M J_{F_M} \|p\|_{0,M} \|\widetilde{P}p\|_{0,M}, \end{aligned}$$

where we have used that

$$\int_{K_j} (\widetilde{P}p)^2 = c_j^2 \int_{K_j} \frac{1}{J_{F_M}^2} \geq \frac{|K_j|}{\max\limits_M J_{F_M}^2} c_j^2$$

and that $|M| \leq C|K_j|$, $j = 1, 2, 3, 4$, with C only depending on σ and ϱ. Now, using the inequalities above and noting that, for a quadrilateral regular mesh, $\max_M J_{F_M} \leq C \min_M J_{F_M}$ with a constant C independent of h, we obtain (3.6).

To verify (3.7), let $P : \mathrm{L}^2(\Omega) \longrightarrow \widetilde{Q}_h$ be the orthogonal projection onto $\widetilde{Q}_h$. Let $v \in \mathrm{H}_0^1(\Omega)$ be such that $\|v\|_{1,\Omega} = 1$. By the definition of P, (3.6), and the fact that $\widetilde{Q}_h$ contains the piecewise constants over $\mathcal{T}_{2h}$, we have

$$\begin{aligned} (p - \widetilde{P}p, v) &= (p - \widetilde{P}p, v - Pv) \leq \|p - \widetilde{P}p\|_{0,\Omega} \|v - Pv\|_{0,\Omega} \\ &\leq C\|p\|_{0,\Omega} \|v - Pv\|_{0,\Omega} \leq Ch\|p\|_{0,\Omega} \|v\|_{1,\Omega}. \end{aligned}$$

Thus we conclude (3.7). $\square$

Now we are in order to define an operator Π as required in Theorem 3.1:

LEMMA 3.4. *Let (β, w, γ) be the solution of equations (2.1) and $\widetilde{\beta} \in \widetilde{H}_h^1$ be as in Lemma 3.2. Then, there exists an operator $\Pi : \mathrm{H}_0(\mathrm{rot}, \Omega) \cap \mathrm{H}^1(\Omega)^2 \longrightarrow \Gamma_h$ such that (3.2) and (3.3) hold true.*

Proof. For $\eta \in \mathrm{H}_0(\mathrm{rot}, \Omega) \cap \mathrm{H}^1(\Omega)^2$, let $\Pi\eta := R(\eta - L\eta)$, where $L\eta := \mathbf{curl}\,\phi := (-\partial\phi/\partial x_2, \partial\phi/\partial x_1)$, with $\phi \in \mathrm{H}^1(\Omega)$ being a solution of

$$-\Delta\phi = \mathrm{rot}\, R\eta - \widetilde{P}(\mathrm{rot}\, R\eta) \quad \text{in } \Omega,$$

with homogeneous Neumann boundary conditions. Note that this problem is compatible since its right hand side belongs to $\mathrm{rot}\,\Gamma_h \subset \mathrm{L}_0^2(\Omega)$. Then, the standard estimates for the Neumann problem yield

$$\|L\eta\|_{m+1,\Omega} \leq \|\,\mathrm{rot}\, R\eta - \widetilde{P}(\mathrm{rot}\, R\eta)\|_{m,\Omega}, \qquad m = -1, 0. \tag{3.8}$$

Also note that

$$\mathrm{rot}\, L\eta = -\Delta\phi = \mathrm{rot}\, R\eta - \widetilde{P}(\mathrm{rot}\, R\eta). \tag{3.9}$$

From the definition of the operator Π, we have

$$\|\eta - \Pi\eta\|_{0,\Omega} \leq \|\eta - R\eta\|_{0,\Omega} + \|RL\eta\|_{0,\Omega}.$$

The first term in the right hand side is bounded by (2.9), while for the second term we use again (2.9), (3.8), Lemma 3.3, and (2.8), to obtain

$$\begin{aligned}
\|RL\eta\|_{0,\Omega} &\leq \|L\eta - RL\eta\|_{0,\Omega} + \|L\eta\|_{0,\Omega} \leq Ch\|L\eta\|_{1,\Omega} + \|L\eta\|_{0,\Omega} \\
&\leq Ch\|\,\mathrm{rot}\, R\eta - \widetilde{P}(\mathrm{rot}\, R\eta)\|_{0,\Omega} + C\|\,\mathrm{rot}\, R\eta - \widetilde{P}(\mathrm{rot}\, R\eta)\|_{-1,\Omega} \\
&\leq Ch\|\,\mathrm{rot}\, R\eta\|_{0,\Omega} \leq Ch\|\eta\|_{1,\Omega}.
\end{aligned}$$

Thus, we conclude (3.2).

To prove (3.3), note that (2.7) together with Lemma 3.2 yield

$$\int_\Omega \mathrm{rot}\left[R(\widetilde{\beta} - \beta)\right] q_h = 0 \qquad \forall q_h \in \widetilde{Q}_h,$$

whereas, since $\widetilde{\beta} \in \widetilde{H}_h^1$, from (2.7) and the definition of $\widetilde{H}_h^1$ we have

$$\int_\Omega \mathrm{rot}\, R\widetilde{\beta}\, q_h = \int_\Omega \mathrm{rot}\,\widetilde{\beta}\, q_h = 0 \qquad \forall q_h \in Q_{h4}.$$

Hence, since $\mathrm{rot}\, R\widetilde{\beta} \in \mathrm{rot}\,\Gamma_h$, from (3.5) we conclude that $\widetilde{P}(\mathrm{rot}\, R\beta) = \mathrm{rot}\, R\widetilde{\beta}$. Therefore,

$$\mathrm{rot}\, R\widetilde{\beta} = \widetilde{P}(\mathrm{rot}\, R\beta) = -\frac{t^2}{\kappa}\widetilde{P}(\mathrm{rot}\, R\gamma), \tag{3.10}$$

because of the definition of γ in (2.1) and the fact that $\mathrm{rot}\, R(\nabla w)$ vanishes, as a consequence of Lemma 2.1.

On the other hand, note that

$$\mathrm{rot}\, RL\gamma = \mathrm{rot}\, L\gamma. \tag{3.11}$$

Indeed, $\operatorname{rot} RL\gamma$ and $\operatorname{rot} L\gamma$ both belong to $\operatorname{rot}\Gamma_h$ (the latter because of (3.9)). Then, from the characterization (3.4) of this space, it is enough to verify that $\int_K \operatorname{rot} RL\gamma = \int_K \operatorname{rot} L\gamma\ \forall K \in \mathcal{T}_h$, which in its turn is a consequence of (2.7). Therefore, from the definition of Π, (3.11), and (3.9), we obtain

$$\operatorname{rot}\Pi\gamma = \operatorname{rot} R(\gamma - L\gamma) = \operatorname{rot} R\gamma - \operatorname{rot} L\gamma = \widetilde{P}(\operatorname{rot} R\gamma),$$

which together with (3.10) allow us to conclude (3.2). ∎

3.2. ***DL4*.** The convergence of this method follows immediately from that of *MITC4*. However, we have an alternative proof valid for any regular mesh without the need of Assumption 3.1.

In this case, to define the approximation $\widetilde{\beta}$ of β and the operator Π satisfying the hypotheses of Theorem 3.1, we use some known results for the Stokes problem (see Girault and Raviart[14]).

LEMMA 3.5. *There exists $\widetilde{\beta} \in H_h^2$ such that (3.1) holds true. Furthermore, $R\widetilde{\beta} = R\beta$.*

Proof. By using results from [14] (section 3.1, chapter II) and taking into account a rotation of the space $\mathrm{H}(\operatorname{div},\Omega)$, it follows that for $\beta \in \mathrm{H}_0^1(\Omega)^2$ there exists $\widetilde{\beta} \in H_h^2$ such that

$$\int_\ell (\widetilde{\beta} - \beta)\cdot \tau_\ell = 0 \qquad \forall \ell \in \mathcal{T}_h,$$

and

$$|\widetilde{\beta} - \beta|_{m,\Omega} \le C h^{k-m} |\beta|_{k,\Omega}, \qquad m = 0, 1, \quad k = 1, 2.$$

Then $R(\widetilde{\beta} - \beta) = 0$ because of the definition of R, whereas (3.1) corresponds to the inequality above for $k = 2$ and $m = 1$. ∎

LEMMA 3.6. *There exists an operator $\Pi : \mathrm{H}_0(\operatorname{rot},\Omega) \cap \mathrm{H}^1(\Omega)^2 \longrightarrow \Gamma_h$ such that (3.3) and (3.2) hold.*

Proof. Because of the previous lemma we have $R(\widetilde{\beta} - \beta) = 0$. On the other hand, $\operatorname{rot} R(\nabla w) = 0$, because of Lemma 2.1. Then, it is enough to take $\Pi = R$ to obtain (3.3), whereas (3.2) follows from (2.9). ∎

3.3. Main result in H^1 norm. Now we are in position to establish the error estimates. As above, in the case of *MITC4*, we consider meshes satisfying Assumption 3.1.

THEOREM 3.7. *Given $(\theta, f) \in \mathrm{L}^2(\Omega)^2 \times \mathrm{L}^2(\Omega)$, let (β, w) and (β_h, w_h) be the solutions of Problems 2.1 and 2.3, respectively. Then, there exists a constant C, independent of t and h, such that*

$$\|\beta - \beta_h\|_{1,\Omega} + \|w - w_h\|_{1,\Omega} \le Ch|(\theta, f)|_t.$$

Proof. It is a direct consequence of Lemmas 3.2, 3.4, 3.5, 3.6, Theorem 3.1, and the *a priori* estimate (2.2). ∎

4. L^2 error estimates. Our next goal is to prove L^2 error estimates optimal in order and regularity. To do this, we follow the techniques in [13] where a triangular element similar to *DL4* is analyzed, although the arguments therein cannot be directly applied to our case. Let us remark that, in the case of *MITC4*, this result completes the analysis carried out in [10, 18] for higher order methods.

Our proofs are based on a standard Nitsche's duality argument. However, since the methods are non-conforming, additional consistency terms also arise. Then, higher order estimates must be proved for these terms too, which is the most delicate part of the paper.

First, we introduce the dual problem corresponding to equations (2.1). Let $(\varphi, u) \in \mathrm{H}^1_0(\Omega)^2 \times \mathrm{H}^1_0(\Omega)$ be the solution of

$$\left\{\begin{array}{l} a(\eta,\varphi) + (\nabla v - \eta, \delta) = (v, w - w_h) + (\eta, \beta - \beta_h) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall (\eta, v) \in \mathrm{H}^1_0(\Omega)^2 \times \mathrm{H}^1_0(\Omega), \\ \delta = \dfrac{\kappa}{t^2}(\nabla u - \varphi). \end{array}\right. \tag{4.1}$$

By taking $\eta = 0$ in (4.1), we have

$$\operatorname{div} \delta = w_h - w. \tag{4.2}$$

An *a priori* estimate analogous to (2.2) yields for this problem:

$$\|\varphi\|_{2,\Omega} + \|u\|_{2,\Omega} + \|\delta\|_{0,\Omega} + t\,\|\delta\|_{1,\Omega} \le C\left(\|\beta - \beta_h\|_{0,\Omega} + \|w - w_h\|_{0,\Omega}\right). \tag{4.3}$$

The arguments in the proof of Lemma 3.4 in [13] can be used in our case leading to the following result:

LEMMA 4.1. *Given $(\theta, f) \in \mathrm{L}^2(\Omega)^2 \times \mathrm{L}^2(\Omega)$, let (β, w, γ) and (β_h, w_h, γ_h) be the solutions of equations (2.1) and (2.10), respectively. Let (φ, u, δ) be the solution of (4.1). Let $\widetilde{\varphi} \in H_h$ be the vector field associated to φ by Lemmas 3.2 or 3.5, for MITC4 or DL4, respectively. Then, there exists a constant C, independent of t and h, such that*

$$\|\beta - \beta_h\|_{0,\Omega} + \|w - w_h\|_{0,\Omega} \le Ch^2 |(\theta, f)|_t + \frac{|(\beta_h - R\beta_h, \delta)| + |(\gamma, \widetilde{\varphi} - R\widetilde{\varphi})|}{\|\beta - \beta_h\|_{0,\Omega} + \|w - w_h\|_{0,\Omega}}.$$

Our next step is to prove that the last term in the inequality above is $\mathcal{O}(h^2)$ too. A similar result has been proved in [13] in the case of triangular meshes. That proof relies on a technical result for the rotated Raviart-Thomas interpolant R (Lemma 3.3 of that reference). It is easy to check that the arguments given there do not apply for quadrilateral elements. Therefore, we need to introduce new arguments and this is the aim of the following lemma.

LEMMA 4.2. *Given $\zeta \in H(\operatorname{div}, \Omega)$ and $\psi \in \mathrm{H}^1_0(\Omega)^2$, there holds*

$$|(\zeta, \psi - R\psi)| \le Ch^2 \left(\sum_K |R\psi - \psi|^2_{1,K}\right)^{1/2} \|\operatorname{div}\zeta\|_{0,\Omega} + Ch\|\operatorname{rot}(R\psi - \psi)\|_{0,\Omega}\,\|\zeta\|_{0,\Omega}.$$

Proof. For $K \in \mathcal{T}_h$, let $s_K \in \mathrm{H}^1(K)$ be a solution of

$$-\Delta s_K = \operatorname{rot}(R\psi - \psi) \quad \text{in } K,$$

with homogeneous Neumann boundary conditions. By virtue of (2.7) we know that the above problem is compatible. Hence, s_K satisfies

$$\|\operatorname{curl} s_K\|_{m+1,K} \le C\|\operatorname{rot}(R\psi - \psi)\|_{m,K}, \qquad m = -1, 0. \tag{4.4}$$

The Laplace equation above can be equivalently written

$$\operatorname{rot}\left[\operatorname{curl} s_K - (R\psi - \psi)\right] = 0$$

and, hence, there exists $r_K \in \mathrm{H}^1(K)$ (unique up to an additive constant) such that

$$\nabla r_K = \operatorname{curl} s_K - (R\psi - \psi). \tag{4.5}$$

Moreover, from the homogeneous Neumann boundary condition satisfied by s_K, we have $\nabla r_K \cdot \tau_\ell = -R\psi \cdot \tau_\ell + \psi \cdot \tau_\ell$ for each edge ℓ of K. Thus, if we define $G \in \mathrm{L}^2(\Omega)^2$ such that $G|_K = \nabla r_K$, then $G \in \mathrm{H}_0(\operatorname{rot}, \Omega)$ and $\operatorname{rot} G = 0$.

Hence, there exists $r \in \mathrm{H}^1(\Omega)/\mathbb{R}$, such that $G = \nabla r$ in Ω. Furthermore, since $G \in \mathrm{H}_0(\operatorname{rot}, \Omega)$, r can be chosen in $\mathrm{H}^1_0(\Omega)$ and the additive constants defining r_K on each $K \in \mathcal{T}_h$ can be fixed as to satisfy $r|_K = r_K$.

Let A and B be as in Figure 2.2. Then, because of (2.6), we have

$$r(B) = r(A) + \int_\ell \nabla r_K \cdot \tau_\ell = r(A) + \int_\ell (-R\psi + \psi) \cdot \tau_\ell = r(A).$$

Thus r vanishes at all nodes of $\mathcal{T}_h$, since $r|_{\partial\Omega} = 0$. Hence, a standard scaling argument on each element K yields $\|r\|_{0,K} \le Ch^2 |r_K|_{2,K}$ (see for instance [11]) and, then, by using (4.5) and (4.4), we have

$$\begin{aligned} \|r\|_{0,K} &\le Ch^2 \left|\nabla r_K\right|_{1,K} \le Ch^2 \left(\left|\operatorname{curl} s_K\right|_{1,K} + \left|R\psi - \psi\right|_{1,K}\right) \\ &\le Ch^2 \left[\left\|\operatorname{rot}(R\psi - \psi)\right\|_{0,K} + \left|R\psi - \psi\right|_{1,K}\right] \le Ch^2 \left|R\psi - \psi\right|_{1,K}. \end{aligned} \tag{4.6}$$

On the other hand, let $(\cdot,\cdot)_K$ be the usual inner product in $\mathrm{L}^2(K)$ and P the orthogonal projection onto the constant functions. Because of (2.7), we have $\forall \eta \in \mathrm{H}^1_0(\Omega)$,

$$\frac{\left(\operatorname{rot}(R\psi - \psi), \eta\right)_K}{\|\eta\|_{1,K}} = \frac{\left(\operatorname{rot}(R\psi - \psi), \eta - P\eta\right)_K}{\|\eta\|_{1,K}} \le \frac{\left\|\operatorname{rot}(R\psi - \psi)\right\|_{0,K} \|\eta - P\eta\|_{0,K}}{\|\eta\|_{1,K}}.$$

Hence,

$$\left\|\operatorname{rot}(R\psi - \psi)\right\|_{-1,K} \le Ch \left\|\operatorname{rot}(R\psi - \psi)\right\|_{0,K}.$$

Now, let $S \in \mathrm{L}^2(\Omega)^2$ be such that $S|_K = \operatorname{curl} s_K$. Therefore, because of (4.4) we have

$$\begin{aligned} \|S\|^2_{0,\Omega} &= \sum_{K \in \mathcal{T}_h} \left\|\operatorname{curl} s_K\right\|^2_{0,K} \le \sum_{K \in \mathcal{T}_h} \left\|\operatorname{rot}(R\psi - \psi)\right\|^2_{-1,K} \\ &\le Ch^2 \sum_{K \in \mathcal{T}_h} \left\|\operatorname{rot}(R\psi - \psi)\right\|^2_{0,K} \le Ch^2 \left\|\operatorname{rot}(R\psi - \psi)\right\|^2_{0,\Omega}. \end{aligned} \tag{4.7}$$

Finally, from (4.5) we obtain

$$\left|(\zeta, \psi - R\psi)\right| = \left|\int_\Omega \zeta \cdot \nabla r + \int_\Omega \zeta \cdot S\right| \le \left\|\operatorname{div} \zeta\right\|_{0,\Omega} \|r\|_{0,\Omega} + \|\zeta\|_{0,\Omega} \|S\|_{0,\Omega},$$

and the lemma follows by using (4.6) and (4.7). $\square$

To obtain a bound of the consistency term in Lemma 4.1, there only remains to estimate the terms involving $(R\psi - \psi)$ of the previous lemma. To this aim, we use the analogue of Theorem 4.3 in [24] applied to our situation in the space $\mathrm{H}(\mathrm{rot}, \Omega)$, which reads:

$$|\psi - R\psi|_{1,K} \leq C \left(|\psi|_{1,K} + h_K \,|\mathrm{rot}\,\psi|_{1,K} \right) \leq \|\psi\|_{2,K} \tag{4.8}$$

and

$$\|\mathrm{rot}(\psi - R\psi)\|_{0,K} \leq C \left(\frac{\delta_K}{h_K} \,|\mathrm{rot}\,\psi|_{0,K} + h_K \,|\mathrm{rot}\,\psi|_{1,K} \right), \tag{4.9}$$

where δ_K is a measure of the deviation of the quadrilateral K from a parallelogram, as defined in Figure 4.1.

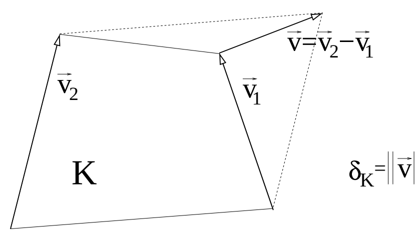

Fig. 4.1. *Geometrical definition of δ_K.*

Note that for shape-regular meshes clearly $\delta_K/h_K \leq C \;\forall K \in \mathcal{T}_h$. On the other hand, $\{\mathcal{T}_h\}$ is said to be a family of *asymptotically parallelogram* meshes when there exists a constant C such that $\max_{K\in\mathcal{T}_h} (\delta_K/h_K) \leq Ch$ for all the meshes.

Now we are in position to estimate the consistency term in Lemma 4.1:

Lemma 4.3. *Let β_h, δ, γ and $\widetilde{\varphi}$ be as in Lemma 4.1. Then, there holds*

$$\frac{|(\beta_h - R\beta_h, \delta)| + |(\gamma, \widetilde{\varphi} - R\widetilde{\varphi})|}{\|\beta - \beta_h\|_{0,\Omega} + \|w - w_h\|_{0,\Omega}} \leq Ch \left(h + t \max_{K\in\mathcal{T}_h} \frac{\delta_K}{h_K} \right) |(\theta, f)|_t.$$

Proof. First, we have

$$|(\beta_h - R\beta_h, \delta)| \leq \left| \big((\beta_h - \beta) - R(\beta_h - \beta), \delta \big) \right| + |(\beta - R\beta, \delta)|\,.$$

By using (2.9), Theorem 3.7, and (4.3), we obtain

$$\begin{aligned} \left| \big((\beta_h - \beta) - R(\beta_h - \beta), \delta \big) \right| &\leq Ch \,\|\beta_h - \beta\|_{1,\Omega} \,\|\delta\|_{0,\Omega} \leq Ch^2 \,|(\theta, f)|_t \,\|\delta\|_{0,\Omega} \\ &\leq Ch^2 \,|(\theta, f)|_t \left(\|\beta - \beta_h\|_{0,\Omega} + \|w - w_h\|_{0,\Omega} \right). \end{aligned}$$

On the other hand, by the definition of γ in (2.1) and the estimate (2.2), we have

$$|\mathrm{rot}\,\beta|_{0,K} = \frac{t^2}{\kappa} \,|\mathrm{rot}\,\gamma|_{0,K} \leq Ct \,|(\theta, f)|_t\,.$$

Then, by using Lemma 4.2, (4.8), (4.9), the estimate above, (2.2), (4.2), and (4.3), we have

$$
\begin{aligned}
|(\beta - R\beta, \delta)| &\le Ch^2 \left(\sum_K |R\beta - \beta|_{1,K}^2 \right)^{1/2} \|\operatorname{div} \delta\|_{0,\Omega} + Ch \|\operatorname{rot}(R\beta - \beta)\|_{0,\Omega} \|\delta\|_{0,\Omega} \\
&\le Ch^2 \|\beta\|_{2,\Omega} \|\operatorname{div} \delta\|_{0,\Omega} + Ch \left(\max_{K \in \mathcal{T}_h} \frac{\delta_K}{h_K} |\operatorname{rot} \beta|_{0,\Omega} + h |\operatorname{rot} \beta|_{1,\Omega} \right) \|\delta\|_{0,\Omega} \\
&\le Ch^2 |(\theta, f)|_t \|\operatorname{div} \delta\|_{0,\Omega} + Ch \left(h + t \max_{K \in \mathcal{T}_h} \frac{\delta_K}{h_K} \right) |(\theta, f)|_t \|\delta\|_{0,\Omega} \\
&\le Ch \left(h + t \max_{K \in \mathcal{T}_h} \frac{\delta_K}{h_K} \right) |(\theta, f)|_t \left(\|\beta - \beta_h\|_{0,\Omega} + \|w - w_h\|_{0,\Omega} \right).
\end{aligned}
$$

The term $|(\gamma, \widetilde{\varphi} - R\widetilde{\varphi})|$ can be bounded almost identically, by using Lemma 3.2 for *MITC4* or Lemma 3.5 for *DL4* to estimate $\|\widetilde{\varphi} - \varphi\|_{1,\Omega}$ and the fact that

$$
-\operatorname{div} \gamma = f \qquad \text{in } \Omega,
$$

which follows by taking $\eta = 0$ in the first equation of (2.1). Therefore, we obtain

$$
|(\gamma, \widetilde{\varphi} - R\widetilde{\varphi})| \le Ch \left(h + t \max_{K \in \mathcal{T}_h} \frac{\delta_K}{h_K} \right) |(\theta, f)|_t \left(\|\beta - \beta_h\|_{0,\Omega} + \|w - w_h\|_{0,\Omega} \right),
$$

which allows us to conclude the proof. ∎

Finally, we can establish an $\mathrm{L}^2(\Omega)$ error estimate. As above, in case of *MITC4* elements, we consider meshes satisfying Assumption 3.1.

THEOREM 4.4. *Given $(\theta, f) \in \mathrm{L}^2(\Omega)^2 \times \mathrm{L}^2(\Omega)$, let (β, w) and (β_h, w_h) be the solutions of Problems 2.1 and 2.3, respectively. Then, there exists a constant C, independent of t and h, such that*

$$
\|\beta - \beta_h\|_{0,\Omega} + \|w - w_h\|_{0,\Omega} \le Ch \left(h + t \max_{K \in \mathcal{T}_h} \frac{\delta_K}{h_K} \right) |(\theta, f)|_t .
$$

Proof. It is a direct consequence of Lemmas 4.1 and 4.3. ∎

COROLLARY 4.5. *The following error estimate holds for any family of asymptotically parallelogram meshes:*

$$
\|\beta - \beta_h\|_{0,\Omega} + \|w - w_h\|_{0,\Omega} \le Ch^2 |(\theta, f)|_t .
$$

Remark 4.1. *The asymptotically parallelogram assumption on the meshes is not necessary as long as $h > \alpha t$, for α fixed. Indeed, according to Theorem 4.4, for general regular meshes with $h > \alpha t$, we have*

$$
\|\beta - \beta_h\|_{0,\Omega} + \|w - w_h\|_{0,\Omega} \le C_\alpha h^2 |(\theta, f)|_t.
$$

Note that the condition $h > \alpha t$ is fulfilled in practice for reasonably large values of α.

5. The spectral problem. The aim of this section is to study how the eigenvalues and eigenfunctions of Problem 2.4 approximate those of Problem 2.2. We do this in the framework of the abstract spectral approximation theory as stated, for instance, in the monograph by Babuška and Osborn[5]. In order to use this theory, we define operators T and T_h associated to the continuous and discrete spectral problems, respectively.

We consider the operator

$$T : \mathrm{L}^2(\Omega)^2 \times \mathrm{L}^2(\Omega) \longrightarrow \mathrm{L}^2(\Omega)^2 \times \mathrm{L}^2(\Omega),$$

defined by $T(\theta, f) := (\beta, w)$, where $(\beta, w) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$ is the solution of Problem 2.1. Note that T is compact, as a consequence of estimate (2.2). Since the operator is clearly self-adjoint with respect to $(\cdot, \cdot)_t$, then, apart from $\mu = 0$, its spectrum consists of a sequence of finite multiplicity real eigenvalues converging to zero. Note that λ is an eigenvalue of Problem 2.2 if and only if $\mu := 1/\lambda$ is an eigenvalue of T, with the same multiplicity and corresponding eigenfunctions.

As shown in [13], each eigenvalue μ of Problem 2.1 converges to some limit μ_0, when the thickness $t \to 0$. Indeed, μ_0 is an eigenvalue of the operator associated with the Kirchhoff model of the same plate (see Lemma 2.1 in [13]). From now on, for simplicity, we assume that $\mu = 1/\lambda$ is an eigenvalue of T which converges to a simple eigenvalue μ_0, as t goes to zero (see section 2 in [13] for further discussions).

Now, analogously to the continuous case, we introduce the operator

$$T_h : \mathrm{L}^2(\Omega)^2 \times \mathrm{L}^2(\Omega) \longrightarrow \mathrm{L}^2(\Omega)^2 \times \mathrm{L}^2(\Omega),$$

defined by $T_h(\theta, f) := (\beta_h, w_h)$, where $(\beta_h, w_h) \in H_h \times W_h$ is the solution of Problem 2.3. The operator T_h is also self-adjoint with respect to $(\cdot, \cdot)_t$. Clearly, the eigenvalues of T_h are given by $\mu_h := 1/\lambda_h$, with λ_h being the strictly positive eigenvalues of Problem 2.4, and the corresponding eigenfunctions coincide.

As a consequence of Theorem 3.7, for each simple eigenvalue μ of T, there is exactly one eigenvalue μ_h of T_h converging to μ as h goes to zero (see for instance [16]). The following theorem shows optimal t-independent error estimates. Let us remark that the results of this theorem are valid for both methods, *MITC4* and *DL4*, although, for the former, under Assumption 3.1 on the meshes as in the previous section.

THEOREM 5.1. *Let λ and λ_h be simple eigenvalues of Problems 2.2 and 2.4, respectively, such that $\lambda_h \to \lambda$ as $h \to 0$. Let (β, w) and (β_h, w_h) be corresponding eigenfunctions normalized in the same manner. Then, under the assumptions stated above, there exists $C > 0$ such that, for t and h small enough, there holds*

$$\|\beta - \beta_h\|_{1,\Omega} + \|w - w_h\|_{1,\Omega} \leq Ch.$$

Furthermore, for any family of asymptotically parallelogram meshes, there hold

$$\|\beta - \beta_h\|_{0,\Omega} + \|w - w_h\|_{0,\Omega} \leq Ch^2$$

and

$$|\lambda - \lambda_h| \leq Ch^2.$$

Proof. The proof, which relies on Theorem 3.7 and Corollary 4.5, are essentially the same as those of Theorem 2.1, 2.2, and 2.3 in [13]. ∎

6. Numerical experiments. In this section we report some numerical experiments carried out with both methods applied to the spectral problem 2.2.

First, we have tested the two methods by using different meshes, not necessarily satisfying the assumptions in the theorems above. We have considered a square clamped moderately thick plate of side-length L and thickness-to-span ratio $t/L = 0.1$. We report the results obtained with both types of elements using the following three families of meshes:

$\mathcal{T}_h^{\mathrm{U}}$: It consists of uniform subdivisions of the domain into $N \times N$ sub-squares, for $N = 4, 8, 16, \dots$ (see Figure 6.1). Clearly, these are parallelogram meshes satisfying Assumption 3.1.

$\mathcal{T}_h^{\mathrm{A}}$: It consists of "uniform" refinements of a non-uniform mesh obtained by splitting the square into four quadrilaterals. Each refinement step is obtained by subdividing each quadrilateral into other four, by connecting the midpoints of the opposite edges. Thus we obtain a family of $N \times N$ asymptotically parallelogram shape regular meshes as shown in Figure 6.2. Furthermore, for $N = 4, 8, 16, \dots$, these meshes satisfy Assumption 3.1.

$\mathcal{T}_h^{\mathrm{T}}$: It consists of partitions of the domain into $N \times N$ congruent trapezoids, all similar to the trapezoid with vertices $(0,0)$, $(1/2,0)$, $(1/2,2/3)$ and $(0,1/3)$, as shown in Figure 6.3. Clearly, these are not asymptotically parallelogram meshes and they do not satisfy Assumption 3.1.

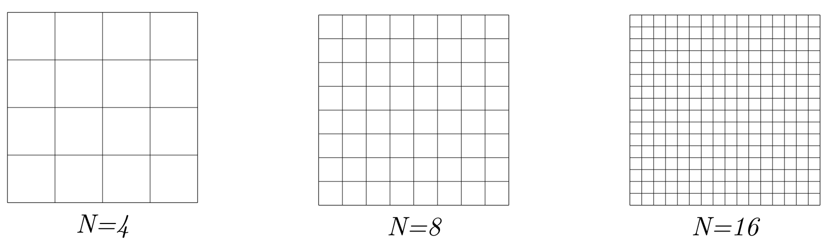

FIG. 6.1. *Uniform square meshes* $\mathcal{T}_h^{\mathrm{U}}$.

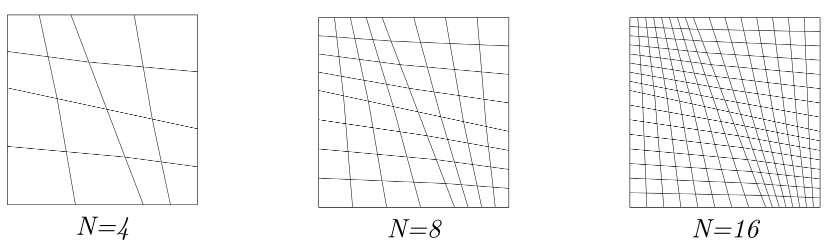

FIG. 6.2. *Asymptotically parallelogram meshes* $\mathcal{T}_h^{\mathrm{A}}$.

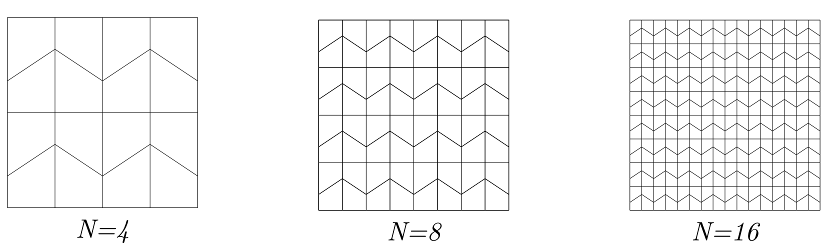

FIG. 6.3. *Trapezoidal meshes* $\mathcal{T}_h^{\mathrm{T}}$.

Let us remark that the third family was used in [3, 4] to show that the order of convergence of some finite elements deteriorate on these meshes, in spite of the fact that they are shape regular.

We have computed approximations of the free vibration angular frequencies $\omega = t\sqrt{\lambda/\rho}$ corresponding to the lowest-frequency vibration modes of the plate. In order to compare the obtained results with those in [1] we present the computed frequencies ω_{mn}^h in the following non-dimensional form:

$$\hat{\omega}_{mn} := \omega_{mn}^h L \left[\frac{2(1+\nu)\rho}{E} \right]^{1/2},$$

m and n being the numbers of half-waves occurring in the mode shapes in the x and y directions, respectively.

Tables 6.1 and 6.2 show the four lowest vibration frequencies computed by our method with successively refined meshes of each type, $\mathcal{T}_h^{\mathrm{U}}$, $\mathcal{T}_h^{\mathrm{A}}$, and $\mathcal{T}_h^{\mathrm{T}}$. Each table includes also the values of the vibration frequencies obtained by extrapolating the computed ones as well as the estimated order of convergence. Finally, it also includes in the last column the results reported in [1]. In every case we have used a Poisson ratio $\nu = 0.3$ and a correction factor $k = 0.8601$. The reported non-dimensional frequencies are independent of the remaining geometrical and physical parameters, except for the thickness-to-span ratio.

Table 6.1
Scaled vibration frequencies $\hat{\omega}_{mn}$ computed with MITC4.

Mesh	Mode	$N=16$	$N=32$	$N=64$	Extrap.	Order	[1]
$\mathcal{T}_h^{\mathrm{U}}$	$\hat{\omega}_{11}$	1.6055	1.5946	1.5919	1.5910	2.01	1.591
	$\hat{\omega}_{21}$	3.1042	3.0550	3.0429	3.0389	2.03	3.039
	$\hat{\omega}_{12}$	3.1042	3.0550	3.0429	3.0389	2.03	3.039
	$\hat{\omega}_{22}$	4.3534	4.2850	4.2681	4.2625	2.02	4.263
$\mathcal{T}_h^{\mathrm{A}}$	$\hat{\omega}_{11}$	1.6073	1.5951	1.5921	1.5911	2.01	1.591
	$\hat{\omega}_{21}$	3.1094	3.0563	3.0433	3.0390	2.02	3.039
	$\hat{\omega}_{12}$	3.1190	3.0586	3.0438	3.0390	2.03	3.039
	$\hat{\omega}_{22}$	4.3711	4.2894	4.2692	4.2626	2.02	4.263
$\mathcal{T}_h^{\mathrm{T}}$	$\hat{\omega}_{11}$	1.6112	1.5961	1.5923	1.5910	1.99	1.591
	$\hat{\omega}_{21}$	3.1129	3.0575	3.0436	3.0388	1.99	3.039
	$\hat{\omega}_{12}$	3.1306	3.0618	3.0446	3.0388	2.00	3.039
	$\hat{\omega}_{22}$	4.3916	4.2955	4.2708	4.2622	1.96	4.263

Table 6.2
Scaled vibration frequencies $\hat{\omega}_{mn}$ computed with DL4.

Mesh	Mode	$N=16$	$N=32$	$N=64$	Extrap.	Order	[1]
$\mathcal{T}_h^{\mathrm{U}}$	$\hat{\omega}_{11}$	1.5956	1.5922	1.5913	1.5910	1.98	1.591
	$\hat{\omega}_{21}$	3.0711	3.0470	3.0409	3.0388	1.99	3.039
	$\hat{\omega}_{12}$	3.0711	3.0470	3.0409	3.0388	1.99	3.039
	$\hat{\omega}_{22}$	4.3136	4.2754	4.2657	4.2624	1.98	4.263
$\mathcal{T}_h^{\mathrm{A}}$	$\hat{\omega}_{11}$	1.5929	1.5915	1.5912	1.5910	1.94	1.591
	$\hat{\omega}_{21}$	3.0592	3.0441	3.0402	3.0388	1.96	3.039
	$\hat{\omega}_{12}$	3.0732	3.0476	3.0411	3.0389	1.98	3.039
	$\hat{\omega}_{22}$	4.3136	4.2756	4.2658	4.2624	1.96	4.263
$\mathcal{T}_h^{\mathrm{T}}$	$\hat{\omega}_{11}$	1.5927	1.5914	1.5911	1.5910	2.21	1.591
	$\hat{\omega}_{21}$	3.0606	3.0445	3.0403	3.0388	1.94	3.039
	$\hat{\omega}_{12}$	3.0654	3.0453	3.0405	3.0390	2.05	3.039
	$\hat{\omega}_{22}$	4.3131	4.2754	4.2657	4.2623	1.96	4.263

It can be clearly seen that both methods converge for the three types of meshes with an optimal $\mathcal{O}(h^2)$ order. Hence, none of the two particular assumptions on the meshes (Assumption 3.1 and to be asymptotically parallelogram) seem to be actually necessary.

As a second test, we have made a numerical experiment to assess the stability of the methods as the thickness t goes to zero. We have used a sequence of clamped plates with decreasing values of the thickness-to-span ratios: $t/L = 0.1$, 0.01, 0.001, 0.0001. All the other geometrical and physical parameters have been taken as in the previous test.

We have computed again approximations of the free vibration angular frequencies $\omega = t\sqrt{\lambda/\rho}$. The quotients ω/t are known to converge to the corresponding vibration frequencies of an identical Kirchhoff plate (i.e., to the frequencies obtained from the Kirchhoff model for the deflection of a similar zero-thickness ideal plate; see Lemma 2.1 from [13]). Because of this, we present now the computed frequencies ω_{mn}^h scaled in the following manner:

$$\tilde{\omega}_{mn} := \omega_{mn}^h \frac{L}{t}\left[\frac{2(1+\nu)\rho}{E}\right]^{1/2}.$$

The obtained results have been qualitatively similar for both methods. We only report those obtained with *DL4*, since the performance of *MITC4* has been assessed in many other papers (see for instance [8], as well as [15] for the vibration problem).

We present in Table 6.3 the results for the lowest-frequency vibration mode, with the same meshes as in the previous test. In each case, for each thickness-to-span ratio t/L, we have computed again an extrapolated more accurate value of the scaled vibration frequency and the estimated order of convergence. Finally we have also estimated by extrapolation the limit values of the scaled frequencies $\tilde{\omega}_{mn}$ as t goes to zero.

Table 6.3
Scaled vibration frequency $\tilde{\omega}_{11}$ computed with DL4 for different thickness-to-span ratios t/L.

Mesh	t/L	$N = 16$	$N = 32$	$N = 64$	Extrap.	Order
$\mathcal{T}_h^{\mathrm{U}}$	0.1	15.9561	15.9220	15.9133	15.9104	1.98
	0.01	17.5778	17.5485	17.5412	17.5387	1.99
	0.001	17.5975	17.5685	17.5612	17.5588	2.00
	0.0001	17.5976	17.5687	17.5614	17.5590	2.00
	0 (extrap.)	17.5977	17.5687	17.5614	17.5590	2.00
$\mathcal{T}_h^{\mathrm{A}}$	0.1	15.9286	15.9151	15.9116	15.9104	1.94
	0.01	17.5368	17.5382	17.5385	17.5387	1.87
	0.001	17.5563	17.5580	17.5586	17.5588	1.74
	0.0001	17.5565	17.5582	17.5588	17.5590	1.74
	0 (extrap.)	17.5565	17.5582	17.5588	17.5590	1.76
$\mathcal{T}_h^{\mathrm{T}}$	0.1	15.9272	15.9141	15.9113	15.9105	2.21
	0.01	17.5681	17.5450	17.5395	17.5377	2.05
	0.001	17.5901	17.5671	17.5608	17.5585	1.89
	0.0001	17.5903	17.5673	17.5611	17.5588	1.89
	0 (extrap.)	17.5903	17.5674	17.5611	17.5588	1.89

Note that the extrapolated values for each thickness-to-span ratio are almost identical for the three meshes. Moreover, although the estimated orders of convergence seem to deteriorate a bit as t/L goes to zero for the non-uniform meshes, the values obtained with these meshes are better than those computed with the uniform mesh (i.e. closer to the extrapolated ones), even for the coarser meshes. Therefore, this test suggests that the method is locking-free for any kind of regular meshes.

Finally, we report in Table 6.4 the corresponding extrapolated values as t/L goes to zero for the four lowest scaled vibration frequencies. It can be seen from this table that the results are essentially the same as for $\tilde{\omega}_{11}$. Furthermore, the computed orders of convergence are even closer to 2.

Table 6.4

Extrapolated values as $(t/L) \to 0$ of the scaled vibration frequencies $\tilde{\omega}_{mn}$ computed with DL4.

Mesh	Mode	$N = 16$	$N = 32$	$N = 64$	Extrap.	Order
$\mathcal{T}_h^{\mathrm{U}}$	$\hat{\omega}_{11}$	17.5977	17.5687	17.5614	17.5590	2.00
	$\hat{\omega}_{21}$	36.2064	35.9115	35.8374	35.8125	1.99
	$\hat{\omega}_{12}$	36.2064	35.9115	35.8374	35.8126	1.99
	$\hat{\omega}_{22}$	53.4123	52.9570	52.8428	52.8045	1.99
$\mathcal{T}_h^{\mathrm{A}}$	$\hat{\omega}_{11}$	17.5565	17.5583	17.5588	17.5590	1.76
	$\hat{\omega}_{21}$	35.9947	35.8590	35.8243	35.8123	1.97
	$\hat{\omega}_{12}$	36.2003	35.9102	35.8371	35.8124	1.99
	$\hat{\omega}_{22}$	53.3174	52.9353	52.8374	52.8037	1.97
$\mathcal{T}_h^{\mathrm{T}}$	$\hat{\omega}_{11}$	17.5904	17.5673	17.5611	17.5588	1.89
	$\hat{\omega}_{21}$	36.0770	35.8823	35.8303	35.8113	1.90
	$\hat{\omega}_{12}$	36.2500	35.9259	35.8412	35.8112	1.94
	$\hat{\omega}_{22}$	53.5074	52.9936	52.8526	52.7993	1.87

Further experiments with *MITC4* have been reported in [15], including other boundary condition and the extension of this method to compute the vibration modes of Naghdi shells.

REFERENCES

[1] B. S. Al Janabi and E. Hinton, *A study of the free vibrations of square plates with various edge conditions*, in Numerical Methods and Software for Dynamic Analysis of Plates and Shells, E. Hinton, ed., Pineridge Press, Swansea, 1987.

[2] D. N. Arnold and R. S. Falk, *A uniformly accurate finite element method for the Reissner-Mindlin plate*, SIAM J. Numer. Anal., 26 (1989) 1276–1290.

[3] D. N. Arnold, D. Boffi, and R. S. Falk, *Approximation by quadrilateral finite elements*, Math. Comp., 71 (2002) 909–922.

[4] D. N. Arnold, D. Boffi, R. S. Falk, and L. Gastaldi, *Finite element approximation on quadrilateral meshes*, Comm. Numer. Methods Engrg., 17 (2001) 805–812.

[5] I. Babuška and J. Osborn, *Eigenvalue problems*, in Handbook of Numerical Analysis, Vol. II, P. G. Ciarlet and J. L. Lions, eds., North Holland, Amsterdam, 1991, pp. 641–787.

[6] K. J. Bathe and F. Brezzi, *On the convergence of a four-node plate bending element based on Mindlin/Reissner plate theory and a mixed interpolation*, in The Mathematics of Finite Elements and Applications V, J. R. Whiteman, ed., Academic Press, London, 1985, pp. 491–503.

[7] K. J. Bathe and E. N. Dvorkin, *A four-node plate bending element based on Mindlin/Reissner plate theory and a mixed interpolation*, Internat. J. Numer. Methods Engrg., 21 (1985) 367–383.

[8] K. J. Bathe, A. Iosilevich, and D. Chapelle, *An evaluation of the MITC shell elements.* Comp. Struct., 75 (2000) 1–30.

[9] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York, 1991.

[10] F. Brezzi, M. Fortin, and R. Stenberg, *Quasi-optimal error bounds for approximation of shear-stresses in Mindlin-Reissner plate models*, Math. Models Methods Appl. Sci, 1 (1991) 125–151.

[11] P. G. Ciarlet, *Basic error estimates for elliptic problems*, in Handbook of Numerical Analysis, Vol. II, P. G. Ciarlet and J. L. Lions, eds., North-Holland, Amsterdam, 1991, pp. 17–351.

[12] R. Durán and E. Liberman, *On mixed finite element methods for the Reissner-Mindlin plate model*, Math. Comp., 58 (1992) 561–573.

[13] R. Durán, L. Hervella-Nieto, E. Liberman, R. Rodríguez, and J. Solomin, *Approximation of the vibration modes of a plate by Reissner-Mindlin equations*, Math. Comp., 68 (1999) 1447–1463.

[14] V. Girault and P. A. Raviart, Finite Element Methods for Navier-Stokes Equations, Springer-Verlag, Berlin, 1986.

[15] E. Hernández, L. Hervella-Nieto, and R. Rodríguez, *Computation of the vibration modes of plates and shells by low-order MITC quadrilateral finite elements*, Comp. Struct., 81 (2003) 615–628.

[16] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1995.

[17] E. Liberman, *Sobre la convergencia de métodos de elementos finitos para el modelo de placas de Reissner-Mindlin*, Ph.D. Thesis, Universidad Nacional de La Plata, Argentina, 1995.

[18] P. Peisker and D. Braess, *Uniform convergence of mixed interpolated elements for Reissner-Mindlin plates*, M^2AN, 26 (1992) 557–574.

[19] J. Pitkäranta and R. Stenberg, *Error bounds for the approximation of the Stokes problem using bilinear/constant elements on irregular quadrilateral meshes*, in The Mathematics of Finite Elements and Applications V, J. R. Whiteman, ed., Academic Press, London, 1985, pp. 325–334,

[20] J. Pitkäranta and M. Suri, *Design principles and error analysis for reduced-shear plate-bending finite elements*, Numer. Math., 75 (1996) 223–266.

[21] P. A. Raviart and J. M. Thomas, *A mixed finite element method for second order elliptic problems*, in Mathematical Aspects of Finite Element Methods, I. Galligani and E. Magenes, eds., Lecture Notes in Mathematics, Vol. 606, Springer, Berlin, 1977, pp. 292–315.

[22] R. Stenberg, *Analysis of mixed finite element methods for the Stokes problem: a unified approach*, Math. Comp., 42 (1984) 9–23.

[23] R. Stenberg and M. Suri, *An hp error analysis of MITC plate elements*, SIAM J. Numer. Anal., 34 (1997) 544-568.

[24] J. M. Thomas, *Sur l'analyse numérique des méthodes d'éléments finis hybrides et mixtes*, Thèse de Doctorat d'Etat, Université Pierre et Marie Curie, Paris 6, France, 1977.