A posteriori error estimators for mixed approximations of eigenvalue problems

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Abstract

In this paper we introduce and analyze an a posteriori error estimator for the approximation of the eigenvalues and eigenvectors of a second order elliptic problem obtained by the mixed finite element method of Raviart-Thomas of the lowest order.

We define an error estimator of the residual type which can be computed locally from the approximate eigenvector and prove that the estimator is equivalent to the norm of the error in the approximation of the eigenvector up to higher order terms. The constants involved in this equivalence depend on the corresponding eigenvalue but are independent of the mesh size, provided the meshes satisfy the usual minimum angle condition.

Moreover, the square root of the error in the approximation of the eigenvalue is also bounded by a constant times the estimator.

1 Introduction

The object of this paper is to introduce and analyze a-posteriori error estimators for Raviart-Thomas mixed finite element approximation of second order elliptic eigenvalue problems. The interest of the mixed formulation for eigenvalue problems comes from its applications in the computation of vibration modes of a fluid in a displacement formulation. In particular, in problems involving fluid-structure interaction, the use of the displacement formulation for the fluid is convenient because it leads to standard symmetric eigenvalue problems while the use of the pressure or potential as variable leads to non symmetric eigenvalue problems due to the interface condition (see for example [19]).

Several approaches have been used to compute the vibration modes of a fluid in the displacement formulation (see [11, 17, 3, 4, 15]). Among them, the use of the Raviart-Thomas method seems one of the most natural choices since it can be proved that it does not produce spurious eigenvalues and it has optimal order convergence. This has been proved also for fluid-structure interaction problems and very good numerical results have been obtained [3, 4, 8, 21].

The bibliography on a posteriori error estimators for finite element approximations is now very extensive (we refer to the book [26] and references therein). For standard finite element approximations of eigenvalue problems, error estimators were first introduced by Babuska and Rheinboldt [7] for one dimensional equations and more recently by Verfürth [25] as an application of his general results for non linear problems.

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As it is well known, the mixed finite element method of Raviart and Thomas is strongly related with the non conforming piecewise linear approximation of Crouzeix and Raviart (see [2, 18]). We will use this relation in our arguments which are based in some of the ideas introduced in [14, 13] for a posteriori error estimators in non-conforming methods which were also used in [1, 9] for a family of mixed finite element approximations.

We define an error estimator of the residual type for both eigenvectors and eigenvalues and prove that it is equivalent to the error up to higher order terms. The results are valid for a general simply connected polygonal domain and general meshes satisfying the usual regularity assumption and the estimator is easy to compute locally.

An outline of the rest of the paper is as follows. In Section 2 we prove some auxiliary results by using the equivalence between the mixed approximation and a non conforming method and in Section 3 we introduce the error estimator and prove the main results.

2 Statement of the problem and relation with the non conforming method

Let $\Omega \subset \mathbb{R}^2$ be a simply connected polygonal domain. We consider the eigenvalue problem

$$\begin{cases} -\Delta p = \lambda p & \text{in } \Omega \\ p = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Introducing $\mathbf{u} = -\nabla p$ we obtain the usual mixed formulation of problem (2.1) which in weak form is given by: find $\mathbf{u} \in H(\mathrm{div}\,,\Omega)$ and $p \in L^2(\Omega)$ such that

$$\begin{cases} \int_\Omega \mathbf{u}\cdot\mathbf{v} - \int_\Omega p\,\mathrm{div}\;\mathbf{v} = 0 & \forall\;\; \mathbf{v} \in H(\mathrm{div}\,,\Omega) \\ \int_\Omega \mathrm{div}\;\mathbf{u}\,q = \lambda \int_\Omega p\,q & \forall\;\; q \in L^2(\Omega). \end{cases} \tag{2.2}$$

Let $\{\mathcal{T}_h\}$ be a regular family of triangulations (i.e., satisfying the minimum angle condition), where h denotes the mesh size. The lowest order space of Raviart-Thomas is defined as [22, 5]

$$RT_0(K) = \mathcal{P}_0(K)^2 + (x_1, x_2)\,\mathcal{P}_0(K)$$

where $\mathcal{P}_k(K)$ denotes the space of restrictions to K of polynomials of total degree not greater than k. Define V_h and Q_h as

$$V_h = \{\mathbf{v} \in H(\mathrm{div}\,,\Omega) : \mathbf{v}|_K \in RT_0(K)\,,\,\forall K \in \mathcal{T}_h\}$$

and,

$$Q_h = \{q \in L^2(\Omega) : q|_K \in \mathcal{P}_0(K)\,,\,\forall K \in \mathcal{T}_h\}$$

then, the mixed finite element approximation of problem (2.2) reads: find $\mathbf{u}_h \in V_h$ and $p_h \in Q_h$ such that

$$\begin{cases} \int_\Omega \mathbf{u}_h\cdot\mathbf{v} - \int_\Omega p_h\,\mathrm{div}\;\mathbf{v} = 0 & \forall\;\; \mathbf{v} \in V_h \\ \int_\Omega \mathrm{div}\;\mathbf{u}_h\,q = \lambda_h \int_\Omega p_h\,q & \forall\;\; q \in Q_h. \end{cases} \tag{2.3}$$

Assume, for simplicity, that λ is a simple eigenvalue. Then, taking $\|p\|_0 = \|p_h\|_0 = 1$, it follows from the abstract theory [20] and known a priori estimates that, for h small enough (depending on λ),

$$\|\mathbf{u} - \mathbf{u}_h\|_0 + \|p - p_h\|_0 = O(h^\alpha)$$

and

$$|\lambda - \lambda_h| = O(h^{2\alpha})$$

where $\alpha = 1$ if Ω is convex and $\alpha = \pi/\omega - \varepsilon$ for a non convex Ω, $\omega < 2\pi$ being the maximum interior angle of Ω.

Following the arguments given in [2, 18] it can be seen that problem (2.3) is equivalent to a non conforming approximation of the standard formulation of (2.1). Indeed, let us introduce the non conforming space of Crouzeix and Raviart enriched by local bubbles. Denoting by $\mathcal{B}(K)$ the space of cubic polynomials vanishing on ∂K we define

$$\begin{aligned} X_h = \;& \{\phi \in L^2(\Omega) : \; \phi|_K \in \mathcal{P}_1(K), \; \forall K \in \mathcal{T}_h, \; \phi \text{ is continuous at interior} \\ & \text{midside points and vanishes at boundary midside points}\}, \\ B_h = \;& \{b \in H^1_0(\Omega) : \; b|_K \in \mathcal{B}(K) \; \forall K \in \mathcal{T}_h\}, \\ W_h = \;& X_h \oplus B_h. \end{aligned} \tag{2.4}$$

Let

$$V_h^d = \{\mathbf{v} \in L^2(\Omega)^2 : \mathbf{v}|_K \in RT_0(K) \; \forall \, K \in \mathcal{T}_h\} \tag{2.5}$$

then we introduce the following L^2-projection operators:

$$\begin{array}{ll} P : L^2(\Omega) \to Q_h & \Pi : L^2(\Omega)^2 \to V_h^d \\ P\,p \in Q_h \text{ such that} & \Pi\mathbf{u} \in V_h^d \text{ such that} \\ \int_\Omega (p - P\,p)\,q = 0 \quad \forall q \in Q_h & \int_\Omega (\mathbf{u} - \Pi\,\mathbf{u})\,\mathbf{v} = 0 \quad \forall \mathbf{v} \in V_h^d. \end{array} \tag{2.6}$$

Then, it can be seen that problem (2.3) is equivalent to: find $\phi_h \in W_h$ such that

$$\sum_{K \in \mathcal{T}_h} \int_K \Pi(\nabla\phi_h) \cdot \nabla\psi = \lambda_h \int_\Omega P\phi_h\,\psi \quad \forall \psi \in W_h, \tag{2.7}$$

in the sense that they have the same eigenvalues λ_h and the eigenvectors are related by $\mathbf{u}_h = -\Pi(\nabla\phi_h)$ and $p_h = P\phi_h$. We omit the details which are as in [2, 18].

Applying to problem (2.7) the general theory developed for example in [6] we can prove the following a priori error estimate for the eigenvectors

$$\|p - \overline{\phi}_h\|_0 = O(h^{2\alpha}) \tag{2.8}$$

where $\overline{\phi}_h = \phi_h / \|\phi_h\|_0$.

In fact, let T and T_h be defined as the inverse operators of the continuous and discrete problems, i.e., for $f \in L^2(\Omega)$, $Tf = \tilde{p} \in H^1_0(\Omega)$ and $T_h f = \tilde{\phi}_h \in W_h$ where $\tilde{p}$ and $\tilde{\phi}_h$ satisfy respectively

$$\int_\Omega \nabla\tilde{p} \cdot \nabla q = \int_\Omega f\,q \quad \forall q \in H^1_0(\Omega) \tag{2.9}$$

and,

$$\sum_{K \in \mathcal{T}_h} \int_K \Pi(\nabla\tilde{\phi}_h) \cdot \nabla\psi = \int_\Omega Pf\,\psi \quad \forall \psi \in W_h. \tag{2.10}$$

Well-known results in the theory of spectral approximation yield the following error estimate for eigenvectors provided the sequence of operators $\{T_h\}_h$ converges uniformly to T in $\mathcal{L}(L^2, L^2)$, see e.g. [6],

$$\|p - \overline{\phi}_h\|_0 \le C \|(T - T_h)|_E\|_{\mathcal{L}(L^2,L^2)} \tag{2.11}$$

where E denotes the eigenspace corresponding to λ. Then (2.8) is a consequence of (2.11) and of the lemmas below.

Before proving the lemmas let us introduce some notation and recall some approximation results that we will use. Let $S = S^i \cup S^b$ be the set of all edges of the triangulation, with S^i being the interior ones and S^b those on the boundary. For $\psi \in H^1(\Omega)$, $K \in \mathcal{T}_h$ and $\ell \in S$, $\ell \subset \partial K$ there exists a piecewise linear continuous approximation ψ_I [12, 23] which satisfies

$$\|\psi - \psi_I\|_{0,\ell} \le C h_\ell^{1/2} \|\psi\|_{1,\widetilde{K}} \tag{2.12}$$

and,

$$\|\psi - \psi_I\|_{0,K} + h_K \|\psi - \psi_I\|_{1,K} \le C h_K^{1+\gamma} \|\psi\|_{1+\gamma,\widetilde{K}} \quad , \quad 0 \le \gamma \le 1 \tag{2.13}$$

where $\widetilde{K}$ is the union of all neighbors of K, h_ℓ is the length of ℓ and, h_K is the diameter of K. Moreover, for $\psi \in H_0^1(\Omega)$ we can take $\psi_I \in H_0^1(\Omega)$.

Recall also that the L^2-projection satisfies

$$\|q - Pq\|_{0,K} \le C h_K^\gamma |q|_{\gamma,K} \quad , \quad 0 \le \gamma \le 1. \tag{2.14}$$

We will also make use of the non conforming piecewise linear edge average interpolant. For $p \in H^1(\Omega)$ let p_A linear on each K and defined by

$$\int_\ell p_A = \int_\ell p \quad \forall \ell \in S.$$

Note that integrating by parts we have

$$\int_K \nabla(p - p_A) \cdot \nabla q = 0 \quad \forall q \in \mathcal{P}_1(K). \tag{2.15}$$

Standard arguments show that,

$$\|\nabla(p - p_A)\|_{0,K} \le h^\gamma |p|_{1+\gamma} \ , \quad 0 \le \gamma \le 1. \tag{2.16}$$

Lemma 2.1

$$\|(T - T_h)|_E\|_{\mathcal{L}(L^2,L^2)} = O(h^{2\alpha})$$

Proof. We have to show that

$$\|Tf - T_h f\|_0 \le C h^{2\alpha} \|f\|_0 \quad \forall f \in E.$$

Let $\tilde{p} = Tf$ and $\tilde{\phi}_h = T_h f$. Since $\tilde{\phi}_h \in W_h$ it can be written as $\tilde{\phi}_h = \tilde{p}_h + \tilde{\beta}_h$ with $\tilde{p}_h \in X_h$ and $\tilde{\beta}_h \in B_h$.

To simplify notation let us call $\nabla_h q$ the L^2 vector function such that $\nabla_h q|_K = \nabla q$. Notice that $\nabla_h X_h \subset V_h^d$. Taking a piecewise linear test function $\psi = q \in X_h$ in (2.10) and observing that $\Pi(\nabla \tilde{p}_h) = \nabla \tilde{p}_h$ and, $\int_\Omega \Pi(\nabla \tilde{\beta}_h) \cdot \nabla q = 0$ we obtain,

$$\int_\Omega \nabla_h \tilde{p}_h \cdot \nabla_h q = \int_\Omega P f \, q. \tag{2.17}$$

Denoting by $e = \tilde{p} - \tilde{p}_h$ and subtracting (2.17) from (2.9) we have

$$\int_\Omega \nabla_h e \cdot \nabla_h q = \int_\Omega (f - Pf) \, q + \sum_{\ell \in S^i} \int_\ell \frac{\partial \tilde{p}}{\partial n} [\![q]\!]_\ell + \sum_{\ell \in S^b} \int_\ell \frac{\partial \tilde{p}}{\partial n} q \tag{2.18}$$

where $[\![q]\!]_\ell$ denotes the jump of q along the edge ℓ. Let us now take $\tilde{q} = \tilde{p}_A - \tilde{p}_h$ in the error equation (2.18) and using (2.15), then we have,

$$\begin{aligned}
\|\nabla_h \tilde{q}\|_0^2 &= \int_\Omega \nabla_h e \cdot \nabla_h \tilde{q} \\
&= \int_\Omega (f - Pf)\,\tilde{q} + \sum_{\ell\in S} \int_\ell \tfrac{\partial \tilde{p}}{\partial n}\,[\![\tilde{q}]\!]_\ell + \sum_{\ell\in S^b} \int_\ell \tfrac{\partial \tilde{p}}{\partial n}\,\tilde{q} \\
&= \int_\Omega (f - Pf)\,(\tilde{q} - P\tilde{q}) + \sum_{\ell\in S} \int_\ell \tfrac{\partial \tilde{p}}{\partial n}\,[\![\tilde{q}]\!]_\ell + \sum_{\ell\in S^b} \int_\ell \tfrac{\partial \tilde{p}}{\partial n}\,\tilde{q} \\
&\le C\{h\|f - Pf\|_0 \|\nabla_h \tilde{q}\|_0 + h^\alpha \|\tilde{p}\|_{1+\alpha} \|\nabla_h \tilde{q}\|_0\}
\end{aligned}$$

where the last terms can be bounded in the usual way: namely, for each $\ell \in S^i$ (resp. $\ell \in S^b$), $[\![\tilde{q}]\!]_\ell$ (resp. $\tilde{q}|_\ell$) is a linear function vanishing at the midpoint of ℓ, hence a trace theorem (which can be applied because $\alpha > 1/2$) and a scaling argument (see for example [10]) yield the estimate.

Therefore, by using the a priori estimate $\|\tilde{p}\|_{1+\alpha} \le \|f\|_0$ (see [16]) and (2.16) we obtain,

$$\|\nabla_h e\|_0 \le C h^\alpha \|f\|_0. \tag{2.19}$$

In order to conclude the proof we have to obtain a double order error estimate in L^2. Let $r \in H_0^1(\Omega)$ be the solution of $-\Delta r = e$. Then, a standard duality argument gives,

$$\|e\|_0^2 = \int_\Omega \nabla_h e \cdot \nabla r - \sum_{\ell\in S^i} \int_\ell \frac{\partial r}{\partial n}\,[\![e]\!]_\ell - \sum_{\ell\in S^b} \int_\ell \frac{\partial r}{\partial n}\,e. \tag{2.20}$$

Using (2.20) and (2.18) with $q = r_I$ and observing that the last terms in (2.18) vanishes since r_I is continuous we have,

$$\|e\|_0^2 = \int_\Omega \nabla_h e \cdot \nabla(r - r_I) - \sum_{\ell\in S} \int_\ell \frac{\partial r}{\partial n}\,[\![e]\!]_\ell + \int_\Omega (f - Pf)\,(r_I - Pr_I). \tag{2.21}$$

Then, due to (2.13), (2.14) and again to the a priori estimate $\|r\|_{1+\alpha} \le C\|e\|_0$ we obtain,

$$\|e\|_0 \le C\{h^\alpha \|\nabla_h e\|_0 + h^2 \|f\|_1\}$$

where the terms involving the integrals on ℓ have been bounded as before. This estimate combined with (2.19) yields,

$$\|e\|_0 \le C h^{2\alpha} \|f\|_1$$

but, since $f \in E$ there exists a constant C depending on λ such that $\|f\|_1 \le C\|f\|_0$ and so,

$$\|e\|_0 \le C h^{2\alpha} \|f\|_0.$$

Now,

$$\|Tf - T_h f\|_0 = \|\tilde{p} - \tilde{\phi}_h\|_0 \le \|\tilde{p} - \tilde{p}_h\|_0 + \|\tilde{\beta}_h\|_0$$

so, it remains to estimate $\|\tilde{\beta}_h\|_0$. Observe that $\tilde{\beta}_{h,K} = \tilde{\beta}_h|_K$ satisfies,

$$\int_K \Pi(\nabla \tilde{\beta}_{h,K}) \cdot \nabla \tilde{\beta}_{h,K} = \int_K Pf\,\tilde{\beta}_{h,K}$$

hence, writing $\tilde{\beta}_{h,K} = c_K b_K$ with b_K being a bubble function such that $b_K = 1$ at the barycenter $\mathbf{x}_K$ of K it follows that,

$$c_K = \frac{\int_K Pf\,b_K}{\|\Pi(\nabla b_K)\|_{0,K}^2}.$$

Now, since

$$\int_K \Pi(\nabla b_K) = \int_K \nabla b_K = 0$$

we have

$$\Pi(\nabla b_K) = \mu(\mathbf{x} - \mathbf{x}_K)$$

and using

$$\int_K \Pi(\nabla b_K)\cdot(\mathbf{x}-\mathbf{x}_K) = \int_K \nabla b_K\cdot(\mathbf{x}-\mathbf{x}_K) = -2\int_K b_K$$

we obtain

$$\mu = -\frac{2\int_K b_K}{\int_K |\mathbf{x}-\mathbf{x}_K|^2}.$$

Therefore, the regularity assumption on the meshes yields,

$$\frac{\|b_K\|_{0,K}}{\|\Pi(\nabla b_K)\|_{0,K}^2} \leq C h_K$$

and so,

$$|c_K| \leq C h_K \|f\|_{0,K};$$

hence it follows that,

$$\|\nabla \beta_h\|_0 \leq C h \|f\|_0.$$

and the Poincaré inequality on each element implies

$$\|\beta_h\|_0 \leq C h^2 \|f\|_0$$

which concludes the proof. ∎

Let us now prove the uniform convergence of T_h to T, which is required to have (2.11).

Lemma 2.2 *The sequence $\{T_h\}_h$ converges uniformly to T in $\mathcal{L}(L^2, L^2)$ as h goes to 0.*

Proof. We show that for all $f \in L^2(\Omega)$ we have

$$||Tf - T_h f||_0 \leq Ch||f||_0.$$

The proof follows the same lines as the previous one. The fact that f belonged to the eigenspace E was used only once to estimate (2.21) with the desired order. When f is taken in $L^2(\Omega)$ we can only obtain from (2.21) the following bound, using similar arguments as before

$$\begin{aligned}
\|e\|_0^2 &= \textstyle\int_\Omega \nabla_h e\cdot\nabla(r - r_I) - \sum_{\ell\in S}\int_\ell \frac{\partial r}{\partial n}[\![e]\!]_\ell + \int_\Omega (f - Pf)(r_I - Pr_I) \\
&= \textstyle\int_\Omega \nabla_h e\cdot\nabla(r - r_I) - \sum_{\ell\in S}\int_\ell \frac{\partial r}{\partial n}[\![e]\!]_\ell + \int_\Omega f(r_I - Pr_I) \\
&\leq C\{h^\alpha||\nabla_h e||_0 + h||f||_0\}||e||_0 \\
&\leq Ch||f||_0||e||_0.
\end{aligned}$$

And this implies the desired bound. ∎

3 A posteriori error estimator

Let us go back to the Raviart-Thomas approximation (2.3) of the mixed problem (2.2). We start by giving an estimate for the error in the approximation of the eigenvector in terms of the discrete solution. We will denote with h.o.t a "higher order term" and for a vector field $\mathbf{v} \in L^2(\Omega)^2$,

$$d(\mathbf{v}, \nabla H_0^1) = \inf_{s\in H_0^1(\Omega)} \|\mathbf{v} - \nabla s\|_0$$

The generic constant C appearing in our estimates depends on λ. Although it is not difficult to trace this dependence, we will not state it explicitely to simplify notation.

Lemma 3.1 *The following estimate holds:*

$$\|\mathbf{u} - \mathbf{u}_h\|_{H(div,\Omega)} + \|p - p_h\|_0 \leq d(\mathbf{u}_h, \nabla H_0^1) + C\|\phi_h - P\phi_h\|_0 + \ h.o.t \tag{3.22}$$

where,

$$h.o.t \ = C\{|\lambda - \lambda_h| + \|p - \overline{\phi}_h\|_0\}$$

with $\overline{\phi}_h = \phi_h/\|\phi_h\|_0$ *and* ϕ_h *solution of (2.7).*

Proof. First we estimate the error in the scalar variable. Since $p_h = P\phi_h$ we have,

$$p - p_h = p - \overline{\phi}_h + \overline{\phi}_h - P\phi_h \tag{3.23}$$

but, since $\|P\phi_h\|_0 = 1$, we have also

$$\|\phi_h - \overline{\phi}_h\|_0 = \|\phi_h\|_0 - \|P\phi_h\|_0 \leq \|\phi_h - P\phi_h\|_0$$

and so,

$$\|\overline{\phi}_h - P\phi_h\|_0 \leq 2\|\phi_h - P\phi_h\|_0$$

which combined with (3.23) yields the estimate for the scalar variable.

Now observe that from the equations (2.2) and (2.3) we have,

$$\text{div } (\mathbf{u} - \mathbf{u}_h) = \lambda p - \lambda_h p_h = \lambda(p - p_h) + (\lambda - \lambda_h)p_h$$

and so, since $\|p_h\|_0 = 1$, we obtain,

$$\|\text{div } (\mathbf{u} - \mathbf{u}_h)\|_0 \leq \lambda\|(p - p_h)\|_0 + |(\lambda - \lambda_h)| \tag{3.24}$$

Therefore, it only remains to estimate the L^2-norm of the error in the vector variable. Decompose the error as,

$$\mathbf{u} - \mathbf{u}_h = \nabla s + \text{rot } \psi$$

with $s \in H_0^1(\Omega)$ and $\psi \in H^1(\Omega)$ with zero mean value. Since this is an orthogonal decomposition and $\mathbf{u} = \nabla p$ we have,

$$\|\text{rot } \psi\|_0 = d(\mathbf{u}_h, \nabla H_0^1). \tag{3.25}$$

Therefore, to conclude the proof we have to estimate $\|\nabla s\|_0$. Now,

$$\|\nabla s\|_0^2 = \int_\Omega (\mathbf{u} - \mathbf{u}_h) \cdot \nabla s = -\int_\Omega \text{div } (\mathbf{u} - \mathbf{u}_h)\, s \leq C\|\text{div } (\mathbf{u} - \mathbf{u}_h)\|_0 \|\nabla s\|_0$$

which, in view of (3.24) concludes the proof. ∎

Next we give a relation between the error for the eigenvalue and the error for the eigenvector.

Lemma 3.2 *The following equation holds,*

$$\lambda - \lambda_h = \|\mathbf{u} - \mathbf{u}_h\|_0^2 - \lambda_h\|p - p_h\|_0^2 \tag{3.26}$$

and consequently,

$$|\lambda - \lambda_h|^{1/2} \leq C\{\|\mathbf{u} - \mathbf{u}_h\|_0 + \|p - p_h\|_0\} \tag{3.27}$$

Proof. Taking $\mathbf{v} = \mathbf{u}_h$ in the first equation of (2.2) and using that div $\mathbf{u}_h = \lambda_h p_h$ we obtain that $\int_\Omega \mathbf{u} \cdot \mathbf{u}_h = \lambda_h \int_\Omega p\, p_h$. In a similar way, using (2.2) and (2.3), we can see that $\|\mathbf{u}\|_0^2 = \lambda \|p\|_0^2$ and that $\|\mathbf{u}_h\|_0^2 = \lambda_h \|p_h\|_0^2$. Therefore, since $\|p\|_0 = \|p_h\|_0 = 1$ we have $\|\mathbf{u}\|_0^2 = \lambda$ and $\|\mathbf{u}_h\|_0^2 = \lambda_h$ and therefore,

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_0^2 &= \lambda - 2\textstyle\int_\Omega \mathbf{u} \cdot \mathbf{u}_h + \lambda_h = \lambda - 2\lambda_h \int_\Omega p\, p_h + \lambda_h \\ &= \lambda - \lambda_h + \lambda_h (2 - 2\textstyle\int_\Omega p\, p_h) \end{aligned}$$

and the proof of (3.26) concludes by observing that, since $\|p\|_0 = \|p_h\|_0 = 1$, we have

$$2 - 2\int_\Omega p\, p_h = \|p - p_h\|_0^2.$$

Now, the estimate (3.27) follows immediately from (3.26) since $\lambda_h \to \lambda$ and so we can bound λ_h by a constant depending on λ. ∎

In order to define a practical estimator we have to bound the terms appearing on the right hand side of (3.22) in terms of computable quantities. With this goal we give in the next lemma an explicit expression of $\phi - P\phi_h$ in terms of $\mathbf{u}_h$. We will denote with $\phi_{h,K}$ and $p_{h,K}$ the restrictions to K of ϕ_h and p_h.

Lemma 3.3 *On each element K we have*

$$\phi_{h,K} - P\phi_{h,K} = -\frac{1}{|K|}\Big(\int_K \mathbf{u}_h\Big) \cdot (\mathbf{x} - \mathbf{x}_K) + \frac{1}{2}\Big(\frac{b_K}{\int_K b_K} - \frac{1}{|K|}\Big) \int_K (\mathbf{u}_h \cdot (\mathbf{x} - \mathbf{x}_K)) \tag{3.28}$$

where b_K is the bubble function such that $b_K = 1$ at the barycenter $\mathbf{x}_K$ of K.

Proof. Since $\mathbf{u}_h = -\Pi(\nabla \phi_h)$ we have

$$\int_K \nabla \phi_h = -\int_K \mathbf{u}_h \tag{3.29}$$

and,

$$\int_K \nabla \phi_h \cdot (\mathbf{x} - \mathbf{x}_K) = -\int_K \mathbf{u}_h \cdot (\mathbf{x} - \mathbf{x}_K). \tag{3.30}$$

Writing $\phi_h = \overline{p}_h + \beta_h$ with $\overline{p}_h \in X_h$ and $\beta_h \in B_h$ we have

$$\phi_{h,K} - P\phi_{h,K} = \nabla \overline{p}_{h,K} \cdot (\mathbf{x} - \mathbf{x}_K) + \alpha_K \Big(b_K - \frac{1}{|K|}\int_K b_K\Big). \tag{3.31}$$

Now, observing that $\int_K \nabla \beta_h = 0$, it follows from (3.29) that

$$\nabla \overline{p}_{h,K} = -\frac{1}{|K|}\int_K \mathbf{u}_h$$

which combined with (3.31) yields

$$\phi_{h,K} - P\phi_{h,K} = -\frac{1}{|K|}\Big(\int_K \mathbf{u}_h\Big) \cdot (\mathbf{x} - \mathbf{x}_K) + \alpha_K \Big(b_K - \frac{1}{|K|}\int_K b_K\Big). \tag{3.32}$$

On the other hand, since $\int_K \nabla \overline{p}_h \cdot (\mathbf{x} - \mathbf{x}_K) = 0$ we obtain from (3.30) that

$$\int_K \nabla \beta_h \cdot (\mathbf{x} - \mathbf{x}_K) = -\int_K \mathbf{u}_h \cdot (\mathbf{x} - \mathbf{x}_K) \tag{3.33}$$

and so, if we write $\beta_h = \sum_K \alpha_K b_K$ we have

$$\alpha_K \int_K \nabla b_K \cdot (\mathbf{x} - \mathbf{x}_K) = -\int_K \mathbf{u}_h \cdot (\mathbf{x} - \mathbf{x}_K) \tag{3.34}$$

but $\int_K \nabla b_K \cdot (\mathbf{x} - \mathbf{x}_K) = -2\int_K b_K$ and therefore,

$$\alpha_K = \frac{\int_K \mathbf{u}_h \cdot (\mathbf{x} - \mathbf{x}_K)}{2\int_K b_K} \tag{3.35}$$

which together with (3.32) concludes the proof. ∎

Now we can define the error estimator and, in the next theorem, we prove that it is a bound for the error up to higher order terms.

For an element K let S_K be the set of edges of K and t be the unit tangent on ℓ oriented clockwise. Define $J_\ell = [\![\mathbf{u}_h \cdot t]\!]_\ell$ if ℓ is an interior edge and $J_\ell = 2\mathbf{u}_h \cdot t$ if ℓ is on the boundary of Ω.

Then, the local error estimator is defined by,

$$\eta_K^2 = \sum_{\ell \in S_K} \|J_\ell\|_{0,\ell}^2 h_\ell + \|\mathbf{u}_h\|_{0,K}^2 h_K^2$$

and the global one by,

$$\eta^2 = \sum_{K \in \mathcal{T}_h} \eta_K^2 .$$

Theorem 3.1 *The following estimate holds:*

$$\|\mathbf{u} - \mathbf{u}_h\|_{H(div,\Omega)} + \|p - p_h\|_0 \leq C\eta + h.o.t$$

and,

$$|\lambda - \lambda_h|^{1/2} \leq C\eta + h.o.t$$

where h.o.t is as in Lemma 3.1.

Proof. We use the notation of Lemma 3.1. In view of Lemmas 3.1 and 3.2 it is enough to prove that,

$$d(\mathbf{u}_h, \nabla H_0^1) = \|\text{rot } \psi\|_0 \leq C\{\sum_{\ell \in S} \|J_\ell\|_{0,\ell}^2 h_\ell\}^{1/2} \tag{3.36}$$

and,

$$\|\phi_h - P\phi_h\|_0 \leq C\{\sum_{K \in \mathcal{T}_h} \|\mathbf{u}_h\|_{0,K}^2 h_K^2\}^{1/2}. \tag{3.37}$$

Let ψ_I as in Section 2. Then, rot $\psi_I \in V_h$ and is divergence free. So, from (2.3) we see that,

$$\int_\Omega \mathbf{u}_h \cdot \text{rot } \psi_I = 0$$

and therefore, since $\mathbf{u} = \nabla p$ and $\mathbf{u}_h \in V_h$, we have,

$$\begin{aligned} \|\text{rot } \psi\|_0^2 &= \int_\Omega (\mathbf{u} - \mathbf{u}_h) \cdot \text{rot } \psi \\ &= -\int_\Omega \mathbf{u}_h \cdot \text{rot } \psi = -\int_\Omega \mathbf{u}_h \cdot \text{rot } (\psi - \psi_I) \\ &= -\sum_{K \in \mathcal{T}_h} \int_K \text{rot } \mathbf{u}_h (\psi - \psi_I) - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \mathbf{u}_h \cdot t\, (\psi - \psi_I) \\ &= -\tfrac{1}{2} \sum_{\ell \in S} \int_\ell J_\ell\, (\psi - \psi_I) \end{aligned} \tag{3.38}$$

and so, (3.36) follows from the Schwartz inequality and (2.12).

Finally, in order to obtain (3.37) we compute $\|\phi_h - P\phi_h\|_{0,K}$for each element K using Lemma 3.3. Since the meshes satisfy the regularity assumption, we can use an inverse inequality to estimate the last term in (3.28) as follows

$$\left\| \frac{b_K}{\int_K b_K} - \frac{1}{|K|} \right\|_{0,K}^2 \leq \frac{C}{|K|},$$

and therefore, (3.37) follows easily. ∎

In order to have an estimator useful for practical adaptive refinement, a property usually required is that the estimator bounds the error also by below. Our next theorem gives a proof of this fact for our case. The argument is based on the technique introduced in [24].

Theorem 3.2 *For any element $K \in \mathcal{T}_h$,*

$$\|\mathbf{u}_h\|_{0,K} h_K \leq C\{\|\mathbf{u} - \mathbf{u}_h\|_{0,K} + \|p - p_h\|_{0,K}\},$$

for any interior edge $\ell \in S_{K_1} \cap S_{K_2}$,

$$\|J_\ell\|_{0,\ell} h_\ell^{1/2} \leq C\|\mathbf{u} - \mathbf{u}_h\|_{0,K_1 \cup K_2}$$

and for any boundary edge $\ell \in S_K$,

$$\|J_\ell\|_{0,\ell} h_\ell^{1/2} \leq C\|\mathbf{u} - \mathbf{u}_h\|_{0,K}.$$

In particular,

$$\eta_K \leq C\{\|\mathbf{u} - \mathbf{u}_h\|_{0,K^*} + \|p - p_h\|_{0,K}\}$$

where K^* denotes the union of all elements sharing an edge with K.

Proof. First observe that in view of (2.2) and (2.3), for any $\mathbf{v} \in H(div, \Omega)$ vanishing on ∂K we have,

$$\int_K (\mathbf{u} - \mathbf{u}_h) \cdot \mathbf{v} - \int_K (p - p_h) \operatorname{div} \mathbf{v} = - \int_K \mathbf{u}_h \cdot \mathbf{v} \tag{3.39}$$

and so, choosing $\mathbf{v} = -\nu \mathbf{u}_h b_K$ with $\nu = \frac{\int_K |\mathbf{u}_h|^2 h_K^2}{\int_K |\mathbf{u}_h|^2 b_K}$ we get,

$$- \int_K \mathbf{u}_h \cdot \mathbf{v} = \|\mathbf{u}_h\|_{0,K}^2 h_K^2$$

therefore, it follows from (3.39) that,

$$\|\mathbf{u}_h\|_{0,K}^2 h_K^2 \leq \{\|\mathbf{u} - \mathbf{u}_h\|_{0,K} + \|p - p_h\|_{0,K}\} \|\mathbf{v}\|_{H(div,K)}. \tag{3.40}$$

Now, standard arguments based on the equivalence of norms in $\mathcal{P}_1^2$ on a reference element show that $\nu \leq C h_K^2$ and so, by inverse estimates we obtain

$$\|\mathbf{v}\|_{1,K} \leq C\|\mathbf{v}\|_{L^\infty(K)} \leq C\nu \|\mathbf{u}_h\|_{L^\infty(K)} \leq C\|\mathbf{u}_h\|_{0,K} h_K.$$

which together with (3.40) gives

$$\|\mathbf{u}_h\|_{0,K} h_K \leq C\{\|\mathbf{u} - \mathbf{u}_h\|_{0,K} + \|p - p_h\|_{0,K}\}. \tag{3.41}$$

On the other hand, for $\psi \in H^1(\Omega)$ we have,

$$
\begin{aligned}
\int_\Omega(\mathbf{u}-\mathbf{u}_h)\cdot \mathrm{rot}\ \psi &= -\int_\Omega \mathbf{u}_h\cdot \mathrm{rot}\ \psi = -\sum_{K\in\mathcal{T}_h}\int_{\partial K}\mathbf{u}_h\cdot t\,\psi\\
&= -\frac{1}{2}\sum_{K\in\mathcal{T}_h}\sum_{\ell\in S_K}\int_\ell J_\ell\,\psi.
\end{aligned}
$$

Now, let $\ell\in S_{K_1}\cap S_{K_2}$ (with obvious modifications for ℓ on the boundary) and take ψ such that supp $\psi\subset K_1\cup K_2$. Then,

$$\int_{K_1\cup K_2}(\mathbf{u}-\mathbf{u}_h)\cdot \mathrm{rot}\ \psi = \int_\ell J_\ell\,\psi. \tag{3.42}$$

Let b_ℓ be the piecewise quadratic basis function on $K_1\cup K_2$ which is equal to one at the midpoint of ℓ and vanishes on $\partial(K_1\cup K_2)$. Extend J_ℓ to $K_1\cup K_2$ as a linear function in such a way that it is constant along directions orthogonal to ℓ and define, $\psi=\delta J_\ell b_\ell$ with $\delta=\frac{\int_\ell J_\ell^2\,h_\ell}{\int_\ell J_\ell^2\,b_\ell}$. Then,

$$\int_\ell J_\ell\,\psi = \|J_\ell\|_{0,\ell}^2 h_\ell. \tag{3.43}$$

Again, standard arguments show that $\delta\le Ch_\ell$ and by inverse estimates we obtain,

$$\|\psi\|_{1,K_1\cup K_2}\le C\|J_\ell\|_{0,\ell}h_\ell^{1/2}.$$

This together with (3.42) and (3.43) yields,

$$\|J_\ell\|_{0,\ell}h_\ell^{1/2}\le C\|\mathbf{u}-\mathbf{u}_h\|_{0,K_1\cup K_2}$$

which completes the proof. ∎

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